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Isotropic Turbulence

DERIVATION OF FOURTH ORDER VELOCITY-CORRELATIONS IN TERMS OF BASIC DEFINING SCALAR 'f' OF SECOND ORDER VELOCITY-CORRELATIONS IN THE CASE OF HOMOGENEOUS AND ISOTROPIC TURBULENCE UNDER QUASINORMALITY HYPOTHESIS

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It is well known that in the case of homogeneous and isotropic turbulence relating to incompressible fluids, second order velocity-correlations are given by

$$\overline{u_i u_j'} = \overline{u^2}\left[-\frac{1}{2r}f'(r,t)\,\xi_i\xi_j + \{f(r,t) + \frac{r}{2}f'(r,t)\}\,\delta_{ij}\right],$$

where $\overline{u_1 u_1'} = \overline{u^2} f(r, t)$. Further, due to M. Millionschtchikov's hypothesis (1941*a,b*) fourth order velocity-correlations are related to second-order velocity-correlations as in a normal distribution. This hypothesis tends to resolve the indeterminancy as evident in the well-known equation due to T. v̈. Karman and L. Howarth (1938). Due to this hypothesis, the expression relating to the fourth order velocity-correlations in regard to four points P, Q, R, S is found to be

$$\overline{u_i u_j' u_k'' u_l'''} = \overline{u_i u_j'}\,.\,\overline{u_k'' u_l'''} + \overline{u_i u_k''}\,.\,\overline{u_j' u_l'''} + \overline{u_i u_l'''}\,.\,\overline{u_j' u_k''} \qquad ...(1)$$

(cf. deduction in Introduction that follows)

In the present work, the above fourth order velocity-correlations, as expressed by (1), have been deduced in terms of $f(r, t)$ alone.

Keywords : Turbulence—Homogeneous & Isotropic; Second & Fourth-Order Velocity-Correlations

INTRODUCTION

It is well known that equation due to T. v̈. Karman and L. Howarth (1938) involving two basic defining scalars $f(r, t)$ and $h(r, t)$ is given by

$$\frac{\partial}{\partial t}\{\overline{u^2}f(r,t)\} + 2(\overline{u^2})^{3/2}\left\{\frac{\partial h(r,t)}{\partial r} + \frac{4h(r,t)}{r}\right\}$$

$$= 2\nu\,\overline{u^2}\left\{\frac{\partial^2 f(r,t)}{\partial r^2} + \frac{4}{r}\,\frac{\partial f(r,t)}{\partial r}\right\}. \qquad \ldots(2)$$

The aforesaid equation is derived for an isotropic and homogeneous incompressible turbulent fluid-field at the expense of the equations of motion and the equation of continuity. Hence, all the principles of conservation being taken care of, for a homogeneous and isotropic turbulent fluid field, this is an equation containing two unknowns $f(r, t)$ and $h(r, t)$. As such, the entire problem becomes indeterminate and we cannot uniquely solve $f(r, t)$ and $h(r, t)$ from a single equation at our disposal. It is useful to note that if we try with an equation containing one higher order moment, it will give rise to a triple-correlation related to fourth order velocity-correlations. The scalar form of this equation for a homogeneous and isotropic turbulent fluid will relate one basic defining scalar of the third order velocity-correlations to that of fourth order velocity-correlations. So, in all, we get two equations involving three basic defining scalars pertaining to second order, third order and fourth order velocity-correlations. Under such circumstances, the problem would still remain indeterminate, because we are unable to obtain general solutions for three unknowns from two independent equations in a unique manner. It is to be emphasised that for the representation of a physical picture, we need have a unique set of general solutions for the above-mentioned defining scalars defining different orders of velocity-correlations. To eradicate the inherent difficulty so mentioned, M. Millionschtchikov (1941*a, b*) introduces his well-known quasi-normality hypothesis in the form,

$$\overline{u_i u'_j u''_k u'''_l} = \overline{u_i u'_j}\cdot\overline{u''_k u'''_l} + \overline{u_i u''_k}\cdot\overline{u'_j u'''_l} + \overline{u_i u'''_l}\cdot\overline{u'_j u''_k}$$

$$+\,\overline{u_i}\,\overline{u'_j u''_k u'''_l} + \overline{u'_j}\cdot\overline{u_i u''_k u'''_l} + \overline{u''_k}\cdot\overline{u_i u'_j u'''_l}$$

$$+\,\overline{u'''_l}\cdot\overline{u_i u'_j u''_k}$$

$$= \overline{u_i u'_j}\cdot\overline{u''_k u'''_l} + \overline{u_i u''_k}\cdot\overline{u'_j u'''_l} + \overline{u_i u''_l}\cdot\overline{u'_j u''_k}$$

