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Statistical Mechanics

HOMI JEHANGIR BHABHA MEDAL LECTURE—1981

APPLICATIONS OF STATISTICAL MECHANICS TO SOME PROBLEMS IN PHYSICS AND ASTROPHYSICS

F. C. AULUCK, F. N. A.

Professor Emeritus, Department of Physics & Astrophysics, University of Delhi, Delhi-110 007, India

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There is a deep connection between problems in statistical mechanics and problems in the partition theory of numbers. The number-theoretic methods are elegant and also more powerful. The theory of pressure ionization leads to a mass-radius relation for white dwarf stars and planets. There is a shift in the energy levels of an atom surrounded by an electromagnetic radiation, which is dependent on the intensity of the radiation field. The Thomas-Fermi theory with exchange and correlation can be used to explain the observed properties of atoms, their periodic features and their behaviour under high pressures. A gravitating fluid sphere becomes spheroidal under the influence of magnetic field and rotation and exhibits various modes of oscillation. The theory of random fragmentation can be used to explain the mass function for stars and galaxies.

Keywords : Statistical Mechanics; Partition of Numbers; Pressure Ionization; Thomas-Fermi-Dirac-Correlation Theory; Stability Problem in Magnetohydrodynamics; Random Fragmentation

MR PRESIDENT, DISTINGUISHED FELLOWS OF THE ACADEMY AND FRIENDS !

It is a great privilege to be awarded the *Homi Jehangir Bhabha Medal* for Physical Sciences by INSA. Professor Bhabha made outstanding contributions to Theoretical Physics. It was at his initiative that the Atomic Energy Establishment at Trombay was started as a centre of atomic research and development. In his Presidential Address at the International Conference for Peaceful Uses of Atomic Energy held at Geneva in 1955, he made the prophesy that a method would be found for liberating fusion energy in a controlled manner and with that the energy problems of the world would be solved forever. The experimental work in Plasma Physics done in various laboratories indicates that this goal is likely to be achieved in the very near future and Bhabha's prediction will prove to be correct. Bhabha was largely responsible for setting up the Indian National Committee for Space Research. He was also the Chairman of the Government Electronics Committee. The Tata Institute of Fundamental Research was established by him and is now one of the leading centres of research in mathematics, theoretical and experimental physics.

(§I) Statistical Mechanics and Partitions of Numbers

The central problem in statistical mechanics is that of distributing a given amount of energy among a given number of particles of systems. This problem is essentially the same as that of determining the number of ways in which a positive integer can be written as the sum of integral positive integers under certain restrictions. The theory of partitions of numbers has been extensively developed by Hardy and Ramanujan (1918). An asymptotic formula for $p(n)$, the number of unrestricted partitions of a positive integer n into positive integral parts given by Hardy and Ramanujan is

$$p(n) \sim \frac{1}{(4\sqrt{3})n} \exp\,(\pi\sqrt{2n/3}),$$

Rademacher (1937) has given an exact formula for $p(n)$.

The first application of $p(n)$, due to Böhr and Kalckar (1937), is for estimating the density of energy levels for a heavy nucleus. Assuming that the energy levels are equidistant with a spacing equal to Δ, they obtained for the number of levels with energies between E and $E + dE$ of a nucleus the formula

$$\rho(E)\,dE = \frac{1}{(4\sqrt{3})E} \exp\,(\pi\sqrt{2E/3\Delta})\,dE.$$

Following the statistical theory of nuclear structure, as propounded by Böhr, Auluck and Kothari (1946) studied the thermodynamical properties of an assembly of n identical (non-interacting) linear simple harmonic oscillators. For Bose assemblies they obtain the expression for $p_m(n)$, the number of ways a given number n can be written as the sum of exactly m parts; and for Fermi assemblies they obtain the expresion for $q_m(n)$, the number of partitions of n in m summands which are all different.

This problem can be generalised in a number of ways. The energy levels can be taken to be proportional to r^s for $r = 1, 2, 3, \ldots$, where s is a positive integer. We (Auluck & Kothari, 1946) obtained for $p(n; s)$ the number of ways of writing n, the sum of the sth powers of positive integers, the expression due to Hardy and Ramanujan

$$p(n; s) \sim (2\pi)^{-(s+1)/2} \left(\frac{s}{s+1}\right)^{1/2} k n^{[1/(s+1)]-3/2} \exp\,\{(s+1)\,k n^{1/(s+1)}\}$$

where $$k = \left[\frac{1}{s}\,\Gamma\left(1 + \frac{1}{s}\right)\zeta\left(1 + \frac{1}{s}\right)\right]^{s/(s+1)}.$$

For practical problems s is not an integer. In this case, we define

$$P_h(u) = \frac{P(u+h) - P(u)}{h}$$

where $P(u)$ is the number of solutions, in integers $n_r \geqslant 0$ of

$$n_1\lambda_1 + n_2\lambda_2 + \ldots + n_r\lambda_r < u,$$

where the λ_i s are a given set of numbers, not necessarily integers, such that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \ldots$, h is a positive number. Asymptotic expressions for $P_h(u)$ have been obtained by Ingham (1941) and Auluck and Haselgrove (1952). If λ_is are all integers, $h = 1$. For other cases h is a positive non-integral number for which $P_h(u)$ is an increasing function of u.

