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Theoretical Physics

MACROSCOPIC FLUCTUATION FOR SLOWLY VARYING PARAMETERS

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The paper deals with the fluctuation of macrovariables for the slowly varying environmental parameters of a macrosystem.

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Introduction

The macroscopic theory of fluctuation plays an important role both in the problems of equilibrium and non-equilibrium phenomena. In the case of non-equilibrium system far from equilibrium it is more significant giving rise to the phenomena of symmetry breaking, non-equilibrium instabilities etc. (Glansdroff & Prigogine, 1971) The object of this paper is to study the fluctuation of macrovariables of a system in the quasi-steady state which is far from the equilibrium state but very near to the steady (non-equilibrium) state.

Dynamical Equation of Evolution

Let us consider for simplicity a macrosystem with the single macrovariable $x(t)$. We take $x(t)$ to represent the deviation of the macrovariable from the steady state value. The basic dynamical equation for the evolution of the macrovariable $x(t)$ is given by (Lax, 1962)

$$\frac{dx}{dt} = X(x, \alpha) \qquad \ldots(1)$$

where X is a function of the macrovariable $x(t)$ and the external (or environmental) parameter α. The parameter α is in general a function of the time t and in the steady states is constant. In the quasi-steady state which corresponds to the situation far from equilibrium and very near to the steady (non-equilibrium) state, the parameter α is a slowly varying function of time. In this paper, we shall study the effect of such a slowly varying environmental condition on the evolution of the macrovariable $x(t)$ and such effect will be manifested in the fluctuation or dispersion of the macrovariable $x(t)$ due to the random effect of the environment.

Linearization and Random Perturbation

Let us linearize the equation (1) as

$$\frac{dx}{dt} = \left(\frac{\partial x}{\partial x}\right)_0 x$$

or

$$\frac{dx}{dt} + a(t)\,x = 0 \qquad \ldots(2)$$

where $a(t) = -\left(\frac{\partial X}{\partial x}\right)_0$ is a function of the parameter $\alpha(t)$ and for the quasi-steady state it is also a slowly varying function of the time t. The random effect of the environment is taken into account by addition of a random perturbation term $F(t)$ and extending the macroscopic equation of evolution (2) to the stochastic differential equation (Nicolis & Prigogine, 1977)

$$\frac{dx}{dt} + a(t)x = F(t) \qquad \ldots(3)$$

where $F(t)$ is assumed to be a white noise to satisfy

$$< F(t) > = 0 \qquad \ldots(4)$$

$$< F(t)\, F(t_1) > = 2D(t)\, \delta(t - t_1) \qquad \ldots(5)$$

where $< >$ represents the average over the ensemble of the stochastic process concerned and $D(t)$ which is the variance of $F(t)$ is assumed to be a slowly varying function of time. The equation (3) is the generalization of Langevin equation. A microscopic deviation of the equation (3) on the method of statistical mechanics was done by Furukawa (1976). Our interest, however, is on the macroscopic method of extending the dynamical equation of evolution of $x(t)$ to the stochastic regime.

Spectral Decomposition and Fluctuation

If the system described by the stochastic differential equation (3) receives the input $F(t)$ which is near to exponential, the output is also near to exponential. The requirement for this is that in addition to the coefficient $a(t)$ in the equation (3), the response or admittance function $Z(i\omega, t)$ should also be a slowly varying function of time (Pugachev, 1963). The random perturbation has the Fourier representation

$$F(t) = \int_{-\infty}^{\infty} V(\omega)\, e^{i\omega t}\, d\omega \qquad \ldots(6)$$

Then for the exponential input $e^{i\omega t}$, the output $x(t)$ is given by

$$x(t) = Z(i\omega t)\, e^{i\omega t} \qquad \ldots(7)$$

Replacing $F(t)$ and $x(t)$ by $e^{\lambda t}$ and $Z(\lambda, t)\, e^{\lambda t}$ (respectively) in equation (3), we get

$$\frac{d}{dt} Z(\lambda, t) + (a(t) + \lambda)\, Z(\lambda, t) = 1 \qquad \text{...(8)}$$

which is a linear inhomogeneous equation. To determine $Z(\lambda, t)$ we, in conformity with the requirement of $Z(\lambda, t)$ being a slowly varying function of time, resort to the approximate evaluation of the admittance function $Z(\lambda, t)$ as follows:

