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I. PHYSICS

Fluid Mechanics

NON-STEADY AND STEADY FLOW OF MICRODEFORMABLE FLUIDS PAST A ROTATING CYLINDER

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The theory of microdeformable fluids is applied to analyse the steady and non-steady flow past a rotating circular cylinder. The velocity distributions are computed for non-steady case for different values of time and different volume concentration of inner structure. In steady state case, the torque transmitted by the fluid on the cylinder is calculated. The influence of inner structure of the fluid is significant on the flow and torque transmitted. Dissolution of a vortex filament is discussed.

Keywords : Microdeformation; Structured Fluids; Flow past a Rotating Cylinder—Steady Case; Torque Transmitted.

INTRODUCTION

THE theory of microfluids was introduced by Eringen (1964, 1965). It exhibits certain microscopic effects due to local structure and micro-motions of the fluid elements. As a subclass of these fluids Eringen (1966) later analysed the theory of micropolar fluids which exhibit the micro-rotational effects and micro-rotational inertia, possessing certain simplicity. Recently, Kirwan Jr. *et al.* (1976) developed the theory of microdeformable fluids as another subclass of microfluids, imposing two physical constraints on general microfluid theory. One of the two constraints being an 'Onsager' type of symmetry condition (Onsager, 1952) which was applied to the viscosity and gyroviscosity coefficient tensors. Other constraint restricted the constitutive equations so that gradients of the trace, deviator and skew part of sub-structure strain rate are related to comparable elements of the stress moments. The result of these two constraints when applied to the theory of microfluids, became a theory which was general enough to allow deforming structure, yet contained eight fewer gyroviscosities than the general theory, thus enabling simplification without much loss of generality and physical significance.

Ackert (1952) studied flow past a rotating cylinder. He observed that potential solution constitutes a meaningful solution to Navier-Stokes equations. Kirde (1962) made an analytic study of the case when the velocity distribution in the vortex differs from that imposed by potential theory. In present work, we study flow of microdeformable fluids past a rotating cylinder and dissolution of a vortex filament. Hankel and Laplace transforms are used to solve the reduced governing equations. The velocity distributions are computed for different volume concentrations of inner structure of the fluid. The steady state part of the solution is also given and torque transmitted by the fluid on the rotating cylinder is calculated. We observe that in steady state case, the torque on the cylinder increases with the volume concentration of inner structure of the fluid.

BASIC EQUATIONS

The following balance laws (Eringen, 1964) are taken.

Conservation of mass

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0, \qquad \ldots(1)$$

Conservation of microinertia moment

$$\dot{i}_{km} - i_{rm} \nu_{rk} - i_{kr} \nu_{rm} = 0, \qquad \ldots(2)$$

Balance of momentum

$$\rho \dot{v}_k = \rho f_k + t_{nk,m}, \qquad \ldots(3)$$

Balance of first-stress moments

$$\rho \dot{\sigma}_{km} + S_{km} - t_{mk} - \lambda_{nkm,n} = \rho l_{km}, \qquad \ldots(4)$$

Conservation of energy

$$\rho \dot{\epsilon} - t_{mk} v_{m,k} + (t_{km} - S_{km})\, \nu_{km} - \lambda_{rkm} \nu_{mk,r} - q_{r,r} + \rho h = 0,$$

In addition, the following constitutive equations (Kirwan Jr. *et al.*, 1976) characterize the microdeformable fluids:

$$t_{km} = [-\pi + (\lambda + 2\mu/3)\, d + \{\eta - \lambda + 2(\mu_0 + \mu_1)/3\}\nu]\, \delta_{km}$$

$$+ 2\mu \overset{*}{d}_{km} + 2(\mu_0 - \mu_1)\, [\nu_{[km]} + v_{[k,m]}] + 2(\mu_0 + \mu_1)\overset{*}{\nu}_{km}, \qquad \ldots(6)$$

$$S_{km} = [-\pi + \{\eta + 2(\mu + \mu_0 + \mu_1)/3]\, d + (\eta_0 + 2\eta/3)\nu]\, \delta_{km}$$

$$+ 2(\mu + \mu_0 + \mu_1)\overset{*}{d}_{km} + 2\eta \overset{*}{\nu}_{km}, \qquad \ldots(7)$$

$$\lambda_{k[lm]} = \gamma_1\{\delta_{kl}\nu_{[nm],n} + \delta_{km}\nu_{ln],n}\} + \gamma_2\{\nu_{[km],l} + \nu_{[lk],m}\}$$
$$+ \gamma_3 \nu_{[ml],k}, \qquad \ldots(8)$$