For assemblies characterised by the conservation of two parameters, say energy E and angular momentum P, one can define $p(m, n)$ as the distinct number of ways, in which an assembly of particles corresponding to given values of $E = m\epsilon_0$ and $P = n\eta_0$ can be realised, assuming that each particle of the assembly can occupy the levels (r, s) (r, s are non-negative integers) where the contribution of the level (r, s) to E is $r\epsilon_0$ and to P is $s\eta_0$. One can obtain the asymptotic formula for $p(m, n)$ which reduces to for $m = n$ (Auluck, 1953).

$$p(n, n) \sim \frac{c^{19/12}}{(2\pi)^{3/4} 3^{1/2} n^{55/36}} \exp\left[3cn^{2/3} + 3dn^{1/3} + \zeta'(-1) + \tfrac{9}{4}d^2 + \tfrac{1}{4}\right],$$

where $c = [\zeta(3)]^{1/3}$, $d = \zeta(2)/3c$.

We discuss some properties of $p_m(n)$ for large values of n. It was shown by Auluck *et al.* (1942) that if m_0 is the value of m for which $p_m(n)$ is maximum

$$m_0 \sim \frac{\sqrt{6n}}{\pi} \ln \frac{\sqrt{6n}}{\pi},$$

A more exact expression has been given by Szekeres (1953) which proves a conjecture of Auluck *et al.* (1942) on the uniqueness of the maximum of $p_m(n)$. If $\bar{m}$ denotes the mean value of m defined according to

$$\bar{m} = \sum_{n=1}^{n} m p_m(n) / \sum_{m=1}^{n} p_m(n),$$

an expression for $\bar{m}$ can be obtained for large values of n (Husimi, 1938). It is seen that $\bar{m} \neq m_0$ but

$$\bar{m}/m_0 \to 1 \quad \text{as} \quad n \to \infty,$$

i.e., the mean and the most probable states are the same when the number of particles is large.

Haselgrove and Temperley (1954) have obtained an asymptotic formula for the number of partitions $p_m(n; \lambda_1, \lambda_2 \ldots)$ of a large positive integer n into m parts $(\lambda_1, \lambda_2, \ldots)$. They have shown that the most probable value m_0 is different from the average value $\bar{m}$ if $\sum \frac{1}{\lambda_r}$ is convergent, in this case $\bar{m}/m_0$ does not tend to unity as n tends to infinity. For example, if we take the summands to be the squares of the integers 1, 2, 3 ... then

$$m_0/\bar{m} \to 0.5504.$$

We have referred to the problem of partitions into non-integral numbers. These summands, in any partition, can be arranged in an ascending order of magnitude and the nearest-neighbour spacings determined. The study of the distribution of these spacings is rather a difficult problem. The corresponding problem in the statistical theory of the distribution functions of the nearest-neighbour spacings in the energy levels of the nuclei has been dealt with by Wigner (1951), Mehta (1960), Gaudin (1961), and Porter and Thomas (1956). According to Wigner, the probability density of $x = S/D$ is given by

$$f(x) = \tfrac{1}{2}\pi x \exp(-\tfrac{1}{2}\pi x^2),$$

where S describes any nearest-neighbour spacing and D the average spacing of the nuclear levels. This function shows a maximum and tends to zero for large values of x. This result is in good agreement with the results of Gaudin and Mehta who have discussed the eigen-values of a random matrix under certain restrictions. We have studied the nearest-neighbour spacings of the numbers of the form $m + n\lambda$ less than are equal to N, where m and n are positive integers (including zero) and λ is an irrational. If all the numbers $m + n\lambda < N$ are put in an ascending order, the number of the nearest-neighbour spacings are $\frac{N^2}{2\lambda} + 0(N)$. The average spacing is $D = 2\lambda/N$. The frequency distribution here also shows a behaviour similar to that described above (Auluck & Ahluwalia, 1966).

In the theoretical study of molecular size distribution in polymers, it is customary to obtain the desired results following either the probability method or Darwin Fowler method of the steepest descent or by solving the appropriate set of coupled rate equations. Invariably, the results obtained by these conventional methods are identical. However, the theory can be developed exactly for all degrees of freedom in terms of the partition theory of numbers (Auluck *et al.*, 1957). In a series of papers, Nanda and Pathria (1959, 1962) have employed the theory for the purpose of studying theoretically the molecular size distribution in linear polymers on degradation. The number theory procedure is elegant and also more powerful. Nanda (1964) has also shown that the conventional results for molecular size distribution in condensation polymerization become invalid for very high degrees of freedom. Recently, Nanda (1980) has applied the number theory approach to the problem of multi-chain polymerization and derived exact results for the various quantities of interest. Here again, the conventional results cannot be relied upon for very high degrees of polymerization.

(§II) Pressure Ionization and White Dwarf Stars

In degenerate matter, the behaviour of the electron gas is almost independent of the temperature and in consequence the Saha ionization formula is inapplicable for determining the degree of ionization of degenerate matter. In degenerate matter, the ionization is primarily a consequence of high pressure. The theory of ionization is degenerate matter composed of atoms of one kind only has been worked out by Kothari (1938). In this theory, the mean molecular weight μ per free electron

becomes a function of density. The study of the structure of cold stellar bodies incorporating the theory of pressure ionization leads to the mass-radius relation for a spherical configuration. Figure 1 shows the theoretical relation between mass M and radius R for degenerate bodies.

