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Relativity

CHARGED DUST IN EINSTEIN ZERO-MASS SCALAR THEORY

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In this paper, we show that unlike the case in Brans-Dicke theory, the ratio of charge density to mass density has a constant value for a static charged dust distribution in the Einstein zero-mass scalar theory as well as in the Dunn theory of gravitation.

Keywords: Einstein Zero-Mass Scalar; Brans-Dicke Theory; Charge & Mass Densities; Dunn Theory of Gravitation

INTRODUCTION

MAJUMDAR (1947) studied electrostatic fields in Einstein theory with the static metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + g_{44} dt^2 \qquad \ldots(1)$$

where $g_{44} < 0$, and Greek indices run from 1 to 3, and he obtained an important relation

$$- g_{44} = 4\pi\psi^2 + A\psi + B \qquad \ldots(2)$$

which expresses the functional dependence of g_{44} on the electrostatic potential ψ. With a specialised form of (2), a class of solution known as Majumdar-Papapetrou (MP) class was obtained independently by Majumdar (1947) and Papapetrou (1947). Later, a class of black-hole solutions was obtained by Hartle and Hawking following the Majumdar-Papapetrou and Israel-Wilson (1972) techniques.

While finding the analogue of MP class of solutions to the Brans-Dicke theory, Tiwari and Nayak (1976) obtained the following results

$$g_{44} = - 4\pi\Phi^{-1} (\psi \pm \surd 2)^2 \qquad \ldots(2')$$

and

$$\frac{\sigma}{\rho} = \pm \Phi^{-1/2}$$

where σ and ρ are the charge density and mass density respectively of a charged dust distribution and Φ is the Brans-Dicke scalar field. But later it has been found (Raychaudhuri & Bandyopadhyay, 1978; and Nayak & Tiwari, 1978) that Brans-Dicke field equations do not admit any spherically symmetric solution satisfying the above two relations in (2)′.

Recently, Tiwari (1979) derived a class of solutions, similar to the Majumdar-Papapetrou (MP) class for static coupled zero-mass and source-free electromagnetic fields. But, so far, no work has been done on charged dust distribution in Einstein zero-mass scalar theory. So, it is considered worthwhile to extend the investigations into the interior of static charged dust distribution in Einstein zero-mass scalar theory to see whether such a distribution admits a functional form of g_{44} similar to that obtained by Tiwari (1979) or not.

In the present, paper, it is shown that unlike the case in Brans-Dicke theory (Raychaudhuri & Bandyopadhyay, 1978) the same functional form of g_{44} holds in vacuum as well as in the interior of static charged dust distribution in the case of Einstein zero-mass scalar field theory. In this connection, it may be mentioned that there is a conformal relation between Brans-Dicke Maxwell field equations and Einstein-Maxwell zero-mass scalar field equations in vacuum. But in presence of matter no such relation exists.

Our investigations, however, reveal that one gets no better result physically from the Einstein-Maxwell zero-mass scalar field theory than what Einstein-Maxwell theory gives for the stability of electron.

Recently, Dunn (1974) has proposed a scalar-tensor theory of gravitation. In this theory, unlike the Brans-Dicke theory, the scalar field has got an intrinsic geometrical role to play. Field equations of Dunn theory of gravitation are similar to those of Einstein zero-mass scalar field equations, although, the geometrical basis of the two theories are entirely different. As such the methods of solutions of the field equations of the two theories are similar. So the results obtained in this paper would apply to Dunn theory also. Some recent paper (Krori & Nandy, 1980; and Krori *et al.*, 1981), have shown that unlike Brans-Dicke and some other scalar-tensor theories, Birkhoff's theorem and Mach's principle hold in the Dunn theory of gravitation.