$$\overset{*}{\lambda}_{klm} = \gamma_4[\delta_{kl}\overset{*}{\nu}_{nm,n} + \delta_{mk}\overset{*}{\nu}_{ln,n}] + \gamma_5[\overset{*}{\nu}_{km,l} + \overset{*}{\nu}_{lk,m}]$$

$$+ \gamma_6 \overset{*}{\nu}_{ml,k} - 2(\gamma_4 + \gamma_5)\, \delta_{lm}\overset{*}{\nu}_{kn,n}/3, \qquad \ldots(9)$$

$$\lambda_{kpp} = \gamma_7\, \nu_{,k}, \qquad \ldots(10)$$

where ρ is mass density, v_k velocity vector, t_{mk} stress tensor, i_{km} microinertia tensor, ν_{rk} gyration, S_{km} microstress average, λ_{nkm} stress-moments, $\dot{\sigma}_{km}$ inertial spin, f_k body force per unit mass, l_{km} body moment per unit mass, ϵ inertial energy per unit mass, q_k heat flux vector, h heat source per unit mass, $d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ the deformation rate γ's gyroviscosities, π the pressure, λ, μ, μ_0, μ_1, η, η_0

viscosity coefficients, $(\mu_0 + \mu_1)$ viscous shear coefficient, $(\mu_1 - \mu_0)$ spin coupling coefficient, $\overset{*}{d}$ deviator of velocity gradient, $\overset{*}{\nu}$ deviator of gyration, $\nu_{[\ ]}$ skew part of ν, $d = v_{p,p}$, $\nu = \nu_{pp}$.

FORMULATION

We consider the flow of an incompressible microdeformable fluid past a circular cylinder of radius r_0 rotating with angular velocity ω_1. No body forces and body couples are present i.e., $f_k = 0$, $l_{km} = 0$.

We use cylindrical polar coordinates (r, θ, z) with z-axis as the axis of the circular cylinder. All quantities are independent of θ due to axial symmetry.

Components of velocity and gyration are taken in the form:

$$v_i = [0, v(r, t), 0], \quad \nu_{ij} = \nu_{(ij)} + \nu_{[ij]}, \qquad \text{...(11)}$$

where

$$\nu_{(ij)} = \begin{bmatrix} 0 & \overset{*}{\nu} & 0 \\ \overset{*}{\nu} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \nu_{[ij]} = \begin{bmatrix} 0 & \bar{\nu} & 0 \\ -\bar{\nu} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{...(12)}$$

The inertial spin with nonlinear terms neglected is given by (Kirwan Jr. *et al.*, 1976).

$$\dot{\sigma}_{ij} = i_{nj}\,\dot{\nu}_{nj} = I\dot{\nu}_{ji}, \qquad \text{...(13)}$$

where I is the micromoment of inertia for isotropic substructure and is independent of time.

The initial and boundary conditions on velocity, gyration and its gradient are taken as follows :

$$\left.\begin{aligned} (a) \quad \zeta(r, t) &= f(r) && \text{at } t = 0, \\ \bar{\nu}(r, t) &= 0 && \text{at } t = 0, \\ W(r, t) &= 0 && \text{at } t = 0, \end{aligned}\right\} \qquad \text{...(14)}$$

(b) $\zeta(r, t)$, $\partial(\zeta r, t)/\partial r$; $\bar{\nu}(r, t)$, $\partial\bar{\nu}(r, t)/\partial r$; $W(r, t)$, ...(15)

as $\partial W(r, t)/\partial r$ tend to zero as $r \to \infty$.

Here, the vorticity

$$\zeta(r, t) = \frac{\partial v(r, t)}{\partial r} + \frac{v(r, t)}{r}, \qquad \text{...(16)}$$

and

$$W(r, t) = \frac{\partial^2 \overset{*}{\nu}(r, t)}{\partial r^2} + \frac{3}{r}\frac{\partial \overset{*}{\nu}(r, t)}{\partial r} \qquad \text{...(17)}$$

are the assumed transformations of variables.