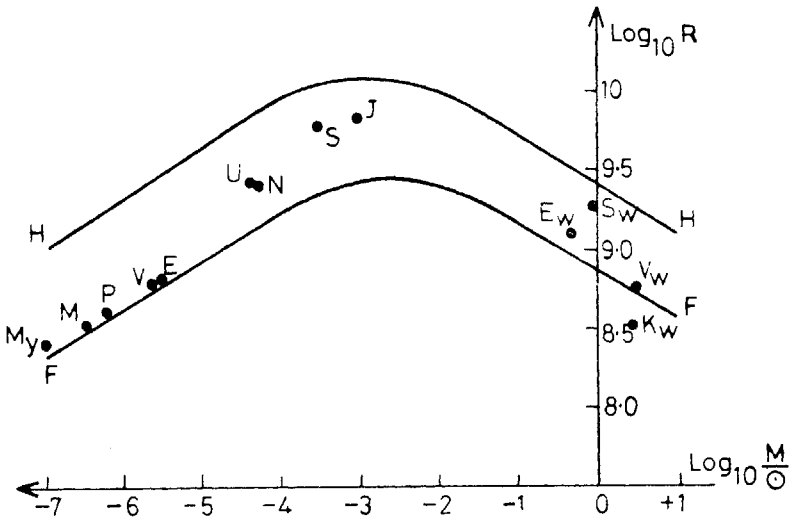

FIG. 1. The curves show the theoretical relation between mass M and radius R. ($\odot$ = mass of the Sun.) The curve HH is for hydrogen and the curve FF is for iron.

M_Y = Mercury. M = Mars. S = Saturn. E_W = O_2 Eridoni B.
V = Venus. J = Jupiter. N = Neptune. S_W = Sirius B.
E = Earth. U = Uranus. P = Pluto. K_W = A.C. 70° 8247.
V_W = Van Maanen's star.

The theory leads to the existence of a maximum radius for cold bodies. The value of the maximum radius R_0 and the corresponding mass M_0 are found to be of the same order of magnitude as the radius and mass of the planet Jupiter. The curves can be approximately fitted into (Auluck, 1939).

$$\frac{R_0}{R} = \tfrac{1}{2}\left[\left(\frac{M}{M_0}\right)^{1/3} + \left(\frac{M_0}{M}\right)^{1/3}\right].$$

For $M >> M_0$ we have the well-known equation of the white dwarf theory

$$R = \frac{l}{\mu_0^{5/3}}\left(\frac{\odot}{M}\right)^{1/3}$$

where $l = 2.79 \times 10^9$ cm.

Another approach to the same problem can be through the study of bounded linear harmonic oscillator. Under certain simplified conditions, the state of an electron inside a cold stellar mass could be represented by the wave functions of a

simple harmonic oscillator; a particle free to move inside a spherical mass would, under the gravitational force, describe an oscillatory motion. The potential energy energy would be $(mu-ev)$ where u is the gravitational potential and v the electrostatic potential. The final mass-radius relation (Auluck, 1942) is practically identical with the formula given above. In the case of Sirius B, the calculated radius for this relation comes out to be for $\mu_0 = 2$ (no hydrogen) 1.6×10^9 cm. This may be compared with the value 0.8×10^9 cm given by Milne's formula. The observed value is 2×10^9 cm. When the relativistic theory of the oscillator is used, it is found (Auluck, 1942) that for a mass exceeding a critical mass, the radius of the equilibrium configuration tends to zero. This critical mass is about three times the Chandrasekhar limit. Again, it can also be shown that for white dwarfs the gravitational fields are more important than the electrostatic fields whereas the electrostatic fields are of great importance in the case of planets (Auluck & Kothari Jr., 1951).

(§III) Effect of Electromagnetic Radiation on the Lamb Shift

The Lamb Shift was discovered by Lamb and Retherford (1947). In the hydrogen atom, the 2S and $2p_{1/2}$ levels are not coincident as required by Dirac's equation for the hydrogen atom but the 2S level is higher than the $2p_{1/2}$ level by (1057.77 ± 0.10) Mc/s. The theory of the effect was first given by Bethe (1947) in the paper entitled 'The E. M. Shift of Energy Levels,' and later elaborated by others. Bethe's explanation of the Lamb shift is based on the change in the electron self-energy brought about by the effect of the coulomb field on the transitions to virtual states which determine the self-energy. In a letter published in *Nature* by Kothari and Auluck (1948) it was pointed out that there is an additional shift if the atom is under the influence of an external electromagnetic radiation. They also pointed out the possible astrophysical significance of this effect.

If the atom is surrounded by black body radiation at temperature T and is subject to the influence of the entire spectrum constituting this radiation, then every level is shifted by an amount depending upon the temperature of the enclosure. For a hydrogen atom the shift in the energy-state $(n, 1)$ is given by (Auluck & Kothari, 1952; and Inderjit Singh, 1953, 1955).

$$W'(T) = -\frac{2\alpha}{\pi\mu}(RT)^2 \int_0^\infty \frac{x\,dx}{e^x - 1} \sum_i \frac{(E_i - E_n)^2 f(n, i)}{(E_i - E_n)^2 - (xRT)^2}$$

where $f(n, i)$ is the oscillator strength for the transition from the initial state n to the state denoted by i. Here $\mu = mc^2$ and α is summerfeld fine structure constant. We can thus determine $W'(T, 2p) - W'(T, 2S)$ at any temperature. To the first order this shift is proportional to $(RT)^4$ i.e., the energy density of radiation. It is in a direction apposite to the Lamb shift. If the atom is contained in the enclosure which is filled with monochromatic radiation lying in the momentum range k to $k + dk$, then the shift due to radiation per unit frequency is