VACUUM FIELD

Einstein-Maxwell zero-mass scalar field equations in presence of matter may be written down as

$$R_{ij} = -\frac{8\pi G}{C^2}\left[\left\{(M_{ij} + E_{ij}) - \frac{1}{2}\, g_{ij}T_m\right\} + \Phi_{,i}\, \Phi_{,j}\right] \quad \ldots(3)$$

$$\Phi^{;s}_{;s} = 0 \quad \ldots(4)$$

$$F^{ij}_{;j} = 4\pi\sigma v^i \quad \ldots(5)$$

and

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \qquad \ldots(6)$$

with

$$E_{ij} = g^{ab}F_{ai}F_{bj} - \tfrac{1}{4}\, g_{ij}F_{mn}F^{mn} \qquad \ldots(7)$$

Here $M_{ij} = \rho v_{,i} v_{,j}$ the energy-momentum tensor of matter distribution, E_{ij} the energy-momentum tensor of the electromagnetic field, ρ the mass density and σ the charge density, T_m the trace of the energy-momentum tensor of matter. Φ is the scalar field of the theory and R_{ij} the usual Riemann curvature tensor.

With a skew-symmetric electromagnetic tensor F_{ij} as

$$F_{ij} = \chi_{j,i} - \chi_{i,j}$$

equation (6) is indentically satisfied. χ_i is the electromagnetic four potential. We shall consider static condition so that only χ_4 exists. We shall henceforth denote χ_4 by ψ.

We now proceed to deduce a functional relationship between g_{44} and ψ in vacuum which will hold irrespective of any symmetry. We consider a line element

$$ds^2 = g_{ik}dx^i dx^k - V^2 dt^2 \qquad \ldots(8)$$

where $g_{44} = -V^2$ and i, k run from 1 to 3. In vacuum, equation (3) in the static condition gives

$$R_{44} = -\frac{8\pi G}{C^2}\, E_{44} \qquad \ldots(9)$$

We now assume that

$$V = V(\psi) \qquad \ldots(10)$$

so that

$$R_{44} = \frac{1}{\sqrt{-g}}\,[\sqrt{-g}\, g^{ik}(2V_\psi^2 - V.V_{\psi\psi})\, \psi_{,i}\psi_{,k}$$

$$- V_\psi V.(\sqrt{-g}\, g^{ik}\psi_{,k})_{,i} - V_\psi^2 \sqrt{-g}\, g^{ik}\psi_{,i}\psi_{,k}] \qquad \ldots(11)$$

$$E_{44} = \tfrac{1}{2}\, g^{ik}\psi_{,i}\psi_{,k} \qquad \ldots(12)$$

Equation (5), in the case of vacuum, gives

$$V_\psi V(g^{ik}\sqrt{-g}\,\psi_{,k})_{,i} - 2g^{ik}\sqrt{-g}\; V_\psi^2\, \psi_{,i}\psi_{,k} = 0 \qquad \ldots(13)$$

Now from equations (9), (11), (12) and (13) we have finally

$$\sqrt{-g}\; g^{ik}\psi_{,i}\psi_{,k}\left(V_\psi^2 + V \cdot V_{\psi\psi} - \frac{4\pi G}{C^2} \right) = 0 \qquad \ldots(14)$$

Now $g^{ik}\psi_{,i}\psi_{,k}$ cannot vanish as this would amount to the electromagnetic field being null.

So we take

$$V_{\psi}^{2} + V \cdot V_{\psi\psi} - \frac{4\pi G}{C^2} = 0 \qquad ...(15)$$

This will be satisfied when we have

$$V^2 = \frac{G}{C^2}(4\pi\psi^2 + A\psi + B) \qquad ...(16)$$

We may express (16) as

$$V^2 = \frac{4\pi G}{C^2}(\psi \pm \sqrt{2})^2 \qquad ...(17)$$

by a proper choice of the arbitrary constants A and B. The relation (17) has been derived by Tiwari (1979) also. But he worked on the assumption that $V = V(\Phi, \psi)$ with Φ and ψ independent of each other. But such an assumption will not be in keeping with spherical symmetry or any symmetry where the orbits of the group of motions are surfaces, because in that case the Φ-constant and ψ-constant surfaces will indicate a functional relationship between them. In fact, such a relationship has been obtained between Φ and ψ as discussed later.