Unknowns are v, $\overset{*}{\nu}$ and $\tilde{\nu}$ to be determined from the equations (1) to (4) with the help of equations (6) to (10) and (11) to (13) whose reduced form is obtained as given below:

$$(\mu+k)\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+2c\left(\frac{\partial\overset{*}{\nu}}{\partial r}+\frac{2\overset{*}{\nu}}{r}\right)$$

$$-2k\frac{\partial\tilde{\nu}}{\partial r}=\rho\frac{\partial v}{\partial t}, \qquad \ldots(18)$$

$$\alpha\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{\nu}}{\partial r}\right)+k\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)-2k\tilde{\nu}=\rho I\frac{\partial\tilde{\nu}}{\partial t} \qquad \ldots(19)$$

$$\beta r\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\frac{\partial\overset{*}{\nu}}{\partial r}+\frac{2\overset{*}{\nu}}{r}\right)\right]-2(\eta-c)\overset{*}{\nu}$$

$$-c\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)=\rho I\frac{\partial\overset{*}{\nu}}{\partial t} \qquad \ldots(20)$$

where

$$K=\mu_1-\mu_0,\ c=\mu_1+\mu_0.$$

SOLUTION

Differentiating eqn. (18) with respect to r and adding $1/r$ times of (18) in resulting expression, we get

$$(\mu+k)\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+\frac{1}{r}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)\right.$$

$$\left.+2c\left[\frac{\partial}{\partial r}\left(\frac{\partial\overset{*}{\nu}}{\partial r}+\frac{2\overset{*}{\nu}}{r}\right)+\frac{1}{r}\left(\frac{\partial\overset{*}{\nu}}{\partial r}+\frac{2\overset{*}{\nu}}{r}\right)\right]\right]$$

$$-2k\left(\frac{\partial^2\tilde{\nu}}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{\nu}}{\partial r}\right)=\rho\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right) \qquad \ldots(21)$$

Similarly, differentiating eqn. (20) with respect to r twice and adding $3/r$ times of its differentiation with respect to r in resulting expression, we get,

$$\beta\left[\frac{\partial^2}{\partial r^2}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\frac{\partial\overset{*}{\nu}}{\partial r}+\frac{2\overset{*}{\nu}}{r}\right)\right]\right\}+\frac{3}{r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\frac{\partial\overset{*}{\nu}}{\partial r}+\frac{2\overset{*}{\nu}}{r}\right)\right]\right\}\right]$$

$$-2(\eta-c)\left(\frac{\partial^2\overset{*}{\nu}}{\partial r^2}+\frac{3}{r}\frac{\partial\overset{*}{\nu}}{\partial r}\right)-c\left[\frac{\partial^2}{\partial r^2}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\right.$$

$$\left.+\frac{3}{r}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\right]=\rho I\frac{\partial}{\partial t}\left(\frac{\partial^2\overset{*}{\nu}}{\partial r^2}+\frac{3}{r}\frac{\partial\overset{*}{\nu}}{\partial r}\right), \qquad \ldots(22)$$

Using eqns. (16) and (17) into the equations (19), (21) and (22), we obtain

$$(\mu + k)\left(\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r}\right) + 2cW - 2k\left(\frac{\partial^2 \bar{\nu}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\nu}}{\partial r}\right)$$

$$= \rho \frac{\partial \zeta}{\partial t} \qquad \ldots(23)$$

$$\alpha\left(\frac{\partial^2 \bar{\nu}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\nu}}{\partial r}\right) + k\zeta - 2k\bar{\nu} = \rho I \frac{\partial \bar{\nu}}{\partial t} \qquad \ldots(24)$$

$$\beta\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r}\right) - 2(\eta - c)W - c\left(\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r}\right)$$

$$= \rho I \frac{\partial W}{\partial t} \qquad \ldots(25)$$

Applying zero-order Hankel Transform to both sides in each of the eqns. (23) to (25) and using boundary conditions (15), we get

$$-(\mu + k)\xi^2\bar{\zeta} + 2c\bar{W} - 2k\xi^2\bar{\bar{\nu}} = \rho \frac{\partial \bar{\zeta}}{\partial t} \qquad \ldots(26)$$

$$-\alpha\xi^2\bar{\bar{\nu}} + k\bar{\xi} - 2k\bar{\bar{\nu}} = \rho I \frac{\partial \bar{\bar{\nu}}}{\partial t} \qquad \ldots(27)$$

$$-\beta\xi^2\bar{W} - 2(\eta - c)\bar{W} + c\xi^2\bar{\zeta} = \rho I \frac{\partial \bar{W}}{\partial t} \qquad \ldots(28)$$

where $\bar{\xi}$, $\bar{\bar{\nu}}$ and $\bar{W}$ are zero-order Hankel

Transforms of ζ, $\bar{\nu}$ and W respectively.