$$W_i = -\frac{8\pi\alpha\hbar^3 I}{\mu k^2}\sum_i \frac{(E_i - E_n)^2 f(n, i)}{(E_i - E_n)^2 - c^2k^2},$$

where I is the intensity of radiation per unit frequency range. The radiation shift varies linearly as the intensity of radiation and is significant only for high intensities of radiation. By using the optical pumping techniques Alfred Kastler and his group (1963) have shown that the energy levels in Hg^{199} are shifted when the atom is surrounded by an intense radiation field, and that the energy shift is linearly proportional to the intensity of radiation. The shift was in the radio-frequency region and of the orders of 1 Hz. This effect was called Lamp Shift by Kastler. Aleksandrov *et al.* (1966) observed that the resonant optical transition frequency in potassium vapour shifts by about 10^9 Hz under the influence of a powerful ruby laser pulse. It may be remarked that the displacements of energy levels are caused by virtual transitions (Lamb Shift) as well as by real transitions to discrete levels (Lamp Shift).

(§IV) Statistical Mechanics of Interacting Systems

It was clear from the early days of Quantum Mechanics that one could not hope to solve the Schrödinger equation for three or more interacting particles. In these cases, one has to use approximate methods to obtain solutions of realistic physical problems. One such approach is the Thomas-Fermi model for an atom. Though initially the model was meant to obtain properties of an atom, it has been extended to the study of molecules and solids. Lieb and Simon (1977) have given a firm mathematical basis for Thomas-Fermi model of the Quantum Theory of atoms, molecules and solids. By taking into account exchange and correlation, one can put the theory of this statistical model of an atom on a realistic basis and explain a large variety of physical phenomena that are dependent upon interatomic interactions.

In order to make such a calculation, one needs a theory of correlation at degenerate-electron-gas densities, and a prescription for interpolating the correlation energy from Wigner's expression valid for low-density electron gas and Gellmann and Brueckner's expression valid for high-density electron gas. Inspite of attempts by various authors to derive a theoretical expression for the correlation energy for all densities, no satisfactory formula is yet available. However, there are a number of interpolation formulae which reproduce the known exact high and low density limits. Taking account of the correlation energy, the TFDC model (Thomas-Fermi-Dirac-Correlation) one obtains the equation

$$\frac{d^2\chi}{dx^2} = \lambda^3/x^{1/2},$$

where $x = r/\mu$, $\mu = \frac{a_0}{4}\left(\frac{9\pi^2}{2Z}\right)^{1/3}$,

$$\chi = \frac{r}{Ze}(V(r) - V_0).$$

and $$\lambda = \mu\left(\frac{4\pi}{Z}\right)^{1/3}\chi^{1/2}\rho^{1/3}.$$

ρ denotes the electron-density, $V(r)$ the electrostatic potential and V_0 is constant which is equal to the maximum value of the potential. The solutions of the TFDC equation have been given for a number of atoms like Argon, Chromium, Krypton, Xenon and Uranium (Erma, 1963). These solutions can be used to derive many physical properties of the atoms. We discuss two such properties. We shall first discuss the problem regarding the periodic features of atoms (Auluck *et al.*, 1968; and Auluck & Jain, 1978). Since the statistical model of the atom is based on the assumption that the electrons surrounding the nucleus constitute a totally degenerate electron gas at zero temperature, in this approximation the usual K, L, M ... peaks of the wave mechanical electron distribution are smoothed out into a plain charge cloud devoid of orbital features. However, the quantum numbers N^*, N_r^* and K^* can be introduced in the TFDC model by using the relations

$$K^* = M/h,$$

$$N_r^* = \frac{1}{2\pi\hbar} \oint p_r dr,$$

and

$$N^* = N_r^* + K^*.$$

Table I gives the energy levels for Argon.

TABLE I

—Enl (in Rydbergs) for argon

	1s	*2s*	*2p*	*3s*	*3p*
TFDC ($K^* = \sqrt{l(l+1)}$)	208.6	19.51	15.63	1.482	0.7214
TFDC ($K^* = l + \frac{1}{2}$)	224.1	20.68	16.16	1.508	0.6107
TFDC ($K^* = l + \frac{1}{2}$) (Relativistic)	230.0	21.31	16.25	1.5666	0.6193
Hartee-Fock-Slater method	232.536	22.8647	18.2072	21.1067	1.0653
Experimental (Landoit & Bornstein)	235.7	24.2	18.0	2.15	0.80

It is seen that the assignment $K^* = l + \frac{1}{2}$ gives slightly better results than the assignment $K^* = \sqrt{l(l+1)}$. Also the relativistic corrections become more important for the s states and higher Z values.

The second problem we shall discuss is the effect of high pressures. Though the Thomas-Fermi method is not suitable for studying crystals under normal conditions whose properties depend upon the details of the electronic shells, at sufficiently high pressures the detailed influence of the outer electronic structure is obliterated and the results of the Thomas-Fermi model become reliable. The information thus obtained has proved very valuable in various problems of astrophysics such as the determination of mass-radius relation of planets (Kothari, 1938; and Auluck, 1939) and in the analysis of gravitational collapse of superdense matter (Harrison *et al.*, 1965). In recent years, however, it has become possible to attain in the laboratory sufficiently