[$\because$ $\overline{u_i} = 0$, $\overline{u'_j} = 0$, $\overline{u''_k} = 0$, $\overline{u'''_l} = 0$, while $\overline{u'_j u''_k u'''_l}$ etc. are assumed to be,

in general, non-zero as per suggestion of M. Millionschtchikov's quasi-normality hypothesis (1941*a, b*), (cf. equation (18.1), p. *242*: *Statistical Hydrodynamics* (Vol. II) — A. S. Monin and A. M. Yaglon (English Version, J. L. Lumley) where u_i, u'_j, u''_k, u'''_l are respective components of fluctuating velocities at four distinct neighbouring points P, Q, R, S having their respective space configurations determined by position vectors **X**, **X′** **X″** **X‴** with respect to a fixed frame of axis (cf. Fig. 1).

Formulation of the Problem and its Calculation

In equation (1) as stated in the previous article, we find that the left-hand side is a fourth-order velocity-correlation, while the right-hand side contains three sets of combinations of paired second order velocity-correlations under the assumption of homogeneity and isotropy in regard to an incompressible fluid. Now, we can proceed to calculate the aforesaid second order velocity-correlations as follows :

Using the technique adopted by H. P. Robertson (1940), let us write down the bilinear form for fluctuating velocities u_a and u_b at the points P and Q respectively for the case of homogeneity and isotropy as

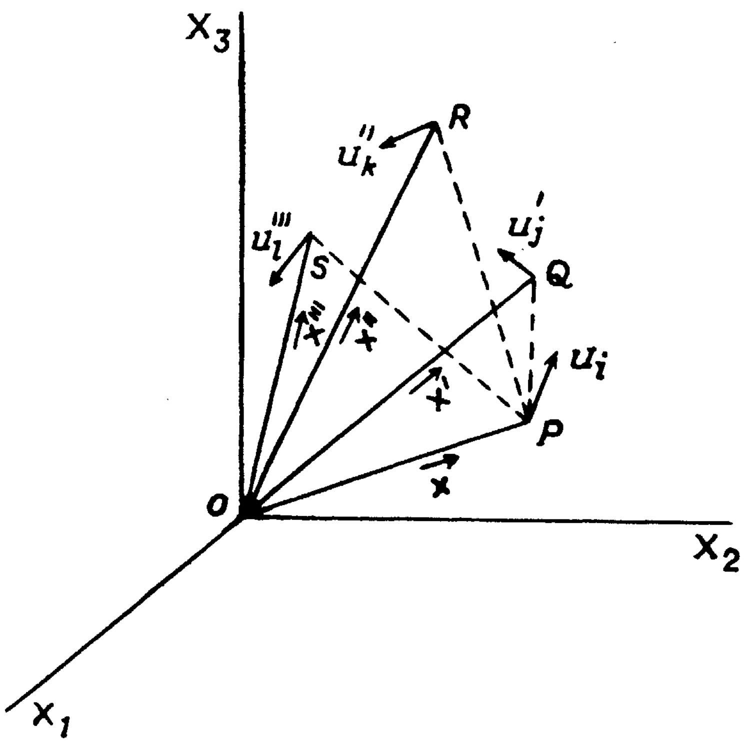

Fig. 1. 1-direction OX_1, 2-direction OX_2, 3-direction OX_3 of the axis of reference. P, Q, R, S four distinct points in the space of turbulent fluid defined by the position vectors **X**, **X′**, **X″**, **X‴** having components of fluctuating velocities as u_i, u'_j, u''_k, u'''_l respectively.

Line joining PQ set forth as **X′**—**X**
Line joining PR set forth as **X″**—**X**
Line joining PS set fourth as **X‴**—**X**

$$\overline{u_a u'_b} = \overline{u_i a_i u'_j b_j} = \overline{u_i u'_j}\; a_i\, b_j \equiv \overline{u^2}\, R_{ij} a_i b_j \qquad \ldots(3)$$

(**a**, **b** being considered as respective unit vectors at P and Q)
For homogeneity and isotropy, the basic invariants are considered as the scalar (or dot) product of two vectors chosen arbitrarily out of the three vectors **a**, **b** and $\xi(=\mathbf{PQ})$ taken in pairs.