Now taking Laplace Transform of equations (26), (27) and (28), we obtain

$$-(\mu + k)\xi^2\bar{\bar{\zeta}} + 2c\bar{\bar{W}} - 2k\xi^2\bar{\bar{\bar{\nu}}} = \rho S\bar{\bar{\zeta}} - \rho\bar{\zeta}(\xi, 0) \qquad \ldots(29)$$

$$-\alpha\xi^2\bar{\bar{\bar{\nu}}} + k\bar{\bar{\zeta}} - 2k\bar{\bar{\bar{\nu}}} = \rho IS\bar{\bar{\bar{\nu}}} - \rho I\bar{\bar{\nu}}(\xi, 0) \qquad \ldots(30)$$

$$-\beta\xi^2\bar{\bar{W}} - 2(\eta - c)\bar{\bar{W}} + c\xi^2\bar{\bar{\zeta}} = \rho I\bar{\bar{W}} - \rho I\bar{W}(\xi, 0), \qquad \ldots(31)$$

where $\bar{\bar{\zeta}} = L[\bar{\zeta}(\xi, t)]$,

$\bar{\bar{\bar{\nu}}} = L[\bar{\bar{\nu}}(\xi, t)]$,

$\bar{\bar{W}} = L[\bar{W}(\xi, t)]$.

Using initial conditions (14), we may write

$$-(\mu+k)\xi^2\bar{\bar{\zeta}} + 2c\bar{\bar{W}} + 2k\xi^2\bar{\bar{\tilde{\nu}}} = \rho S\bar{\bar{\zeta}} - \rho\bar{f}(\xi), \qquad \dots(32)$$

$$-\alpha\xi^2\bar{\bar{\tilde{\nu}}} + k\bar{\bar{\xi}} - 2k\bar{\bar{\tilde{\nu}}} = \rho IS\bar{\bar{\tilde{\nu}}} \qquad \dots(33)$$

$$-\beta\xi^2\bar{\bar{W}} - 2(\eta-c)\bar{\bar{W}} + c\xi^2\bar{\bar{\zeta}} = \rho IS\bar{\bar{W}}, \qquad \dots(34)$$

To solve simultaneous linear algebraic eqns. (32) to (34) for $\bar{\bar{\zeta}}$, $\bar{\bar{\tilde{\nu}}}$ and $\bar{\bar{W}}$, we have from eqns. (33) and (34)

$$\bar{\bar{\tilde{\nu}}} = \frac{k\bar{\bar{\zeta}}}{\rho IS + \alpha\xi^2 + 2k}, \qquad \dots(35)$$

and

$$\bar{\bar{W}} = \frac{c\xi^2\bar{\bar{\zeta}}}{\rho IS + \beta\xi^2 + 2(\eta - c)}, \qquad \dots(36)$$

respectively.

Substituting $\bar{\bar{\tilde{\nu}}}$ and $\bar{\bar{W}}$ from equations (35) and (36) into eqn. (32) and solving for $\bar{\bar{\zeta}}$, we get

$$\bar{\bar{\zeta}} = \bar{f}(\xi)\,.\,[\rho IS + \beta\xi^2 + 2(\eta - c)]\,(\rho IS + \alpha\xi^2 + 2k)/\rho^2 I^2 Q(S), \qquad \dots(37)$$

where

$$\begin{aligned}Q(S) = S^3 &+ \frac{S^2}{\rho I}\,[\xi^2\{\alpha + \beta + I(\mu + k)\} + 2(\eta - c + k)]\\ &+ \frac{S}{\rho^2 I^2}\,[\xi^4(\alpha\beta + I(\mu + k)(\alpha + \beta)] + \xi^2\{2k\beta + 2\alpha(\eta - c)\\ &+ 2I(\mu + k)(\eta - c + k) - 2I(c^2 + k^2)\} + 4k(\eta - c)]\\ &+ \frac{1}{\rho^3 I^2}[\xi^6\{\alpha\beta(\mu + k)\} + \xi^4\{-2k^2\beta - 2\alpha c^2\\ &+ 2(\mu + k)(\beta k + \alpha(\eta - c))\} + 4\xi^2\{(\mu + k)\,k(\eta - c) - c^2 k\\ &- k^2(\eta - c)\}],\end{aligned} \qquad \dots(38)$$