Charged Dust

In this section we consider a static distribution of charged dust. From equation (5) we have

$$V_{\psi}V(g^{ik}\sqrt{-g}\,\psi_{,k})_{,i} - 2g^{ik}\sqrt{-g}\;V_{\psi}^{2}\,\psi_{,i}\psi_{,k} = 4\pi\sigma\sqrt{-g}\,V_{\psi}V^2 \qquad ...(18)$$

Equation (3) now gives

$$R_{44} + \frac{8\pi G}{C^2}E_{44} + \frac{4\pi G}{C^2}g_{44}\,\rho = 0 \qquad ...(19)$$

Equation (11), (12), (18) and (19) now give

$$-\sqrt{-g}\,g^{ik}\psi_{,i}\psi_{,k}\left(V_{\psi}^{2} + V \cdot V_{\psi\psi} - \frac{4\pi G}{C^2}\right)$$
$$- 4\pi\sigma\sqrt{-g}\;V_{\psi}V^2 + \frac{4\pi G}{C^2}g_{44}\sqrt{-g}\,\rho = 0 \qquad ...(20)$$

If we assume that (17) holds in the interior of static charged dust also, then (20) gives

$$- 4\pi\sigma \sqrt{-g}\; V_{\varphi} V^2 - \frac{4\pi G}{C^2} V^2 \rho \sqrt{-g} = 0.$$

This leads to the result

$$\frac{\sigma}{\rho} = \pm \frac{1}{2C\sqrt{\pi}} G^{1/2} \qquad \text{...(21)}$$

By an adjustment of the units of ρ, σ, C and the gravitational constant G, this may be put as

$$\frac{\sigma}{\rho} = \pm 1 \qquad \text{...(22)}$$

which is the same as that obtained from the Einstein theory alone.

Spherically Symmetric Charged Dust Distribution

It has been pointed out under Introduction that the Brans-Dicke field equations do not admit any spherically symmetric solution satisfying (2). So we proceed to see whether (17) and (22) actually hold in the interior of a static charged dust distribution. For this purpose we take up a line element which exhibits spherical symmetry. Our line element is

$$ds^2 = e^{\lambda} dr^2 + r^2 \, d\theta^2 + r^2 \sin^2 \theta d\phi^2 - V^2 dt^2 \qquad \text{...(23)}$$

In the static case, field equations (3)–(5) for the metric (23) may be expressed as

$$\frac{F_{11}}{F} - \frac{\lambda_1 F_1}{2F} - \frac{\lambda_1}{r} = \frac{F_1^2}{F^2} + \frac{4\pi G}{C^2} e^{\lambda}\rho - 2A^2 \frac{e^{\lambda}}{r^4 F^2} \qquad \text{...(24)}$$

$$\frac{1}{r^2} - \frac{\lambda_1}{2r} + \frac{F_1}{rF} - \frac{e^{\lambda}}{r^2} = -\frac{F_1^2}{F^2} + \frac{4\pi G}{C^2} e^{\lambda}\rho \qquad \text{...(25)}$$

$$\frac{F_{11}}{F} - \frac{\lambda_1 F_1}{2F} + \frac{2F_1}{rF} = \frac{F_1^2}{F^2} - \frac{4\pi G}{C^2} e^{\lambda}\rho \qquad \text{...(26)}$$

$$e^{\lambda/2} = \frac{\pm (4\pi G)^{1/2}}{AC} r^2 F \Phi_1 \qquad \text{...(27)}$$

$$\frac{F_{11}}{F} + \left(\frac{2}{r} - \frac{\lambda_1}{2} - \frac{F_1}{F}\right)\frac{F_1}{F} = \pm \frac{8\pi^{3/2} G^{1/2}}{C} e^{\lambda}\sigma \qquad \text{...(28)}$$

where F is given by the relation

$$F = \psi \pm \sqrt{2} \qquad \text{...(29)}$$

and A is an arbitrary constant.