Therefore

$$\begin{aligned}\overline{u_a\, u'_b} \equiv \overline{u^2}\, R_{ij} a_i b_j &= A(\bar{\xi}\cdot\mathbf{a})(\bar{\xi}\cdot\mathbf{b}) + B(\mathbf{a}\cdot\mathbf{b})\\ &= A\xi_i\xi_j a_i b_j + B\, a_i b_i\\ &= A\xi_i\xi_j\, a_i b_j + B\, a_i b_j \delta_{ij}\\ &= (A\xi_i\xi_j + B\delta_{ij})\, a_i b_j,\end{aligned}$$

where the defining scalars A and B are functions of $(\xi \cdot \xi) = r^2$ and relevant time 't'.

Therefore,

$$\overline{u^2}\, R_{ij} = A\xi_i\xi_j + B\, \delta_{ij}, \qquad \ldots(4)$$

Since, for an incompressible fluid,

$$\frac{\partial u'_j}{\partial x'_j} = 0$$

or

$$u_i \frac{\partial u'_j}{\partial x'_j} = 0$$

or

$$\frac{\partial}{\partial x'_j}\,(u_i u'_j) = 0$$

(because of Eulerian concept).

After taking averages over all identical sets of experiments at different locations of the same turbulent fluid, we thus obtain

$$\frac{\partial}{\partial \xi_j}\, \overline{u_i u'_j} = 0$$

where

$$\xi = \mathbf{X}' - \mathbf{X}$$

or

$$\frac{\partial}{\partial \xi_j}\, \overline{u^2}\, R_{ij} = 0. \qquad \ldots(5)$$

Introducing the physical correlations of the form

$$\overline{u_1 u_1'} = \overline{u^2} f(r, t), = \overline{u_2 u_2'} \; \overline{u_3 u_3'} = \overline{u^2}\, g(r, t) \qquad \ldots(6)$$

wherein '1' — direction is chosen along PQ (without any loss of generality), 2 and 3 directions are chosen as two equivalent mutually perpendicular directions, perpendicular to PQ.

Thus, in the context of (4) and (6), with the help of some simple algebraic calculations, we get

$$\overline{u^2}\, R_{ij} = \overline{u^2}\left[\frac{f-g}{r^2}\, \xi_i \xi_j + g\delta_{ij}\right]. \qquad \ldots(7)$$

Using relation (7) in the conditional relation (5), we easily obtain

$$g(r, t) = f(r, t) + \frac{r}{2}\frac{\partial f(r, t)}{\partial r} \qquad \ldots(8)$$

$\therefore$ From (3), (7) and (8), we find

$$\overline{u_i u_j'}\, a_i b_j = \overline{u^2}\,(t)\Big[-\frac{1}{2r}\frac{\partial f(r, t)}{\partial r}\,\xi_i\xi_j + \Big\{f(r, t) + \frac{r}{2}\frac{\partial f(r, t)}{\partial r}\Big\}\,\delta_{ij}\Big]\, a_i b_j$$

to give us, on final analysis,

$$\overline{u_i u_j'} = \overline{u^2}\,(t)\Big[-\frac{1}{2r}\frac{\partial f(r, t)}{dr}\,\xi_i\xi_j + \Big\{f(r, t) + \frac{r}{2}\frac{\partial f(r, t)}{\partial r}\Big\}\,\delta_{ij}\Big]. \qquad \ldots(9)$$

Now, we can write down the second order velocity-correlations for each of the pairs of points (P, Q), (R, S); (P, R), (Q, S); (P, S), (Q, R) as follows:

$$\overline{u_i u_j'} = \overline{u^2}\,(t)\Big[-\frac{1}{2r_1}\frac{\partial f(r, t)}{\partial r_1}\,\xi_i'\,\xi_j' + \Big\{f(r_1, t) + \frac{r_1}{2}\frac{\partial f(r_1, t)}{\partial r_1}\Big\}\,\delta_{ij}\Big] \qquad \ldots(10a)$$

