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I. PHYSICS

Nucleonics

BINDING ENERGY OF TRI-NUCLEONS WITH GOLDHAMMER INTERACTION

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A set of variational calculations are made of shell-model S and D-states. Contribution to the binding energies of ^{3}H and ^{3}He using a Goldhammer interaction consisted of a Serber central force with a repulsive core and a tensor-even component. The upper-bound energies of ^{3}H and ^{3}He are found to be -7.338 MeV and -6.593 MeV respectively. The value of the rms charge radius of ^{3}H obtained from the present calculations is 1.69 fm in good argument with the experimental value.

Keywords : Binding energy; Tri-nucleons; Goldhammer Interaction

INTRODUCTION

JAHN (1958) has defined an important series of even parity $(N)\,^2S_{1/2}$ and $(N)\,^4D_{1/2}$ trial variational oscillator-well wave functions for the ground state of three particles, (^{3}H, ^{3}He) nucleus. Here N is an even integer and corresponds to quanta of excitation in the single, particle wave approach. We obtain here general expressions for $< A \mid A-2 >$ fractional parentage coefficients for ^{3}H and ^{3}He states. The fractional parentage coefficients for an orbital state with N quanta of excitation are shown to factorize, one set of factors being identified with the fractional parentage coefficients of a totally symmetric orbital state of N p-particles, which are here explicitly calculated. By the introduction of symbolic phonon distribution and phonon angular momentum states (the latter carrying the whole orbital angular momentum of the nucleus) a new sub-classification of all possible oscillator-well states of internal orbital motion of an A-particle nucleus is made. The new classification involves the representation of the unitary group U_3 and U_{A-1} in 3 and $A-1$ dimensions respectively.

The states for $A = 3$ are in part characterized as consisting of phonon distributions and phonon angular momentum part which are separately totally symmetric with respect to interchange of the phonon numbers (all possible orbital states are symmetric in the phonon numbers, but the separated phonon distribution and phonon angular momentum parts need not be). The states require the further characterization that one of the phonon distribution state constituents has all the quanta of excitation in a single two-body bond, the states being generated from such a state by application of Young operators. It is this fact which gives the states their importance, as it is only through this constituent, with two-body forces, that the N quantum states may be combined with the $N = 0$ state. For $A = 3$,

the states are unique in this way, so that all other N quantum states have zero coupling with the $N = 0$ state and contribute very little to the total binding energy.

The interaction used in this calculation is the equal range central-tensor forces interaction with a repulsive core given by Goldhammer (1963) and is charge symmetric. The parameters in this potential were chosen by Goldhammer.

Since the nucleons satisfy the Fermi-Dirac Statistics, the wave function of a nucleus must be totally antisymmetric in accordance with the Pauli exclusion principle. The totally antisymmetric wave function for a nucleus is built up from the vector-coupled orbital wave function of the nucleus belonging to a particular symmetry and the corresponding charge and spin wave function belonging to the adjoint symmetry. The vector coupled orbital wave function is an eigenfunction of the total orbital angular momentum operator **L**. Similarly the charge-spin functions are eigenfunctions of the total charge and total spin operator **T** and **S** respectively. Finally, the orbital angular momentum and spin angular momentum are vector coupled to a resultant **J**. In the present paper, we will consider $^{22}S_{1/2}$ and $^{24}D_{1/2}$ states of orbital symmetries [3] and [21] respectively.

The Hamiltonian H for the 3-particle nucleus is $H = T + V$ (here T and V stand for the kinetic energy and the potential energy respectively). In case of ^{3}He, there is an additional term C in the Hamiltonian which stands for the Coulomb interaction between the two protons.

A variational method has been applied to the three nucleon system, in which the wave function Ψ is expanded in terms of a complete set of orthonormal functions Ψ_i, and the energy is minimized with respect to the coefficients occurring in the expansion. Then E is the least eigenvalue of the matrix of **H**, i.e.,

$$E_0 \leqslant E = \int \Psi^* H \Psi \mathrm{d}\tau / \int \Psi^* \Psi \mathrm{d}\tau$$

the integration being taken over all the variables involved. This least eigenvalue is used to obtain an approximate value E to the true energy E_0.

