

## I. PHYSICS

### Biomathematics

# MATHEMATICAL MODELLING ON 'PULL-OUT' FAILURE OF BONE FIBRES

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The most interesting topological approach in mathematical modelling of biological situations is based on the principle of biotopological mapping. The present paper, which is divided into three parts, is an attempt to obtain a model on the biological fact of "pull-out" failure of bone fibres by using concepts of differentiable manifolds and Lie group. An important feature of this analysis is that it seeks to explain propagation of cracks in the context of parallel displacement of fibres, embedded in the geometrical model considered.

**Keywords :** Biotopological Mapping; Fibre-Matrix Interface; Manifold; Crack; Osteons

### INTRODUCTION

OF all studies in mathematical modelling of biology, topological approach dates back to researches of Rashevsky (1954, 1955*a, b, c, d*, 1956). The cardinal point in such studies is based on the principle of what Rashevsky called biotopological mapping, *vide* Rashevsky (1954), which is inferred from some facts of biology. These studies in a way pave the way for developing topological biology from different standpoints. The present paper is an attempt in this direction, seeking to obtain a model on the biological fact of "pull-out" failure of bone fibres by means of geometrizing the situation from the standpoint of topology of fibre bundles. We first set forth the motivation for geometrization on the basis of biological facts about bone; this is followed by axiomatization of these facts in the language of geometry to be associated with the biological situation. The next section deals with the mathematical analysis of the problem. Finally, it is found that the analysis of this sort leads to "pull-out" failure in bones. Another important upshot of this analysis is the explanation of propagation of cracks in the context of parallel displacement of fibres, embedded in the geometrical model considered.

### MOTIVATION

Studies on bone reveal that it is a composite material consisting of some phases viz., crystalline mineral phase, amorphous mineral phase, collagen, protein molecules in the form of gels and sols, liquids etc., *vide* Piekarski (1970). The mineral phase in bone exists in the form of hard and brittle fibres, forming a continuous phase. Collagen also exists in the form of fibres arranged parallel to the long axis of mineral fibres, *vide*, Bonfield and Li (1966). The proportions of collagen and hydroxyapatite in bone are about 50 per cent while that of amorphous phases is

about 1 per cent only, *vide* Piekarski (1970). The mineral fibres have high elastic moduli embedded in a soft protein matrix *vide*, Bonfield and Li (1966). The fibres are joined end to end and are considered as continuous fibres. At lower stresses, the "pull-out" failure as observed by Piekarski (1970) occurs in bones, the individual fibres are pulled out of the organic matrix by shear failure at the fibre-matrix interface. These observational findings enable us to arrive at the following statements:

(1) Bone has been shown to consist of crystalline hydroxyapatite particles, collagen fibres etc., distributed in an organic matrix. We assume that these particles are arranged in this protein matrix in such a way that the organic matrix may be considered as a matrix.

(2) Constituents in the bone in greater and smaller degrees exist so as to allow mutual operations among them.

### MATHEMATICAL MODEL

We take a piece of bone consisting of the above materials and induce a metric  $d$  such that for any two constitutive elements of bone  $C_1$  and  $C_2$  belonging to this specimen (which is an  $R^2$ -space),  $d\{(C_1) - (C_2)\} \geq 0$  etc. Thus, the bone specimen which is an  $R^2$ -space induced by a metric becomes a topological space. Since the constitutive elements are different from one another, they have any two neighbourhoods which are disjoint. Thus it can be taken as a Hausdorff space. We now make the following assumptions:

(1) The matrix referred to above is a matrix with mathematical attributes.

(2) The set of actions and reactions among the different constituents of bone itself is the set of mappings which are required for a Hausdorff's space to become a differentiable manifold.

(3) The inverse of an element can be conceived of by assuming operation between two constituents present in greater and lesser amounts and leading to identity, to be suitably defined after operation among themselves.

By the above stated axiom (2), the idea of a differentiable manifold in the bone specimen is developed. This bone specimen, which we have established as a topological space, consists of a soft protein matrix, which by our previous axiom (1), is a matrix in the mathematical sense. So, it may be called a topological group because it has a group structure and a topology such that for any element  $g$  belonging to this matrix,  $g^{-1}$  is continuous and for any two elements  $g_1$  and  $g_2$ ,  $g_1 g_2$  is continuous simultaneously in  $g_1$  and  $g_2$ . This is possible for the elements of the organic matrix of bone form a continuous phase. From this, a Lie group  $G$  is formed which is a topological group and at the same time a differentiable manifold such that the group operation  $(a, b) \in G \times G \rightarrow ab^{-1} \in G$  is a differentiable map of  $G \times G$  into  $G$ . In biological terms, if we take collagen as the element  $b$ , then 'gel' will denote the element  $b^{-1}$  and thus, the Lie group is well defined *vide*, our axiom (3). Any crack in the bone specimen is represented by a differentiable curve  $\gamma = x_t$  bounded by the conditions  $0 \leq t \leq 1$  on the manifold. The geometry of the bone fibres may be represented as fibres of the principal fibre bundle. The bi-continuity implied by the diffeomorphism, which is required to establish the

structure of a principal fibre bundle, is ensured on account of the continuity of the mineral fibres of bone. The displacement of fibres from one layer to another due to impact leads to lifting mapping of the principal fibre bundle. Thus, bone which behaves more closely to the uniform strain model than that of the stress model, can be geometrised as a principal fibre bundle.