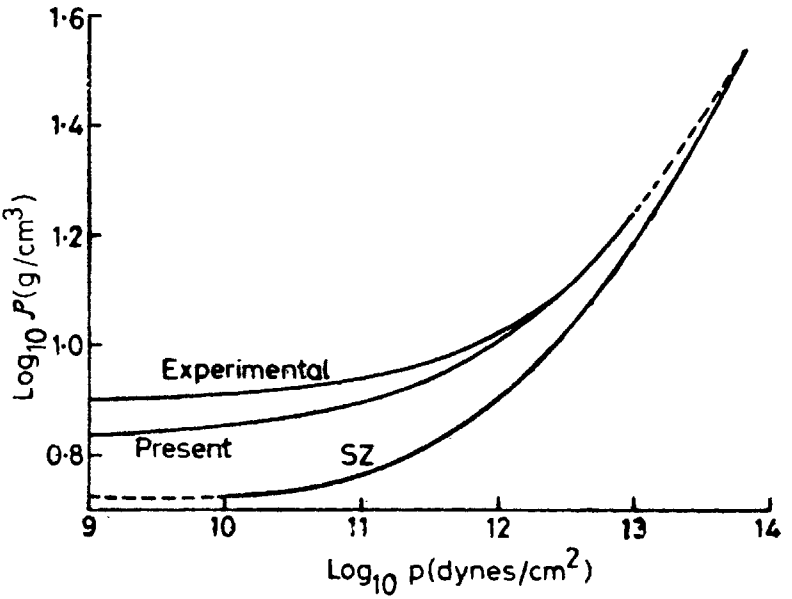

FIG. 2. Relation between density and pressure, taking into account of the exchange, the quantum and the correlation correction terms.

high densities corresponding to several megabars of pressure, by using shock-wave techniques. These experiments (Knopoff, 1963) give us data on the bulk properties of high density matter i.e., the equation of state and the change in the electronic structure at high pressures. The calculation of pressure including the correlation energy term has been done by Salpeter and Zapolsky (1967). However, by using the expression for the correlation energy given by Barnes (1965), one can obtain the relation between pressure and density which is in good agreement with the experimental data. This is shown in Fig. 2.

An interesting feature of these calculations is the prediction of shifts in energy levels at high pressure.

Table II gives the energy levels (in Rydberg) against pressure in Megabars for *Cs* ($Z = 55$). It is seen that there are shifts in the energy levels at high pressures which are more prominent for the higher states. These shifts, as is seen from the table, are of the order of a Rydberg and hence are measurable. Metals possessing internal defects are known to have internal stresses of the order of megabars and hence, if one studies the X-ray absorption or emission spectra of such solids, the predicted shifts may be observed. This would, therefore, give us a method of relating the internal defects directly to X-ray spectra of solids.

(§V) STABILITY PROBLEM IN MAGNETOHYDRODYNAMICS

Chandrasekhar and Fermi, in 1953, considered the problem of the flattening of a gravitating fluid sphere, the fluid being assumed to be incompressible, inviscid and infinitely conducting, under the influence of a magnetic field which is uniform inside

TABLE II

Energy levels (in Rydbergs) against pressure for Cs($Z = 55$)

Pressure (in Megabars) State	1	29	155	1888
1*s*	2420.99	2420.49	2420.10	2419.83
2*s*	404.25	401.50	397.95	370.01
2*p*	376.55	376.03	375.63	375.69
3*s*	113.14	112.41	111.65	109.54
3*p*	103.75	103.14	102.40	100.67
3*d*	88.10	87.40	86.68	83.40
4*s*	39.81	38.90	38.00	35.37
4*p*	36.07	35.37	34.35	31.78
4*d*	30.10	29.29	28.30	25.81
4*f*	23.71	23.27	22.21	19.68
5*s*	15.64	14.75	13.74	12.14
6*s*	6.44	5.45	4.56	1.98

and a dipole field outside. The sphere tends to become an oblate spheroid. Since this pioneering work, several investigators have considered the problem of the equilibrium and stability of liquid and gaseous mass with a prevalent magnetic field and rotation. It can be shown (Auluck & Kothari, 1956, 1957 *a*, *b*) that to the first order, a sphere tends towards an oblate or prolate spheroidal configuration depending upon the type of field : Prendergrast (1956) has constructed a system of currents which keep a uniform fluid sphere in equilibrium. Prendergast (1958) has also investigated its stability and proved that it is unstable if the magnetic energy exceeds two-fifth of the gravitational energy. In this case there is no magnetic field outside the sphere. The field is continuous at the boundary and vanishes identically at the surface of the sphere. It is possible to have other types of magnetic fields and currents which keep a spherical gravitating mass in equilibrium. In the case of gravitating cylinders, the magnetic field increases the stability of cylinder (Auluck & Kothari, 1957 *a*, *b*, 1965; and Auluck & Nayyar, 1960). The basic result is that the normal modes of oscillations are the cylindrical harmonics—one mode far each harmonic and that for $m = 0$, there is a critical wave length above which the cylinder is unstable.

In actual situation the fluid is compressible. Chandrasekhar (1964) established a general variational principle applicable to radial and non-radial oscillations of compressible and gravitating gaseous masses in the absence of a magnetic field. A general variational principle taking into account the magnetic field was developed for the study of stellar pulsations (Singh & Tandon, 1968). The various modes of oscillations (radial and non-radial) have been computed in the presence of a magnetic field (Singh & Tandon, 1969; and Grover, Singh & Tandon, 1973). It is found that the characteristic frequency of the stable modes increases with a low magnetic field. A small magnetic field suppresses the convectively unstable mode.