Internal Coordinate Vectors and their Tranformation under Permutations

We define a complete set of non-dimensional internal coordinate vector $\mathbf{t}_i$ for an A-particle nucleus by

$$\mathbf{t}_i = \sum_{k=1}^{A} B_{ik}\mathbf{q}_k \quad (i = 1, 2, \ldots, A-1) \qquad \ldots(1)$$

($\mathbf{q}_k = \mathbf{r}_{k,b}$, b the oscillator-well parameter, $\mathbf{r}_k$ the vector drawn from the well centre to the k^{th} nucleon), with

$$\sum_{k=1}^{A} B_{ik} = 0, \quad \sum_{k=1}^{A} B_{ik}B_{jk} = \delta_{ij} \qquad \ldots(2)$$

The $\mathbf{t}_i (i = 1, 2, \ldots, A-1)$ are then algebraically orthonormal and algebraically orthogonal to the algebraically normalised non-dimensional centre of mass vector

$$\mathbf{t}_0 = \sum_{k=1}^{A} \frac{\mathbf{q}_k}{A^{1/2}} \qquad \ldots(3)$$

with this definition the kinetic energy remains the sum of squares and the reduced masses are all equal to m, the mass of a nucleon :

$$T = -\hbar^2 \sum_{k=1}^{A} \nabla^2_{q_k}/2mb^2$$

$$= -\hbar^2(\nabla^2_{t_0} + \sum_{i=1}^{A-1} \nabla^2_{t_i})/(2mb^2) \qquad \ldots(4)$$

The vectors $\mathbf{q}_k (k = 1, 2, \ldots, A)$ transform under permutations of the nucleon number k according to the reducible matrix representation

$$S_{[A]} + S_{[A-1,1]} \qquad \ldots(5)$$

of the symmetric group S_A, $\mathbf{t}_0$ is the totally symmetric constituent and the $\mathbf{t}_i (i = 1, 2, \ldots, A - 1)$ transform according to the irreducible matrix representations $S_{[A-1,1]}$:

$$P\mathbf{t}_i = \sum_{j=1}^{A-1} \mathbf{t}_j S^{[A-1,1]}_{ji}(p), \quad (i = 1, 2, \ldots, A - 1). \qquad \ldots(6)$$

(The dimension of the matrix representation $S_{[A-1,1]}$ of S_A is $A - 1$).

The particular choice of the internal coordinate vectors $\mathbf{t}_i$ depends on the particular form chosen, out of equivalent forms, for the representation $S_{[A-1,1]}$. We choose here the standard $\left(P_{A-1,A}, P_{A-3,A-2}, \ldots, \begin{Bmatrix} P_{12} \\ P_{23} \end{Bmatrix}\right)$-diagonal form [Jahn (1954)] for the representation with rows and columns labelled by Yamanouchi symbols, which may be attached to the corresponding internal ccordinate vectors $\mathbf{t}$. Such a choice of vectors is given by

$$\mathbf{t}(1^{a-2}, 21\ 1^{A-a}) = [2/a(a-2)]^{1/2}(\mathbf{q}_1 + \mathbf{q}_2 + \ldots + \mathbf{q}_{a-2}) - [(a-2)/(2a)]^{1/2}(\mathbf{q}_{a-1} + \mathbf{q}_a) \qquad \ldots(7)$$

$$\mathbf{t}(1^{a-2}, 12, 1^{A-a}) = (\tfrac{1}{2})^{1/2}(-\mathbf{q}_{a-1} + \mathbf{q}_a) \qquad \ldots(8)$$

both defined for $a = A, A - 2, \ldots, \begin{cases} 3 \\ 4 \end{cases}$, together with, for A even,

$$\mathbf{t}(12, 1^{A-2}) = (\tfrac{1}{2})^{1/2}(\mathbf{q}_1 - \mathbf{q}_2) \qquad \ldots(9)$$

The vector

$$\mathbf{t}(1^{A-2}, 12) = (\tfrac{1}{2})^{1/2}(-\mathbf{q}_{A-1} + \mathbf{q}_A), \qquad \ldots(10)$$

which is the algebraically normalised non-dimensional vector drawn from particle $A-1$ to particle A, is thus always included in the set; in the fractional parentage coefficient method the energy matrix is evaluated in terms of matrix elements involving this vector alone.