Neglecting in the first approximation the derivative of the slowly varying function $Z(\lambda, t)$, we get from (8)

$$[a(t) + \lambda]\, Z(\lambda, t) = 1$$

So in the first approximation

$$Z(\lambda, t) \simeq Z_1(\lambda, t) = \frac{1}{[a(t) + \lambda]} \qquad \text{...(9)}$$

In the second approximation the first derivative of the slowly varying function $Z(\lambda, t)$ is taken to be equal to the derivative of $Z(\lambda,t)$ from the first approximation

$$\frac{d}{dt} Z(\lambda, t) \simeq \frac{d}{dt} Z_1(\lambda, t) = \frac{-a'(t)}{(a(t) + \lambda)^{2}},\ a'(t) = \frac{d}{dt}\, a(t) \qquad \text{...(10)}$$

So the second approximation is given by

$$[a(t) + \lambda]\, Z_2(\lambda, t) = 1 - \frac{d}{dt} Z_1(\lambda, t) = 1 + \frac{a'(t)}{[a(t) + \lambda]^2}$$

or

$$Z(\lambda, t) \simeq Z_2(\lambda, t) = \left[\frac{1}{[a(t) + \lambda]} + \frac{a'(t)}{(a(t) + \lambda)^3}\right] \qquad \text{...(11)}$$

which is the required admittance function. The mean square deviation of $x(t)$ is given by (Pugachev, 1963)

$$< x^2(t) > = \int s_F(\omega)\ |Z(i\omega, t)|^2\, d\omega \qquad \text{...(12)}$$

where $S_F(w)$ is the spectral density of $F(t)$ and is given by the Fourier transform of the co-variance function

$$S_F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} < F(t)\, F(t_1) > e^{i\omega\tau} d\tau,\ (\tau = t - t_1)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2D(t)\ \delta(\tau)\, e^{-i\omega\tau}\, d\tau = \frac{D(t)}{\pi} \qquad \text{...(13)}$$

So the mean square deviation of the macrovariable $x(t)$ is given by

$$< x^2(t) > = \frac{D(t)}{\pi} \int_{-\infty}^{\infty} \left| \frac{1}{(a(t) + i\omega)} + \frac{a'(t)}{(a(t) + i\omega)^3} \right|^2 d\omega \qquad ...(14)$$

The formula (14) gives the spectral decomposition of the fluctuation of the macro-variable $x(t)$ and is a generalization of Nyquist's relation for the quasi-steady state. In the steady state when $a'(t) = 0$ and $D(t)$ is a constant it reduces to the usual result of irreversible processes (Lax, 1962).

Application

Let us now consider two cases to explore the applicability of the theory. First we consider a simple chemical reaction after Furakawa (1976)

$$A \leftrightarrows B \qquad ...(15)$$

For the case of reaction occurring near the equilibrium state, the fluctuation obeys the Langevin equation

$$(\triangle \dot{A}) = a(\triangle A) + F(t) \qquad ...(16)$$

If the reference state is equilibrium, a and D are contsant. If the reference state slightly is different from the equilibrium, then both a and D may depend on time, but the dependence is very weak so that they can be approximated by the equilibrium values to give the usual results for the fluctuation of the reactant A.

But if the reaction occurs at a state far from equilibrium, then we have to consider the time dependence of $a(t)$ and $D(t)$; if the reaction occur very near to the steady state far from equilibrium, then both $a(t)$ and $D(t)$ are slowly varying function of time and in that case the formula (14) determines the fluctuation of the reactant. The formula (14) is very complicated and different order of approximations is necessary to evaluate. For example, if we take $D(t)$ to be constant in the first approximate and expand $a(t)$ about the steady state by

$$a(t) = a_0 + ta'(t) + ... \qquad ...(17)$$

and put in (14), then the formula (14) will give the expression of fluctuation in powers of time t to different order of approximation Furakawa (1976).

The formula (14), in particular, is of special significance for biological system. The equation (3) for the intrinsic fertility rate $a(t)$ determines the linear growth of population under the random effect of the environment. If the environmental parameters such as the climate, topography etc. which have strong effect on the growth of many populations (e.g., insects) vary slowly, then the formula (14) determines the effect of such variation on the growth of population (Lotka, 1956; and Ludwig, 1977).

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