Applying residue theorem, we find inverse Laplace transforms to get

$$\bar{\zeta} = \frac{\bar{f}(\xi)}{\rho^2 I^2}\,(A_1 e^{r_1 t} + A_2 e^{r_2 t} + A_3 e^{r_3 t}), \qquad \dots(39)$$

$$\bar{\tilde{\nu}} = \frac{k\bar{f}(\xi)}{\rho^2 I^2}\,(B_1 e^{r_1 t} + B_2 e^{r_2 t} + B_3 e^{r_3 t}), \qquad \dots(40)$$

$$W \frac{c\xi^2 \bar{f}(\xi)}{\rho^2 I^2} \; (D_1 e^{r_1 t} + D_2 e^{r_2 t} + D_3 e^{r_3 t}) \qquad \ldots(41)$$

where r_1, r_2, r_3 are the roots of the cubic equation $Q(S) = 0$,

and

$$A_1 = \{\rho I r_1 + \beta\xi^2 + 2(\eta - c)\}(\rho I r_1 + \alpha\xi^2 + 2k)/[(r_1 - r_2)(r_1 - r_3)]$$
$$A_2 = \{\rho I r_2 + \beta\xi^2 + 2(\eta - c)\}(\rho I r_2 + \alpha\xi^2 + 2k)/[(r_2 - r_1)(r_2 - r_3)]$$
$$A_3 = \{\rho I r_3 + \beta\xi^2 + 2(\eta - c)\}(\rho I r_3 + \alpha\xi^2 + 2k)/[(r_3 - r_1)(r_3 - r_2)]$$
$$B_1 = [\rho I r_1 + \beta\xi^2 + 2(\eta - c)]/[(r_1 - r_2)(r_1 - r_3)]$$
$$B_2 = [\rho I r_2 + \beta\xi^2 + 2(\eta - c)]/[(r_2 - r_1)(r_2 - r_3)]$$
$$B_3 = [\rho I r_3 + \beta\xi^2 + 2(\eta - c)]/[(r_3 - r_1)(r_3 - r_2)]$$
$$D_1 = (\rho I r_1 + \alpha\xi^2 + 2k)/[(r_1 - r_2)(r_1 - r_3)]$$
$$D_2 = (\rho I r_2 + \alpha\xi^2 + 2k)/[(r_2 - r_1)(r_2 - r_3)]$$
$$D_3 = (\rho I r_3 + \alpha\xi^2 + 2k)/[(r_3 - r_1)(r_3 - r_2)].$$

By means of the thermodynamic constraints (Kirwan Jr. *et al.*, 1976)

i.e., $k \geqslant 0,\ \mu \geqslant 0,\ \eta - c \geqslant 0,\ \mu(\eta - c) = |\, c\,|^2,\ \alpha \geqslant \gamma_1$ and $\beta \geqslant 0$,

it is easy to show that

$$\xi^6\{\alpha\beta(\mu + k)\} + \xi^4\{-2k^2\beta + 2(\mu + k)(\beta k + \alpha(\eta - c)) - 2c^2\alpha\}$$
$$+ \xi^2\{4(\mu + k)\, k(\eta - c) - 4c^2 k - 4k^2(\eta - c)\} \geqslant 0,$$

i.e.,

$$\xi^6\{\alpha\beta(\mu + k)\} + 2\xi^4[\alpha\{\mu(\eta - c) - c^2\} + k\,\{\alpha(\eta - c) + \beta k\}]$$
$$+ 4\xi^2[k\{\mu(\eta - c) - c^2\}] \geqslant 0,$$

from which it readily follows that at least one of the roots of the eqn. (38) is real and negative. However, we have calculated these roots and found that all the three roots are negative and real which is desirable since all r_1, r_2, r_3 appear as coefficients of time in exponential terms.

To get the complete solution of eqns. (33) to (35), we have to obtain inverse Hankel Transforms of $\bar{\zeta}(\xi, t)$, $\bar{\bar{v}}(\xi, t)$, and $\bar{W}(\xi, t)$ through the Hankel inversion theorem:

$$F(x) = H_n^{-1}(F = \int_0^\infty \xi J_n(\xi r) \,.\, \bar{F}(\xi)\, d\xi.$$

Corresponding to $n = 0$, we get

$$\zeta = \int_0^\infty \xi J_0(\xi r) \frac{\bar{f}(\xi)}{\rho^2 I^2} (A_1 e^{r_1 t} + A_2 e^{r_2 t} + A_3 e^{r_3 t})\, d\xi \qquad \ldots(42)$$

$$\tilde{\nu} = \int_0^\infty k\xi J_0(\xi r) \frac{\bar{f}(\xi)}{\rho^2 I^2} (B_1 e^{r_1 t} + B_2 e^{r_2 t} + B_3 e^{r_3 t}) \, d\xi, \quad \ldots(43)$$

$$W = \int_0^\infty c\xi^3 J_0(\xi r) \frac{\bar{f}(\xi)}{\rho^2 I^2} (D_1 e^{r_1 t} + D_2 e^{r_2 t} + D_3 e^{r_3 t}) \, d\xi. \quad \ldots(44)$$