Thus in the case $A = 3$, we have two vectors

$$\mathbf{t}_1 = \mathbf{t}(1, 21) = (\tfrac{1}{6})^{1/2}\,(2\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3), \qquad \ldots(11)$$

$$\mathbf{t}_2 = \mathbf{t}(1, 12) = (\tfrac{1}{2})^{1/2} - (\mathbf{q}_2 + \mathbf{q}_3), \qquad \ldots(12)$$

transforming, as indicated, under permutations of the nucleon numbers, according to representation $S_{[21]}$ of the symmetric group S_3.

Totally Antisymmetric States

The totally antisymmetric states are obtained in the usual way by linear combination of orbital states symmetrized with respect to the nucleon permutation group S_A with charge spin states of adjoint symmetry and subsequent vector-coupling of **S** and **L** to resultant **J** :

$$\Psi = \left|{}^{2T+1}M_T^{2S+1}\, LJM\right> = \sum_{M_S M_L} \left(SM_S LM_L \,\middle|\, JM\right) \left(\frac{1}{n_{[A]}}\right)^{1/2}$$

$$\times \sum_{r} \left|{}^{2T+1}M_T^{2S+1}\, M_S \middle| \tilde{r}\right> . \,|\, LM_L \,|\, r>, \qquad \ldots(13)$$

where the sum is over the Yamanouchi symbols r or $\tilde{r}$ labelling the rows of the respective representations $S_{[A]}$, or $S_{[\tilde{A}]}$ (of dimension $n_{[A]}$, $[A] = [A_1, A_2, \ldots, A_k]$ a regular partition of A) to which the sommetrized orbital or charge-spin states belong.

The symmetrized charge-spin states may be obtained from certain selected vector-coupled charge-spin states as generating functions by normalization after application of the adjoint Young operators

$$0^{[\tilde{A}]}_{r\tilde{S}} = \left(\frac{n_{[\tilde{A}]}}{A!}\right) \sum_{p} S^{[\tilde{A}]}_{r\tilde{S}}\,(p)\,P$$

$$= \left(\frac{n_{[A]}}{A!}\right) \sum_{p} (-1)^p\, S^{[A]}_{rS}\,(p)\,P \qquad \ldots(14)$$

where p is the parity of P_{24}.

For the ${}^{22}S_{1/2}$, ${}^{24}D_{1/2}$ states for $A = 3$ we require only ${}^{22}[\tilde{3}]$ and ${}^{24}[\tilde{21}]$ charge-spin states and using the P_{23}-diagonal adjoint representation, we find

$$\sqrt{2}\; 0^{[\tilde{3}]}_{\tilde{1},\tilde{11};\tilde{1},\tilde{11}} \left|\, {}^{22}(\Upsilon_1),\, {}^{13}(\Upsilon_2\Upsilon_3);\, {}^{22}M_T M_S >\right.$$

(*continued on next page*)

$$= \frac{1}{\sqrt{2}} \Big| {}^{22}(\gamma_1), {}^{13}(\gamma_2\gamma_3); {}^{22}M_T M_S > - \frac{1}{\sqrt{2}} \Big| {}^{22}(\gamma_1), {}^{31}(\gamma_2\gamma_3); {}^{22}M_T M_S >, \quad \ldots(15)$$

$$0^{[\widetilde{21}]}_{\widetilde{1},\widetilde{21};\widetilde{1},\widetilde{21}} \Big| {}^{22}(\gamma_1), {}^{13}(\gamma_2\gamma_3); {}^{24}M_T M_S > = | {}^{22}(\gamma_1), {}^{13}(\gamma_2\gamma_3), {}^{24}M_T M_S > \quad \ldots(16)$$

$$0^{[\widetilde{21}]}_{\widetilde{1},\widetilde{12};\widetilde{1},\widetilde{21}} \Big| {}^{22}(\gamma_1), {}^{13}(\gamma_2\gamma_3), {}^{24}M_T M_S > = | {}^{22}(\gamma_1), {}^{33}(\gamma_2\gamma_3), {}^{24}M_T M_S >. \quad \ldots(17)$$

For $A = 3$, we associate the charge-spin states of particles 2 and 3 with the vector $\mathbf{t}(1, 12) = \frac{1}{\sqrt{2}}(-\mathbf{q}_2 + \mathbf{q}_3)$ and the charge-spin states of particle 1 with vector $\mathbf{t}(1, 21) = \frac{1}{\sqrt{6}}(2\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3)$ (this latter vector involves only the centre of mass motion for particles 2 and 3).