$$\overline{u_u'' u_k'''} = \overline{u^2}\,(t)\Big[-\frac{1}{2(r_3 - r_2)}\frac{\partial f(r_3 - r_2, t)}{\partial(r_3 - r_2)}(\xi''' - \xi'')_k\,(\xi''' - \xi'')_l + \{f(r_3 - r_2, t) + \frac{r_3 - r_2}{2}\frac{\partial f(r_3 - r_2, t)}{\partial(r_3 - r_2)}\Big\}\,\delta_{kl}\Big] \qquad \ldots(10b)$$

$$\overline{u_i u_k''} = \overline{u^2}(t)\left[-\frac{1}{2r_2}\frac{\partial f(r_2, t)}{\partial r_2}\xi_i''\xi_k'' + \left\{f(r_2, t) + \frac{r_2}{2}\frac{\partial f(r_2, t)}{\partial r_2}\right\}\delta_{ik}\right] \quad \ldots(10c)$$

$$\overline{u_j' u_l'''} = \overline{u^2}(t)\left[-\frac{1}{2(r_3 - r_1)}\frac{\partial f(r_3 - r_1, t)}{\partial(r_3 - r_1)}(\xi''' - \xi')_j(\xi''' - \xi')_l + \left\{f(r_3 - r_1, t) + \frac{r_3 - r_1}{2}\frac{\partial f(r_3 - r_1, t)}{\partial(r_3 - r_1)}\right\}\delta_{jl}\right] \quad \ldots(10d)$$

$$\overline{u_i u_l'''} = \overline{u^2}(t)\left[-\frac{1}{2r_3}\frac{\partial f(r_3, t)}{\partial r_3}\xi_i'''\xi_l''' + \left\{f(r_3, t) + \frac{r_3}{2}\frac{\partial f(r_3, t)}{\partial r_3}\right\}\delta_{il}\partial\right] \quad \ldots(10e)$$

$$\overline{u_j' u_k''} = \overline{u^2}(t)\left[-\frac{1}{2(r_2 - r_1)}\frac{\partial f(r_2 - r_1, t)}{\partial(r_2 - r_1)}(\xi'' - \xi')_j(\xi'' - \xi')_k + \left\{f(r_2 - r_1, t) + \frac{r_2 - r_1}{2}\frac{\partial f(r_2 - r_1, t)}{\partial(r_2 - r_1)}\right\}\delta_{jk}\right]. \quad \ldots(10f)$$

Thus, we can calculate $\overline{u_i u_j' u_k'' u_l'''}$ from the relation (1) which is the outcome of quasi-normality hypothesis due to M. Millionschtchikov (1941*a*,*b*). After substitutions of relations (10*a*–10*f*) in the r.h.s. of (1), we obtain

$$\begin{aligned}\overline{u_i u_j' u_k'' u_l'''} = (\overline{u^2})^2 \Bigg[&\frac{1}{4r_1R_3} f'(r_1, t) f'(R_3, t)\xi_i'\xi_j'\eta_k'\eta_l' \\ &+ \left\{\left(f(r_1, t) + \frac{r_1}{2}f'(r_1, t)\right)\right. \\ &\times \left.\left(f(R_3, t) + \frac{R_3}{2}f'(R_3, t)\right)\right\}\delta_{ij}\delta_{kl} \\ &- \frac{1}{2r_1}f'(r_1, t)\left\{f(R_3, t) + \frac{R_3}{2}f'(R_3, t)\right\} \\ &\times \xi_i'\xi_j'\delta_{kl} - \frac{1}{2R_3}f'(R_3, t)\left\{f(r_1, t)\right. \\ &+ \left.\frac{r_1}{2}f'(r_1, t)\right\}\eta_k^1\eta_l^1\delta_{ij}\Bigg] \\ &+ (\overline{u^2})^2\left[\frac{1}{4r_2R_2}f'(r_2, t)f'(R_2, t)\xi_i''\xi_k''\eta_j''\eta_l'' + \left\{\left(f(r_2, t)\right.\right.\right. \\ &+ \left.\frac{r_2}{2}f'(r_2, t)\right)\end{aligned}$$

$$\times\left(f(R_2, t) + \frac{R_2}{2} f'(R_2, t) \right)\Big\} \delta_{ik}\delta_{jl}$$

$$- \frac{1}{2r_2} f'(r_2, t)\left\{ f(R_2, t) + \frac{R_2}{2} f'(R_2, t) \right\}$$

$$\times \xi_i'' \xi_k'' \delta_{jl} - \frac{1}{2R_2} f'(R_2, t) \left\{ f(r_2, t) \right.$$