Subtracting (26) from (24), one has

$$-\frac{\lambda_1}{r} - \frac{2F_1}{rF} = \frac{8\pi G}{C^2} e^{\lambda}\rho - 2A^2 \frac{e^{\lambda}}{r^4 F^2} \qquad \text{...(30)}$$

Multiplying (25) by 2, one obtains

$$\frac{2}{r^2} - \frac{\lambda_1}{r} + \frac{2F_1}{rF} - \frac{2e^\lambda}{r^2} = -\frac{2F_1^2}{F^2} + \frac{8\pi G}{C^2} e^\lambda \rho \qquad ...(31)$$

Eliminating ρ between (30) and (31) one obtains

$$e^{\lambda/2} = \frac{r(rF_1 + F)}{(r^2F^2 + A^2)^{1/2}} \qquad ...(32)$$

Equations (32) and (27) can be combined to yield

$$\Phi_1 = \frac{a(rF_1 + F)}{rF(r^2F^2 + A^2)^{1/2}} \qquad ...(33)$$

where

$$a = \pm AC\,(4\pi G)^{-1/2} \qquad ...(34)$$

Integrating equation (33) one gets

$$F = (\psi \pm \sqrt{2}) = \frac{A \operatorname{cosec h}(M - N\Phi)}{r} \qquad ...(35)$$

where

$$M = \frac{A\bar{C}}{a}, \; N = \frac{A}{a}, \; \bar{C} = \text{an arbitrary constant.}$$

In view of (35) the metric co-efficients become

$$e^\lambda = \frac{4\pi G}{C^2} r^2 \Phi_1^2 \operatorname{cosech}^2 (M - N\Phi) \qquad ...(36)$$

and

$$-V^2 = -\frac{4\pi G}{C^2} A^2 \frac{\operatorname{cosec h}^2(M - N\Phi)}{r^2} \qquad ...(37)$$

Substitution of (35) and (36) in the field equations (24)-(28) yields

$$\Phi = \alpha r + \beta \qquad ...(38)$$

where α and β are abitrary constants and

$$\rho = \frac{C^2}{4\pi G} \frac{\sinh^2 \{M - N(\alpha r + \beta)\}}{r^2} \left[1 - \frac{2 \coth \{M - N(\alpha r + \beta)\}}{rN\alpha} \right] \qquad ...(39)$$

$$\sigma = \mp \left(\frac{C}{8\pi^{3/2}G^{1/2}}\right) \frac{\sinh^2 \{M - N(\alpha r + \beta)\}}{r^2}$$

$$\times\left[1-\frac{2\coth\{M-N(\alpha r+\beta)\}}{rN\alpha}\right] \qquad \ldots(40)$$

From (39) and (40) one obtains

$$\frac{\sigma}{\rho}=\pm\frac{1}{2C\sqrt{\pi}}G^{1/2} \qquad \ldots(41)$$

which after adjusting units is same as the equation (22). Now for the stability of an electron its charge density should far exceed its mass density so that we must have

$\frac{|\sigma|}{\rho}>1$. This is not satisfied by (41).

Now, equations (35)–(40) constitute the interior solution of the field equations (24)–(28) for a spherically symmetric charged dust distribution. The complete line element is

$$ds^2=N^2r^2\alpha^2\operatorname{cosec}h^2\{M-N(\alpha r+\beta)\}\,dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2)-N^2A^2\frac{\operatorname{cosec}h^2\{M-N(\alpha r+\beta)\}}{r^2}\,dt^2 \qquad \ldots(42)$$

The exterior solution given by Tiwari (1979) is

$$ds^2=e^{-2k}\left(1-\frac{2m}{R}\right)^{-1}\left[dR^2+\left(1-\frac{2m}{R}\right)R^2(d\theta^2+\sin^2\theta d\phi^2)\right] - e^{2k}dt^2 \qquad \ldots(43)$$

with the scalar field

$$\Phi=\left(\frac{2\pi}{k}\right)^{1/2}\log\left(1-\frac{2m}{R}\right)$$

and the electrostatic potential

$$\psi=\left(\frac{8\pi}{k}\right)^{1/2}e^{k}\mp\sqrt{2}$$

where

$$e^{-2k}=1+\frac{a}{2m}\log\left[R(R-2m)b\right]$$

(K, a and b are constatns)

A transformation $e^{-k}R=r$ will bring the metric (43) into a form so that one may match it with our metric (42) at the boundary of the distribution.

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