For the case of a line vortex filament, we have

$f(r) = \chi\delta r/4\pi r$, so that $\bar{f}(\xi) = \chi/4\pi$ and

$$\zeta = \frac{\chi}{4\pi\rho^2 I^2}\int_0^\infty \xi J_0(\xi r) \, . \, (A_1 e^{r_1 t} + A_2 e^{r_2 t} + A_3 e^{r_3 t}) \, d\xi \quad \ldots(45)$$

$$\tilde{\nu} = \frac{\chi k}{4\pi\rho^2 I^2}\int_0^\infty \xi J_0(\xi r) \, . \, (B_1 e^{r_1 t} + B_2 e^{r_2 t} + B_3 e^{r_3 t}) \, d\xi, \quad \ldots(46)$$

and

$$W = \frac{\chi c}{4\pi\rho^2 I^2}\int_0^\infty \xi^3 J_0(\xi r) \, . \, (D_1 e^{r_1 t} + D_2 e^{r_2 t} + D_3 e^{r_3 t}) \, d\xi. \quad \ldots(47)$$

Eqn. (46) represents the solution of skew symmetric part of gyration for the non-steady case of flow past a circular rotating cylinder but equations (45) and (47) give only the transformed variable solutions. Substituting the values of ζ and W in eqns. (16) and (17) respectively and integrating, we obtain

$$\frac{v}{v_0} = \frac{1}{\rho^2 I^2}\int_0^\infty J_1(\xi r) \, . \, (A_1 e^{r_1 t} + A_2 e^{r_2 t} + A_3 e^{r_3 t}) \, d(\xi r_0). \quad \ldots(48)$$

$$\frac{r_0 \overset{*}{\nu}}{v_0} = \frac{c}{\rho^2 I^2}\int_0^\infty r\xi J_2(\xi r) \, . \, (D_1 e^{r_1 t} + D_2 e^{r_2 t} + D_3 e^{r_3 t}) \, d(\xi r_0) + c_2(t), \quad \ldots(49)$$

where $v_0 = \chi/(4\pi r_0)$ is the velocity at time $t = 0$.

Gyration is given by

$$\frac{r_0 \nu}{v_0} = \frac{r_0 \overset{*}{\nu}}{v_0} + \frac{r_0 \tilde{\nu}}{v_0}$$

$$= \frac{1}{\rho^2 I^2}\int_0^\infty \xi[(cD_1 J_2(\xi r) + kB_1 J_0(\xi r))e^{r_1 t}$$

$$+ (cD_2 J_2(\xi r) + kB_2 J_0(\xi r))e^{r_2 t}$$

$$+ (cD_3J_2(\xi r) + kB_3J_0(\xi r))e^{r_3t}]\, d(\xi r_0) + c_2(t) \quad ...(50)$$

where $c_2(t)$ is arbitrary function of time to be determined by specific boundary condition.

In velocity distribution expression (48), as function $g(\xi, t) = A_1e^{r_1t} + A_2e^{r_2t} + A_3e^{r_3t}$ tends to zero very rapidly for large ξ, the usual circulartory flow with a potential is observed. In the region where $g(\xi, t)$ is different from zero, the velocities are smaller than those of corresponding potential flow and flow is rotational.

It is interesting to note that velocity distribution expression (48) reduces to classical Navier-Stokes equations solution putting α, β, k, η and c equal to zero in it. $Q(S)$ in this case reduces to

$$Q(S) = S^3 + S^2\,\mu\xi^2/\rho, \quad ...(51)$$

Therefore, roots of $Q(S) = 0$ (i.e., r_1, r_2 and r_3) becomes 0, 0 and $-\mu\xi^2/\rho$. Substituting these values in eqn. (48), we get

$$\frac{v}{v_0} = \int_0^\infty J_1(\xi r)\,.\exp\left(\frac{-\mu\xi^2}{\rho}\,t\right) d(\xi r_0) \quad ...(52)$$

as $A_1 = A_2 = 0$ and $A_2 = \rho^2 I^2$ in this case.