$$\left. + \frac{r_2}{2} f'(r_2, t) \right\} \eta_j' \eta_l' \delta_{ik} \Big]$$

$$+ \overline{(u^2)}^2 \left[\frac{1}{4r_3R_1} f'(r_3, t) f'(R_1, t)\, \xi_i''' \xi_l''' \eta_j''' \eta_k''' \right.$$

$$+ \Big\{ (f(r_3, t) + \frac{r_3}{2} f'(r_3, t))$$

$$\times\left(f(R_1, t) + \frac{R_1}{2} f'(R_1, t) \right)\Big\} \delta_{il}\delta_{jk}$$

$$- \frac{1}{2r_3} f'(r_3, t)\left\{ f(R_1, t) + \frac{R_1}{2} f'(R_2, t) \right\}$$

$$\times \xi_i''' \xi_l''' \delta_{jk} - \frac{1}{2R_1} f'(R_1, t) \left\{ f(r_3, t) \right.$$

$$\left. + \frac{r_3}{2} f'(r_3, t) \right\} \eta_j''' \eta_k''' \delta_{il}''' \Big] \qquad \ldots(11)$$

where

$$r_2 - r_1 = R_1, \; r_3 - r_1 = R_2, \; r_3 - r_2 = R_3$$

$$\xi''' - \xi'' = \eta', \; \xi''' - \xi' = \eta'', \; \xi'' - \xi' = \eta'''$$

and a dashed symbol over $f(r)$'s denotes a single time differentiation with respect to r.

Deduction and Evaluation of (11) with the Help of $(r, f(r, t))$ Curve

In these days with the help of hot-wire anemometer or laser anemometer, we can easily obtain $(r, f(r, t))$ curve in the form of Fig. 2. Now, with the help of the curve as shown in the diagram represented in Fig. 2, we can proceed to calculate $f(r, t)$ and $\frac{\partial}{\partial r} f(r, t)$ for different values of r. As such, we can have the calculation $\frac{\partial}{\partial r} f(r, t)\Big|_{r=r_1} = \tan \psi_1$ (say) etc. So, in the final form of this full-length calculation for the right-hand side of (11),

$$\overline{u_i u_j' u_k'' u_l'''} = \overline{(u^2)}^2 \left[\frac{1}{4r_1R_3} \tan \psi_1 \tan \psi_2 \, \xi_i' \xi_j' \eta_k' \eta_l' \right.$$

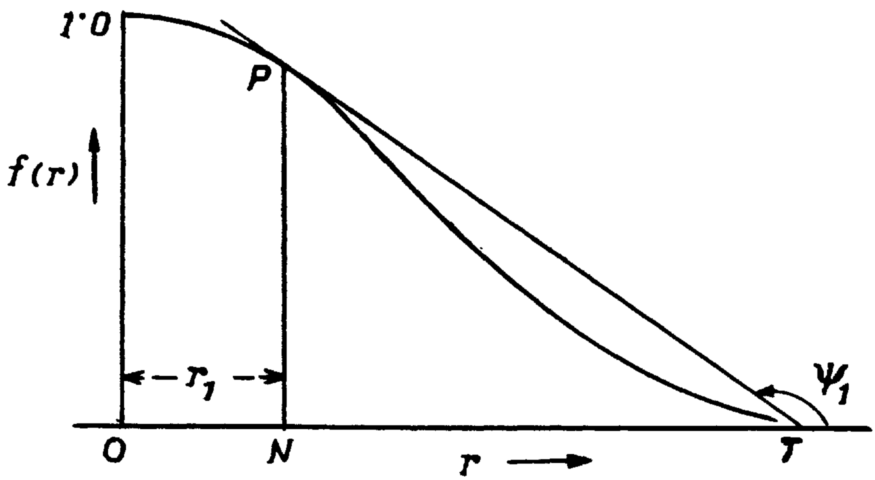

FIG. 2. Arrow with r shows OX-axis, (X not mentioned in diagram) 0 to 1.0 shows OY-axis (Y not mentioned in diagram)
$ON = r_1$, $NP = f(r_1)$
At P tangent drawn to the curve $f(r)$ is shown as PT. PT makes with OX, an angle ψ_1, shown in anticlockwise sense from TX.
[cf. G. K. Batchelor (1953) *The Theory of Homogeneous Turbulence*, p. 48]