Let the bone specimen be defined as the manifold  $M$  of dimension  $n$ . We denote  $T_x(M)$  = tangent space to  $M$  at  $x$ .

Let  $X_1, X_2, \dots, X_n$  form an ordered basis of  $T_x(M)$  [See Fig. 1]. A frame  $u$  at  $x = (X_1, \dots, X_n)$ .

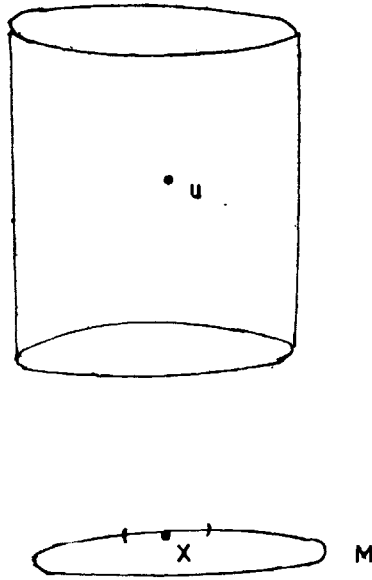


FIG. 1

Let  $a = (a_j^i)$  be the protein matrix, embedded along the fibres of the bone, belonging to the Lie group  $G$ . Then the frame  $ua$  defines  $(Y_1, \dots, Y_n)$  where

$$Y_i = \sum_{j=1}^n a_j^i X_j.$$

We denote the totality of all frames in all points  $x$  by  $L(M)$ . Let  $(x^1, x^2, \dots, x^n)$  be a co-ordinate system in a co-ordinate neighbourhood  $U$  of  $x$ . Then any frame  $u = (X_1, \dots, X_n)$  can be uniquely expressed as

$$X_i = \sum_{k=1}^n X_i^* \frac{\partial}{\partial x^k}$$

where  $(X_i^*)$  is a non-singular matrix.

Let  $\pi$  be the map of  $L(M) \rightarrow M$ . We take  $(x^j)$  and  $(X_i^k)$  as a co-ordinate system in  $\pi^{-1}(U)$ . Then with this choice, we can introduce a topology and a differentiable structure on  $L(M)$ . Thus  $L(M)$  is made to be a differentiable manifold.

Now  $ua = u$  gives  $\Sigma a_i^j X_j = X_i$ .

i.e.,  $a_i^1 X_1 + \dots + a_i^n X_n = X_i$ .

or  $a_1^1 X_1 + \dots + a_1^n X_n = X_1$

which gives  $a_1^1 = 1, a_1^2 = 0, \dots, a_1^n = 0$ .

i.e.,  $\begin{pmatrix} a_i^j \end{pmatrix} = e$ .

Thus  $ua = u \Rightarrow a = e$ .

So, we see that the action of  $G$  is free.

Next,  $\pi(u) = \pi(v)$  if and only if  $Y_i = \Sigma a_i^j X_j$  where  $v = (Y_1, \dots, Y_n)$ . Thus  $\pi(u) = \pi(v)$  if and only if  $v = ua$  for some  $a \in G$ . That is  $M = L(M)/G$ .

Also, we have the canonical projection,  $\pi: P \rightarrow M$  which is differentiable. Every point  $x$  of the manifold composed of bone  $M$  has a neighbourhood  $U$  such that  $\pi^{-1}(U)$ , the lift is isomorphic to  $U \times G$  in the sense that there is a diffeomorphism  $\psi$  of  $\pi^{-1}(U)$  to  $u \times G$  such that  $\psi(u) = (\pi(u), \phi(u))$  where  $\phi(ua) = \phi(u)a$ ,  $a \in G$ .

So  $L(M)$  is a principal fibre bundle over  $M$  with  $G$  as structure group and  $\pi$  as the projection [See Fig. 2]. This bundle of linear frame  $L(M)$  over the base space  $M$  may be called the bundle space and denoted by  $P$ . We give the following model of the above definition which will suit with the physical construction of bone.  $P$  is the cylinder,  $M$  is the circle (manifold); and  $G$  the straight line (Lie group).  $P$  is the principal fibre bundle. Each generator is called a fibre over  $x$  going through  $u$ . This permits the idea of continuity of the fibres in bones to be treated mathematically. Each  $\pi^{-1}(U)$  is a patch of the cylinder. In the fibre-reinforced composite model of bone, this accounts for the "pull-out" failure where the individual fibres are pulled out of the matrix by shear failure at the fibre-matrix interface, *vide* Piekarski (1970).