In the case of ^{3}H and ^{3}He, the ground state is of even parity, hence N (number of quanta of excitation) must be an even integer. Englefield (1956) has obtained general expressions for the ($n \mid n - 2, 2$) fractional parentage coefficients. He has shown that N-quantum fractional parentage coefficients for the orbital states factorise into two parts called A-coefficients and B-coefficients. We may then write our explicit expressions for our series of (^{3}H, ^{3}He) $^{22}S_{1/2}$ and $^{23}D_{1/2}$ states as follows :

$$| (N)\,[3]\,{}^2M_T\,{}^{22}S_{1/2M} > = \sum_{\substack{n_1+n_2=N \\ n_1, n_2 \text{ even}}} \sum_{l} \left(\begin{matrix} n_1 n_2 \\ 1, 11; 1, 11 \end{matrix}\right) B\left(\begin{matrix} n_1 n_2 \\ l\ l\ 0 \end{matrix}\right) \times \frac{1}{\sqrt{2}} \{ | {}^{22}(n_1 l \mid \mathbf{t}_{1,21})\, {}^{13}(n_2 l \mid \mathbf{t}_{1,12}); {}^2M_T\, {}^2S_{1/2M} > - | {}^{22}(n_1 l \mid \mathbf{t}_{1,21}), {}^{31}(n_2 l \mid \mathbf{t}_{1,12}); {}^2M_T\, {}^2S_{1/2M} > \}. \quad \ldots(18)$$

$$| (N)\,[21]\,{}^2M_T\,{}^{24}D_{1/2M} > = \frac{1}{\sqrt{2}} \sum_{\substack{n_1+n_2=N \\ n_1, n_2 \text{ even}}} \sum_{l_1, l_2} A\left(\begin{matrix} n_1 n_2 \\ 1, 21; 1, 21 \end{matrix}\right) B\left(\begin{matrix} n_1 n_2 \\ l_1\, l_2\, 2 \end{matrix}\right) \times | {}^{22}(n_1 l_1 \mid \mathbf{t}_{1,21}), {}^{13}(-n_2 l_2 \mid \mathbf{t}_{1,12}), {}^2M_T\, {}^4D_{1/2M} > + \frac{1}{\sqrt{2}} \sum_{\substack{n_1+n_2=N \\ n_1, n_2 \text{ odd}}} \sum_{l_1, l_2} A\left(\begin{matrix} n_1 n_2 \\ 1, 21; 1, 22 \end{matrix}\right) B\left(\begin{matrix} n_1 n_2 \\ l_1\, l_2\, 2 \end{matrix}\right) \times | {}^{22}(n_1 l_1 \mid \mathbf{t}_{1,21}), {}^{33}(n_2 l_2 \mid \mathbf{t}_{1,12}), {}^2M_T\, {}^4D_{1/2M} > \quad \ldots(19)$$

where

$$A\begin{pmatrix} n_1 n_2 \\ 1,\ 11;\ 1,\ 11 \end{pmatrix} = \frac{2^{N-1}\delta(n_2 N) + 3^{n_1/2}}{[3.2^{N-1}(^{N-1}+1)]^{1/2}} \cdot \left[\frac{N!}{n_1!\, n_2!}\right]^{1/2}, \qquad \ldots(20)$$

n_1 . n_2 are both even, otherwise the coefficient is zero)

$$A\begin{pmatrix} n_1 n_2 \\ 1,\ 21;\ 1,\ 21 \end{pmatrix} = \frac{2^N \delta(n_2 N) - 3^{n_1/2}}{[3.2^{N-1}(2^N - 1)]^{1/2}} \cdot \left[\frac{N!}{n_1!\, n_2!}\right]^{1/2}, \qquad \ldots(21)$$

(n_1, n_2 are both even, otherwise the coefficient is zero)

$$A\begin{pmatrix} n_1 n_2 \\ 1,\ 21;\ 1,\ 21 \end{pmatrix} = \frac{3^{n_1/2}}{[2^{N-1}(2^N - 1)]^{1/2}} \cdot \left[\frac{N!}{n_1!\, n_2!}\right]^{1/2}, \qquad \ldots(22)$$