$$+\left\{\left(f(r_1, t)+\frac{r_1}{2}\tan\psi_1\right)\right.$$

$$\left.\times\left(f(R_3, t)+\frac{R_3}{2}\tan\psi_2\right)\right\}\delta_{ij}\delta_{kl}-\frac{1}{2r_1}\tan\psi_1\left\{f(R_3, t)\right.$$

$$\left.+\frac{R_3}{2}\tan\psi_2\right\}\xi_i'\xi_j'\delta_{kl}$$

$$\left.-\frac{1}{2R_3}\tan\psi_2\left\{f(r_1, t)+\frac{r_1}{2}\tan\psi_1\right\}\eta_k'\eta_l'\delta_{ij}\right]$$

$$+\overline{(u^2)}^2\left[\frac{1}{4r_2R_2}\tan\psi_3\tan\psi_4\,\xi_i''\xi_k''\eta_j''\eta_l''\right.$$

$$+\left\{\left(f(r_2, t)+\frac{r_2}{2}\tan\psi_3\right)\right.$$

$$\left.\times\left(f(R_2, t)+\frac{R_2}{2}\tan\psi_4\right)\right\}\delta_{ik}\delta_{jl}-\frac{1}{2r_2}\tan\psi_3\left\{f(R_2, t)\right.$$

$$\left.+\frac{R_2}{2}\tan\psi_4\right\}\xi_i''\xi_k''\delta_{jl}$$

$$\left.-\frac{1}{2R_2}\tan\psi_4\left\{f(r_2, t)+\frac{r_2}{2}\tan\psi_3\right\}\eta_j''\eta_l''\delta_{ik}\right]$$

$$+ (\overline{u^2})^2 \left[\frac{1}{4r_3R_1} \tan\psi_5 \tan\psi_6 \, \xi_i''' \xi_l'' \eta_j''' \eta_k'' \right.$$

$$+ \left\{ \left(f(r_3, t) + \frac{r_3}{2} \tan \psi_5 \right) \right.$$

$$\times \left. \left(f(R_1, t) + \frac{R_1}{2} \tan \psi_6 \right) \right\} \delta_{il}\delta_{jk} - \frac{1}{2r_3} \tan \psi_5 \left\{ f(R_1, t) \right.$$

$$\left. + \frac{R_1}{2} \tan \psi_6 \right\} \xi_i''' \xi_l'' \delta_{jk}$$

$$\left. - \frac{1}{2R_1} \tan \psi_6 \left\{ f(r_3, t) + \frac{r_3}{2} \tan \psi_5 \right\} \eta_j''' \eta_k'' \delta_{il} \right] \quad \ldots(12)$$

when $\tan \psi_1$, $\tan \psi_2$, $\tan \psi_3$, $\tan \psi_4$, $\tan \psi_5$, $\tan \psi_6$, are obtained from some $(r, f(r))$ curve for different values of r such at $r = r_1, R_3, r_2, R_2, r_3, R_1$ respectively.

Conclusion

For a wind-tunnel experiment or for the atmospheric measurements, we can easily trace a $(r, f(r, t))$ curve with the use of anemometers. Thus, we can take help of a single $(r, f(r, t))$ figure which can amply help us to give whatever is needed for the expression in the r.h.s. of (12).

Evidently, a fourth-order velocity-correlation pertaining to four distinct points can be easily calculated with the help of a diagram like what has been shown in figure 2 and the formula (12) for a relevant time t.

It is to be noted that a field of homogeneous and isotropic turbulence can be known through different order of moments of fluctuating velocity distributions, As such, second order moment being known from $(r, f(r, t))$ curve, one can go for the calculation of its third order moments with the help of Karman-Howarth equation (1938) under relevant assumptions like self-preservation, or dimensional analysis. In our present work, we have incidentally shown how the fourth order moments of such fluctuating velocity distributions can be calculated graphically from the measurements obtained in regard to second order moments relating to fluctuating velocities at four distinct neighbouring points such as P, Q, R, S. As such, to know moments upto the fourth order, we can be sure of the calculations of mean, standard-deviation, skewness, kurtosis of the relevant distribution function.

In the context of four distinct points so considered by us, we can refer to the comments of M. S. Uberoi (1953) wherein the author has found experimental supports of quasi-normality hypothesis of Millionschtchikov, (1941*a*, *b*), when the relevant points are quite distinct (i.e. relevance of the theory in case of more than two distict points).

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