On integration, eqn. (52) gives

$$\frac{v}{v_0} = \frac{r_0}{r}\,\{1 - \exp\,(-r^2/(4\mu_\rho\, t))\}, \quad ...(53)$$

where $\mu_\rho = \mu/\rho$ which is kinematical viscosity coefficient. Equation (53) represent the classical non-steady solution which describes the process of decay of a vortex filament through the action of viscosity.

The steady state part of the solution can be expressed as follows:

$$v = c_1k_1(rl_1) + c_2k_1(rel_2) + c_3/r, \quad ...(54)$$

$$\tilde{v} = m_1l_1c_1k_0(rl_1) + m_2l_2c_2k_0(rl_2) \quad ...(55)$$

and

$$\overset{*}{v} = -\,n_1l_1c_1k_2(rl_1) - n_2l_2c_2k_2(rl_2) + \frac{c}{\eta - c}\,\frac{c_3}{r^2} \quad ...(56)$$

where

$$c_1 = m_2l_2\omega_1\,\frac{c}{\eta - c}\,k_0(r_0l_2)\,.\,G^{-1},$$

$$c_2 = -\,m_1l_1\omega_1\,\frac{c}{\eta - c}\,k_0(r_0l_1)\,.\,G^{-1},$$

$$c_3 = l_1l_2r_0^2\,[n_2m_1k_2(r_0l_2)\,k_0(r_0l_1) - n_1m_2k_2(r_0l_1)\,.\,k_0(r_0l_2)]\,.\,G^{-1},$$

$$m_1 = k(\alpha l_1^2 - 2k)^{-1}, m_2 = k(\alpha l_2^2 - 2k)^{-1},$$

$$n_1 = c(\beta l_1^2 - 2(\eta - c))^{-1},\ n_2 = c(\beta l_2^2 - 2(\eta - c))^{-1}$$

and

$$\begin{aligned} G = {} & l_1 l_2[n_2 m_1 k_2(r_0 l_2) k_0(r_0 l_1) - n_1 m_2 k_2(r_0 l_1) \cdot k_0(r_0 l_2)] \\ & + \frac{c}{\eta - c} \frac{1}{r_0} [l_1 m_1 k_0(r_0 l_1) k_1(r_0 l_2) \\ & - l_2 m_2 k_0(r_0 l_2)\, k_1(r_0 l_1)], \end{aligned}$$

and, l_1^2 and l_2^2 are the roots of the equation,

$$\begin{aligned} \alpha\beta(\mu + k)l^4 - 2[\mu k\beta + \alpha(\eta - c)(\mu + k) - \alpha c^2]l^2 \\ + 4k\,[\mu(\eta - c) - c^2] = 0 \end{aligned}$$

The torque transmitted by the fluid to the cylinder is given by

$$\begin{aligned} M = M_0 [1 & + \left(\frac{cn_1}{\mu} + \frac{1}{2}\right) \frac{l_1 c_1}{\omega_1} k_2(r_0 l_1) \\ & + \left(\frac{cn_2}{\mu} + \frac{1}{2}\right) \frac{l_2 c_2}{\omega_1} k_2(r_0 l_2) + \left\{1 - \frac{c^2}{(\eta - c)\mu}\right\} \frac{c_3^1}{\omega_1 r_0^2} \\ & - \frac{c^2}{(\eta - c)\mu} + \frac{(1 + 2m_1)}{2\mu} \frac{kc_1 l_1}{\omega_1} k_0(r_0 l_1) \\ & + \frac{(1 + 2m_2)}{2\mu} \frac{kc_2 l_2}{\omega_1} k_0(r_0 l_2) \Big], \end{aligned} \qquad \ldots(57)$$

where $c_3^1 = c_3 - \omega_1 r_0^2$ and M_0 is the classical result of the required torque.

From (57), we observe that the torque transmitted in this case is increased in comparison to classical case. This increase in the torque could be attributed to the additional terms due to the inner structure of the fluid.

Numerical Results and Discussion

The velocity distributions for non-steady case are computed for different values of time and different volume concentrations of inner structure of the fluid by solving eqn. (48) numerically. We solve this equation by Runge-Kutta Method for first-order differential equations by taking the suitable limits of integration.

The velocity profiles for 10 per cent and 40 per cent volume concentration of inner structure of the fluid at different times are shown in Figs. 1 and 2 respectively.