(n_1, n_2 are both odd, otherwise the coefficient is zero) and

$$A\begin{pmatrix} 0 & 0 \\ 1,\ 11;\ 1,\ 11 \end{pmatrix} = 1,$$

For $A\begin{pmatrix} n_1 n_2 \\ 1,\ 11;\ 1,\ 11 \end{pmatrix}$, N can take the values, 0, 2, 4, ..., while for $A\begin{pmatrix} n_1 n_2 \\ 1,\ 21;\ 1,\ 21 \end{pmatrix}$ and $A\begin{pmatrix} n_1 n_2 \\ 1,\ 21;\ 1,\ 12 \end{pmatrix}$ N can take the values 2, 4, 6, ... The B-coefficients for N even are given by the following expression

$$B\begin{pmatrix} n_1 n_2 \\ l_1 l_2 L \end{pmatrix} = \frac{\left(\frac{l_1 + l_2 + L}{2}\right)!}{\left(\frac{l_1 + l_2 - L}{2}\right)!\left(\frac{l_1 + L - l_2}{2}\right)!\left(\frac{l_2 + L - l_1}{2}\right)!}$$

$$\times \left[\frac{2^{l_1+l_2-L}(2l_1+1)\,(2l_2+1)\,(l_1+l_2-L)!\,(l_1+L-l_2)!\,(l_2+L-l_1)!}{(l_1+l_2+L+1)!}\right]^{1/2}$$

$$\times \left[\frac{\left(\frac{N-L}{2}\right)!\,(N+L+1)!\left(\frac{n_1+l_1}{2}\right)!\left(\frac{n_2+l_2}{2}\right)!\,n_1!\,n_2!}{N!\left(\frac{N+L}{2}\right)!\left(\frac{n_1-l_1}{2}\right)!\left(\frac{n_2-l_2}{2}\right)!(n_1+l_1+1)!(n_2+l_2+1)!}\right]^{1/2}, \qquad \ldots(23)$$

where $n_1 + n_2 = N$, and $n_1 - l_1$, $n_2 - l_2$ are both even integers greater than or equal to zero and l_1, l_2, L satisfy the three triangular conditions $l_1 + l_2 - L \geqslant 0$, $l_1 + L - l_2 \geqslant 0$, $l_2 + L - l_1 \geqslant 0$. The B-coefficients satisfy the following relation :

$$\sum_{l_1 l_2} \left\{B\begin{pmatrix} n_1 n_2 \\ l_1 l_2 L \end{pmatrix}\right\}^2 = 1 \qquad \ldots(24)$$

Potential-Energy Operator

The interaction operator used by Goldhammer (1963) consisted of a Serber central force with a repulsive core and a tensor-even component :

$$\mathbf{V}_{23} = J_R \exp(-8r_{23}^2/r_c^2) + J_c(P_0 + P_1)\exp(-r_{23}^2/r_c^2)$$

$$+ \tfrac{1}{4}(1 - \vec{\tau}_2, \vec{\tau}_3) S_{23}(r_{23}^2/r_c^2) J_s \exp(-r_{23}^2/r_c^2), \qquad ...(25)$$

where,

$$J_c = -58.65 \text{ MeV}, J_s = -107.29 \text{ MeV}, J_R = 189.75 \text{ MeV}$$

$$r_c = 1.54 \times 10^{-13} \text{ cm}, P_0 = \frac{1}{16}(1 - \vec{\sigma}_2 \cdot \vec{\sigma}_3)(3 + \vec{\tau}_2 \cdot \vec{\tau}_3) \text{ and}$$

$$P_1 = \frac{1}{16}(3 + \vec{\sigma}_2 \cdot \vec{\sigma}_3)(1 - \vec{\tau}_2 \cdot \vec{\tau}_3).$$

The parameters were adjusted to yield a reasonable fit (within 2 per cent of the observed values) to the binding energies of ^{2}H, ^{3}H, and ^{4}He; the rms radii of ^{2}H and ^{4}He; the Coulomb energy in ^{3}He; and the electric quadrapole moment of ^{2}H.