The effect of magnetic field on the equilibrium structure and oscillation of a star assumed to be polytrope has been studied by various authors (Trehan & Billings,

1971, Billings *et al.*, 1973; Trehan & Uberoi, 1972; and Sood & Trehan, 1972) and various modes of oscillation discussed. Das and Tandon (1976 *a*, *b*) have studied the structure and equilibrium magnetic field for a magnetic polytrope. In all these investigations the magnetic field is assumed to be small. The equilibrium structure and oscillation with a large general magnetic field have been studied by Das and Tandon (1976*a*, *b*). They have also taken rotation and radiation pressure into account (1980). It is shown that the contribution of a magnetic field and rotation on the shape of a star is not purely additive. Because of the coupling between rotation and magnetic field, both magnetic field and rotational field change. This interaction, in fact, determines the shape of the equilibrium configuration. However, for uniformly rotating stars the contribution of magnetic field and rotation are additive. The effect of both uniform and differential rotation on various stellar parameters has been investigated and it is shown that the models with large values of central condensation are more are less spherical for sufficiently large values of rotation. Also, the effect of increasing the rotation parameter in various models is significant for various structural papameters e.g., mass, central condensation, oblateness etc.

(§VI) Random Fragmentation and Evolution of Stars and Galaxies

The theory of random fragmentation is of considerable interest in several fields, such as cosmology, statistical thermodynamics, degradation of linear polymers, level distribution in spectra of complex atoms, energy distribution of heavy nuclei, frequency of occurrence of earthquakes according to their magnitude, etc. The study by Auluck and Kothari (1965) begins by considering the random fregmentations of a line of length 1 into N parts. The average number of fragments greater than or equal to x is

$$N(x) = N\left(1 - \frac{x}{l}\right)^{N-1}.$$

This can be applied to the division of a rectangular parallelopiped by planes parallel to the sides intersecting the edges in random lengths. This leads to an expression for the number of fragments $n(V)$ with volumes in unit range about V

$$n(V) = \frac{4N_0}{V_0} \int_0^\infty \frac{K_0(x)}{x} \exp(-4a/x^2)\, dx,$$

where N_0 is the total number of fragments and V_0 the average volume of a fragment. N_0V_0 is the total volume of the cloud, $a = V/V_0$ and $K_0(x)$ is the Bessel function of imaginary argument. Asymptotic expressions are obtained for $a \to 0$ and $a >> 1$. Assuming a uniform mass density in the cloud, the expression given above represents the number of stars formed per unit range in mass i.e., the mass function $F(M)$. The fragmentation formula contains two parameters, the total number of stars and the average mass of a star. If we know the number of stars in any two ranges, we can determine the average mass as well as the total number of stars in a

cluster and hence the total mass in the cluster. Carles Jaschek and Mercedes Jaschek (1961) have applied the fragmentation theory to a number of clusters for which mass frequency data were known. They calculated, by using the above formula, the average mass of a star and the total mass of stars in the cluster. The calculated values of the total masses in the clusters differ from the total masses of the observed stars by a factor of 2. They pointed out that this may be due to the omission of all unidentified cluster members giving, therefore, a lower limit to the real total mass, on the other hand, the total mass obtained from the fragmentation formula is almost certainly too high since the smallest fragments have probably not condensed into stars, consequently a discrepancy of a factor of about two is not unreasonably high.

A number of alternative derivations of the initial mass spectrum have been given. Kruszewki (1961) considers the the formulation of stars as due to prestellar matter. According to him the mass spectrum should depend only on the parameters characterising this primevel matter, namely its density and the velocity of sound in it. He considers a homogeneous infinite medium in which the mean density and velocity of sound do not depend on the coordinates. He also assumes that the probability distribution of density perturbation does not depend on the linear extent of the perturbation. This leads to the frequency distribution per unit mass of proto-star masses

$$F(M) = \frac{1}{2M^2} (\ln M)^2,$$

where M is in units of the mass M_0 given by Jean's criterion. This may be compared with the initial mass function obtained by Salpeter

$$F(M) \sim M^{-2.35},$$

This is in keeping the observation that the relative number of stars of different masses are remarkably similar in different clusters. The relative number of stars decreases with increasing mass from the most numerous at perhaps a few hundredth of a solar mass to the comparatively rare objects of a few hundred solar masses. If a large gas clould breaks into fragments of different sizes, then the fragments contract to become stars. The spectrum of stellar masses, the mass function, can be obtained from count of stars in given ranges of brightness—the luminiosity function and a knowledge of the relation between mass and luminosity. According to Reddish (1978) the fragmentation of a gas cloud is purely a physical process. He assumes that the gravitational energy of the cloud is divided equally among equal ranges of linear fragment size. This leads to the mass function

$$F(M) \simeq AM^{-7/3},$$

when $F(M)\, dM$ is the fraction of fragments in the mass range M and $M + dM$ and A is a constant. This function does not fit in with observation for large masses. If we assume that the gravitational potential energy is randomly distributed among the gravitational potential energies of the various fragments (assumed to be spherical), there being no restriction on the number of fragments of given mass, we obtain

$$F(M)\,dM = C\,\frac{M^{-2/3}\,dM}{\exp\,(M/M_1)^{5/3} - 1},$$

where M_1 and C are constants. C can be expressed in terms of M_1 and M_f the minimum mass. $M_1 \sim 0.85M_b$ where M_b is the brightest member of the assembly. For small masses $F(M)$ varies as $M^{-7/3}$ and for large masses $F(M)$ tends to zero exponentially. This formula is in good agreement with observational data for stars as well as galaxies in different clusters.