#### BIOLOGICAL IMPLICATIONS

From the above geometric model, one can identify the physical failure in fibres of the bones by the action of the lift  $\pi^{-1}(x)$  of the fibre with the principal fibre bundle by action of the Lie group  $G$ .

Any solid material possesses within it a large number of small cracks which are called Griffith cracks, *vide* Griffith (1921). Bone being a brittle substance,

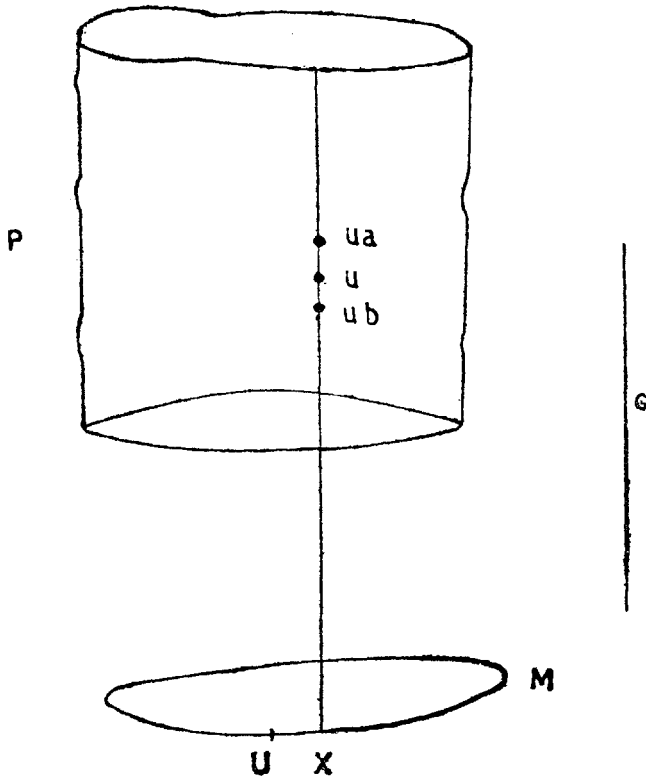


FIG. 2

Griffith crack theory is applicable. Also, there is a constant remodelling among the constituents of bone within the material *vide* Fung (1967). The propagation of cracks in bones can be judged best from the parallel displacement of fibres in the principal fibre bundle.

Let  $\gamma = x_t, 0 \leq t \leq 1$  be a differentiable curve of class  $C^1$  in the base space  $M$ . The curve may be considered as the length of the crack in bone [See Fig. 3]. Let  $u_0$  be an arbitrary point of  $P$  with the canonical projection  $\pi(u_0) = x_0$ . The unique lift  $\gamma^*$  of  $\gamma$  through  $u_0$  has the end point  $u_1$  such that  $\pi(u_1) = x_1$ . By varying  $u_0$  in the fibre  $\pi^{-1}(x_0)$ , we obtain a mapping of the fibre  $\pi^{-1}(x_0)$  onto the fibre  $\pi^{-1}(x_1)$  which maps  $u_0$  into  $u_1$ . This mapping which is the parallel displacement along the curve  $\gamma$ , is denoted by the same letter  $\gamma$ . Thus the propagation of a crack from one layer to another layer inside the bone is explained.

According to Piekarski (1970), fibres are actual Haversian systems pulled out from between the interstitial bones or from tubular interfaces of Haversian lamellae and he has established that osteons and interstitial bone can be represented by fibres and matrix respectively. The experimental findings showing the vertical channel like depressions due to "pull-out" of the osteons, *vide* Saha (1977) are reinforced also from a theoretical point of view as indicated in our analysis.

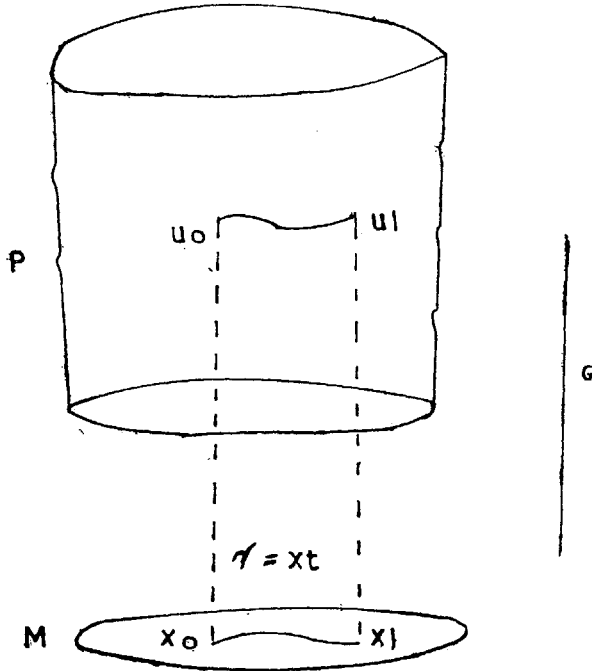


FIG. 3

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