For our case, the interaction operator (25) can be written in the form :

$$\mathbf{V}_{23} = a_0' \exp(-r_{23}^2/a'^2) + \exp(-r_{23}^2/a^2)[a_0 + a_\sigma(\vec{\sigma}_2 \cdot \vec{\sigma}_3)$$

$$+ a_\tau(\vec{\tau}_2 \cdot \vec{\tau}_3) + a_{\sigma\tau}(\vec{\sigma}_2 \cdot \vec{\sigma}_3)(\vec{\tau}_2 \cdot \vec{\tau}_3)] + \frac{Z}{4}(1 - \vec{\tau}_2 \cdot \vec{\tau}_3)$$

$$\times \exp\left(-\frac{r_{23}^2}{a^2}\right) \cdot (L_{23}^{(2)} \cdot S_{23}^{(2)})\left(\frac{r_{23}^2}{a^2}\right). \qquad ...(26)$$

where

$$a_0' = J_R, J_S = Z, r_c = a, a' = \frac{a}{2\sqrt{2}}, a_0 = \tfrac{3}{8} J_c, \text{ and}$$

$$a_\sigma = a_\tau = a_{\sigma\tau} = -\tfrac{1}{8} J_c,$$

and $S_m^{(2)}$ is the mth component of the tensor operator of rank two formed from the vectors $\vec{\sigma}_2, \vec{\sigma}_3$; $L^{(2)}$ is the mth component of the tensor operator of rank two formed from the vectors $\frac{\mathbf{r}_{23}}{r_{23}}, \frac{\mathbf{r}_{23}}{r_{23}}$. The operator $\mathbf{V}_{23}$ commute with J, but not with L and S separately, and of course it operates only on the co-ordinates of the last pair of particles.

Coulomb-Force Operator

The Coulomb-force operator acts only between protons then the operator may be written, in isotopic spin formalism as

$$C_{23} = \tfrac{1}{4}[1 - \tau_z(2)][1 - \tau_z(3)]\frac{e^2}{r_{23}}, \qquad ...(27)$$

with $\tau_z(i)$ being the Z-component of the charge τ_i, and e is the electronic charge.

Elliott (1952) has shown that the operator C_{23} can be written as

$$C_{23} = [\mathbf{T}^{(0)} + \mathbf{T}^{(1)} + \mathbf{T}^{(2)}] \frac{e^2}{r_{23}}, \qquad \ldots(28)$$

where

$\mathbf{T}^{(0)}$, $\mathbf{T}^{(1)}$, $\mathbf{T}^{(2)}$ are tensors of ranks 0, 1, 2 respectively and are given by the relations

$$\mathbf{T}^{(0)} = \tfrac{1}{4}[1 + \tfrac{1}{3}(\tau_2 \cdot \tau_3)], \qquad \ldots(29)$$

$$\mathbf{T}^{(1)} = -\tfrac{1}{4}[\tau_z(2) + \tau_z(3)], \qquad \ldots(30)$$

$$\mathbf{T}^{(2)} = -\frac{1}{12}[(\tau_2 \cdot \tau_3) - 3\tau_z(2)\,\tau_z(3)]. \qquad \ldots(31)$$

Since we are considering only the doublet states in charge ($\mathbf{T} = \frac{1}{2}$), the operator $\mathbf{T}^{(2)}$ does not come into the picture.

Kinetic-Energy Operator

The kinetic-energy operator for the three-particle nucleus is

$$T = \left(-\frac{\hbar^2}{2mb^2}\right) \sum_{i=1}^{3} \triangle_i, \qquad \ldots(32)$$

where

$$\triangle_i = \frac{\partial^2}{\partial q_{ix}^2} + \frac{\partial^2}{\partial q_{iy}^2} + \frac{\partial^2}{\partial q_{iz}^2}, \; q_i = (q_{ix}, q_{iy}, q_{iz}),$$

and m is the nucleon mass.

Using the orthogonal transformations (11) and (12), the kinetic-energy operator remains a sum of squares and the operator for the internal motion is

$$T = \left(-\frac{\hbar^2}{2mb^2}\right) \sum_{t=t_1, t_2} \triangle_t, \qquad \ldots(33)$$

where

$$\frac{\hbar^2}{2mb^2} = \frac{\hbar^2}{2mr_c^2} = 8.745\,\alpha \text{ (in MeV)},$$

$\alpha = r_c^2/b^2$ and r_c is the assumed range of nuclear forces.