REFERENCES

Aleksandrov, E. B., Khodovoi, V. A., and Varilov, S. L. 1966) Frequency shift of optical transition in the field of a light wave. *Zhetif Pisma*, **3**, No. 2, 85–88.

Auluck, F. C. (1939) The theory of pressure ionization and the constitution of white dwarf stars. *Mon. Not. R. astron. Soc.*, **99**, 239–247.

——— (1942) White dwarf and harmonic oscillator. *Proc. natn. Inst. Sci. India*, **8**, 147–156.

——— (1953) On partitions of bipartite numbers. *Proc. Camb. phil. Soc.* **42**, 72–83.

Auluck, F. C. (1967) Random fragmentation. *Pres. Addr. Phys. Sec.*, 54*th* Indian Sci. Congr.

Auluck, F. C., and Ahluwalia, A. S. (1966) On partitions into non-integral numbers and the nearest neighbour spacing distribution function. *J. math. Sci.*, **1**, 9–11.

Auluck, F. C., Chowla, S., and Gupta, H. (1942) On the maximum value of the number of partitions of n into k parts. *J. Indian math. Soc.*, **6**, 105–112.

Auluck, F. C., and Haselgrove, C. B. (1952) On Ingham's Tauberian theorem for partitions. *Proc. Camb. phil. Soc.*, **48**, 566–570.

Auluck, F. C., and Jain, Ashok (1978) *The Statistical Model of Atom and its Applications. Indian nat. Sci. Acad. scient. Rep. Ser.*, **2**, 1–57.

Auluck, F. C., and Kalra, G. L. (1978) Random fragmentation and formation of galaxies In : *Frontiers of Theoretical Physics.* Macmillan & Co. India Ltd., 169–171.

Auluck, F. C., and Kothari, D. S. (1946) Statistical mechanics and the partitions of numbers. *Proc. Camb. phil. Soc.*, **42**, 272–277.

——— (1946–47) Partitions into powers of integers. *Proc. R. Irish Acad.*, April 15 Sessions, 13–16.

——— (1952) Effect of e.m. radiation on the Lamb shift. *Proc. R. Soc.*, **214**, 137–142.

——— (1954) Random fragmentation. *Nature*, **174**, 565–567.

——— (1956) Effect of magnetic field on gravitating spheres. *Z. Astrophys.*, **38**, 237–244.

——— (1957*a*) On the change in shape of a gravitating sphere subject to the influence of a magnetic field. *Proc. natn. Inst. Sci. India*, **23**, 100–107.

——— (1957*b*) The influence of gravitating cylinder. *Z. Astrophys.*, **42**, 101–113.

Auluck, F. C., and Kothari, D. S. (1965) Random fragmentation. *Z. Astrophys.*, **63**, 9–14.

Auluck, F. C., Kothari, D. S., and Luthra, S. M. (1957) The degradation of high polymers and the partition theory of numbers. *Curr. Sci.*, **26**, 173–175.

Auluck, F. C., and Kothari, L. S. (Jr.) (1951) Electrostatic fields in white dwarf stars. *Proc. natn. Inst. Sci. India*, **17**, 15–20.

Auluck, F. C., Kumar, Satish and Jain, Ashok (1968) Periodic features of the statistical atom with exchange and correlation. *Canad. J. Phys.*, **46**, 1–13.

Auluck, F. C., and Nayyar, N. K. (1960*a*) The stability of a gravitating cylinder in the presence of magnetic field. *Z. Astrophys.*, **50**, 7–13.

——— (1960*b*) On the stability of a gravitating cylinder in the presence of a magnetic field. *Proc. natn. Inst. Sci. India*, **26**, 469–477.

Barnes, J. F. (1965) *Los Alamos Report*, LA-3268.

Bethe, H. A. (1947) Ths e.m. shift of energy levels. *Phys. Rev.*, **72**, 339–341.

Billings, D. F., Singh, M., and Trehan, S. K. (1973) Magnetically distorted polytropes. *Astrophys. Space, Sci.*, **25**, 457–469.

Böhr, N., and Kalckar, F. (1937) *Kgl. Danske Vid. Selskab. Math. Phys., Medd.*, **14**, No. 10.

Chandrasekhar, S. (1964) A general variational principle governing the radial and the non-radial oscillations of gaseous masses. *Ap. J.*, **139**, 664–674.

Chandrasekhar, S., and Fermi, E. (1953) Problems of gravitational stability in the presence of a magnetic field. *Ap. J.*, **118**, 116–141.

Das, M. K., Kar, J., and Tandon, J. N. (1981) The effect of a magnetic field on the adiabatic oscillations of convective stellar models with radiation pressure (*In print*).

Das, M. K., and Tandon, J. N. (1976*a*) On the magneto-rotating polytropes. *Ap. J.*, **209**, 233-242.

——— (1976*b*) On the uniformly rotating polytropes. *Astrophys. Space Sci.*, **49**, 261–275, 277–291.

——— (1980) Rapidly rotating stellar model with radiation pressure. *Astrophys. Space. Sci.*, **67**, 409–419.

Erma, V. A. (1963) Extension of statistical model of the atom. *Phys. Rev.*, **132**, 1100–1110.

Feynman, R. P., Metropolis, N., and Teller, E. (1949) Equations of state of elements based on generalized Fermi-Thomas Theory. *Phys. Rev.*, **75**, 1561–1573.

Gaudin, M. (1961) Sur La loi Limit de 1' espacement des valueurs propres d' une matrice aleatoire. *Nucl. Phys.*, **25**, 447–458.