We conclude that volume concentration of inner structure of the fluid influences the flow considerably. In (48) as the radial ditance r diminishes, velocity tends to zero and as time t increases, the vortex core, where the flow is rotational and velocity is less than that of corresponding potential flow, diffuses outward. Thus this makes a case of dissolution of a vortex filament.

The torque transmitted by the fluid to the cylinder is calculated for steady state case for different volume concentrations of inner structure of the fluid. The standard values of various viscosity and material coefficients given by Sevilla (1960) are used for this purpose. We observe that at 10 per cent volume concentration of

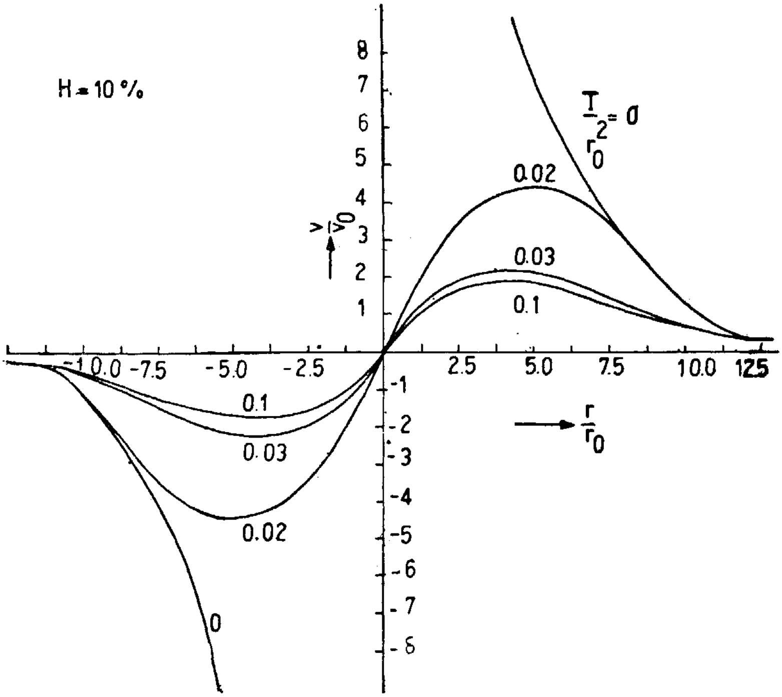

FIG. 1. Calculated velocity distributions at 10% volume concentration of inner structure for different values of time *T*.

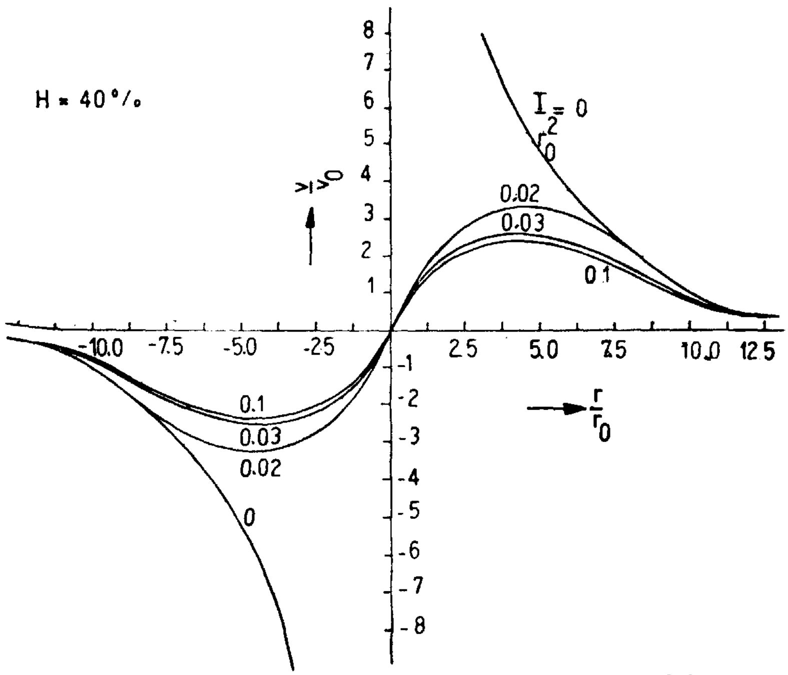

FIG. 2. Calculated velocity distributions at 40% volume concentration of inner structure for different values of time *T*.

inner structure the torque transmitted is almost 6 per cent more than the corresponding classical value of the torque. At 40 per cent volume concentration of inner structure of the fluid the torque transmitted becomes 47 per cent more than the classical value.

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