The *rms* Charge Radius

If $\mathbf{r}_i (i = 1, 2, 3)$ are the position vectors of the three particles in a 3-particle nucleus, and the vector $\mathbf{r}_G$ represents the centre of mass of the three particles, then the distance of the i^{th} particle from the C.G. is

$$\mathbf{r}_{Gi} = -\mathbf{r}_G + \mathbf{r}_i \quad (i = 1, 2, 3).$$

Since the particles are on the same footing it follows that :

$$< \Psi \mid \mathbf{r}_{G3}^2 \mid \Psi > = \tfrac{1}{3} < \Psi \mid \mathbf{r}_{G1}^2 + \mathbf{r}_{G2}^2 + \mathbf{r}_{G3}^2 \mid \Psi >$$

But it can be shown that

$$\mathbf{r}_{G1}^2 + \mathbf{r}_{G2}^2 + \mathbf{r}_{G3}^2 = \tfrac{2}{3}(\mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 - \mathbf{r}_1 \cdot \mathbf{r}_2 - \mathbf{r}_1 \cdot \mathbf{r}_3 - \mathbf{r}_2 \cdot \mathbf{r}_3)$$
$$= b^2(\mathbf{t}_1^2 + \mathbf{t}_2^2) = b^2 R^2$$

where, the vectors $\mathbf{t}_1$, $\mathbf{t}_2$ are functions of the inter-particle distance and

$$R^2 = \mathbf{t}_1^2 + \mathbf{t}_2^2.$$

Hence

$$< \Psi \mid r_{G3}^2 \mid \Psi > = \frac{b^2}{3} < \Psi \mid R^2 \mid \Psi > \qquad \ldots(34)$$

The integral (34) can be evaluated numerically by using Simpson's formula, and the spuare root of the value of this integral is the rms charge radius.

Results

The energy matrices so obtained are symmetric and each element is a function of the variational parameter $\beta \left(= \frac{\alpha}{\alpha + 2} \right)$, occurring in our wave functions. The results of the binding energy calculations for ^{3}H and ^{3}He are given in Table I. The Coulomb energy for ^{3}He obtained as the difference of the binding energies between ^{3}H and ^{3}He.

Table I

Number of quanta N	Binding energy of ^{3}H (MeV)	Binding energy of ^{3}He (MeV)	Coulomb energy of ^{3}He (MeV)	Charge rms radius of ^{3}H (fm)
0	3.527	2.889	0.638	1.94
2	5.141	4.516	0.625	1.74
4	6.793	6.046	0.747	1.67
6	7.101	6.355	0.746	1.68
8	7.338	6.593	0.745	1.69
Experimental values*	8.49		0.764	(1.70 ± 0.05)

*[Mathur and Lagu (1968)]

The lowest state, viz., OS-state (which is totally symmetric in the space coordinates) appears to be the most important of all the states. A separate investigation was made to see the contribution by $2S$, $4S$, $6S$, and $8S$-states towards the binding

eneegy of the ^{3}H. It was found that the inclusion of these states with the *OS*-state raises the binding energy from 3.527 MeV to 3.8403 MeV. Thus it appears that the inclusion of any more higher quantum *S*-states is not likely to increase the binding energy by an appreciable amount.

The inclusion of the *D*-states along with the *S*-states raises the binding energy from 3.527 MeV to 7.338 MeV, which proves the importance of the tensor force in this binding energy of ^{3}H. There might have been some tensor force coupling between ^4D and ^2P, ^4P-states. But since the *P*-states have no coupling with the lowest *OS*-state they are not expected to be of much importance.

The Coulomb energy obtained provides a test for the goodness of the wave-function, and is independent of the interaction used. The result obtained in this calculation is 97.5 per cent of the experimental value and shows that the ground state function used is fairly good approximation,

The value of the rms charge radius of ^{3}H obtained from the present calculations is 1.69 fm., which agree well with the experimental value.

Salem (1969) has considered the same lowest *S*-state shell-model wave function for ^{3}H and modified it by applying a kinetic and potential (Hu and Massey) energy operators on it; thereby deducing the value $-$ 3.08 MeV for the binding of ^{3}H. A new variational method is proposed on the basis of the multiple scattering theory for ^{3}H by Akaishi *et al.* (1974). The Hamada-Johnston (H-J) and Tamagaki (OPEH, OPEG) potentials are employed. The energies for ^{3}H are found to be $-$ 6 MeV for H-J, $-$ 6.59 MeV for OPEH and $-$ 6.64 MeV for OPEG.

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