Grover, S. D., Singh, S., and Tandon. J. N. (1973) The hydromagnetic oscillations, and stability and self-gravitating mass, III. Magnetic polytropes. *Astron. Astrophys.*, **22**, 133-137.

Hardy, G. H., and Ramanujan, S. (1918). Asymptotic formula in combinatory analysis. *Proc. Lond. math. Soc.*, **17**(2), 75-115; also *Collected Papers of S. Ramanujan*. Camb. Univ. Press, 1927.

Harrison, B. K., Thorne, K. S., Wakano, M,, and Wheeler, J. A. (1965) *Gravitational Theory and Gravitational Collapse*. University of Chicago Press, U. S. A.

Haselgrove, C. B., and Temperley H. N. V. (1954) Asymptotic formulae in the theory of partitions. *Proc. Camb. phil. Soc.*, **50**, 225–241.

Husimi, K. (1938) *Partitio numerorum* as occurring in a problem of nuclear physics. *Proc. phys. math. Soc. Japan*, **20**, 912–925.

Ingham, A. E. (1941) A Tauberian theorem for partitions. *Ann. Math.*, **42**, 1075–1090.

Jaschek, C., and Jaschek, M. (1961) Note on the random fragmentation in star formation. *Z. Astrophys.*, **53**, 302–304.

Kastler, A. (1965) Displacement of energy levels of atoms by light. *J. opt. Soc. Am.*, **53**, 902–910.

Knopoff, L. (1963) *High Pressure Physics and Chemistry*. Academic Press.

Kothari, D. S. (1938) The theory of pressure ionization and its applications. *Proc. R. Soc.*, **165**. 486–500.

Kothari, D. S., and Auluck, F. C. (1948) Electromagnetic level shift and effect on radiation field. *Nature*, **162**, 143.

Kruszewski, A. (1961) Expected shape of the mass spectrum for stars formed by gravitational contraction. *Acta Astron.*, **11**, 196–203.

Lamb, W. E. Jr., and Retherford, R. C. (1947) Fine structure of the hydrogen atom by a microwave method. *Phys. Rev.*, **72**, 241–243.

Lieb, E. H., and Simon, B. (1977) The Thomas-Fermi theory of atoms, molecules and solids. *Adv. Math.*, **23**, 22-116.

March, N. H. (1967) The Thomas-Fermi approximation in quantum mechanics. *Adv. in Phys.*, **6**, 1–98.

Mehta, M. L. (1960) On the statistical theory of level spacing in nuclear spectra. *Nuclear Phys.*, **18**, 395–419.

Nanda, V. S. (1964) A theoretical consideration of condensation polymerization in dispersed medium. *J. Polymer Sci.*, **2**, 2275–2282.

——— (1980) A number theoretical approach to the problem of size distribution in polymers. *J. macromol. Sci.*, B **18**(4), 679–689.

Nanda, V. S., and Pathria, R. K. (1959) *Polymers and Theory of Numbers*, **30**, I. *The Single Chain Theory of Degradation*, 27–30; II. *The N-Chain Theory of Degradation*, 31–34; III.

Some Remarks on the Theories of Degradation, 35–36; and IV. *The Problem of Weak and Strong Links*, 1322–1327.

——— (1962) A theoretical study of changes in the statistical character of polydisperse polymer system underlying random degradation. *Proc. R. Soc.*, **A270**, 14–26.

Porter, C. E., and Thomas, R. G. (1956) Fluctuations of nuclear reaction widths. *Phys. Rev.*, **104**, 483–491.

Prendergast, K. H. (1956) The equilibrium of a self-gravitating incompressible fluid with a magnetic field I. *Ap. J.*, **123**, 498–508; (1958) II. *Ap. J.*, **128**, 361–374.

Rademacher, J. (1937) A convergent series for the partition function $p(n)$. *Proc. natn. Acad. Sci. U. S. A.*, **23**, 78–84.

Reddish, V. C. (1978) *Stellar Formation*. Pergamon Press.

Salpeter, E. E., and Zapolsky, H. S. (1967) Theoretical high pressure equation of state including correlation energy. *Phys. Rev.*, **158**, 876–886.

Singh, Inderjit (1953) Effect of electromagnetic radiation on Lamb Shift. *Progr. theor. Phys.*, **10**, 476–477.

——— (1955) Effect of e. m. radiation on Lamb shift, II. *Proc· natn. Inst. India* **21**, 280–289.

Singh, S., and Tandon, J. N. (1965) The hydromagnetic oscillations and stability of self-gravitating masses, I. Compressible fluid cylinder. *Annal. Astrophys.*, **31**, 559–568.

——— (1969) *Ibid.* II. Compressible fluid sphere. *Astron. Astrophys.*, **2**, 266–273.

Sood, N. K., and Trehan S. K. (1972) Non-radial oscilations and convective instability of a polytrope with a toroidal magnetic field. *Astrophys. Space Sci.*, **19**, 411-467.

Szekeres, G. (1953) Some asymptotic formulae in the theory of partitions. *Q. J. Math. Oxford*, **4**, 96–111.

Trehan, S. K., and Billings, D. F. (1971) On the oscillations of a magnetic polytrope. *Ap. J.*, **196**, 567–584.

Trehan, S. K., and Uberoi, M. S. (1972) Magnetically distorted polytropes : structure and radial oscillations. *Ap. J.*, **175**, 161–169.

Wigner, E. P. (1951) On a class of analytic functions from the quantum theory of collisions. *Ann. Math.*, **53**, 36–67.