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OPTIMAL PROPELLANT MASS DISTRIBUTION WITH GIVEN BURNOUT VELOCITY IN A MULTISTAGE ROCKET FLOWN VERTICALLY IN A VACUUM

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The first kind of optimisation is to find the minimum propellant mass and its distribution among the stages, required for a mission of given overall burnout velocity, gross initial mass of the multistage rocket, and equivalent exhaust velocity and structural mass of each stage, neglecting gravity. The second kind of optimisation is to find the same in the gravity-included case where alongwith previous conditions one more condition, i.e., propellant mass flow of each stage is given. Finally, two numerical examples have been cited.

Keywords : Optimal Propellant Mass Distribution; Burnout Velocity; Multistage Rocket; Vacuum; Flight Mechanics; Rocket Propulsion; Payload Mass; Gravity-Free Space

INTRODUCTION

IN almost all textbooks of Rocket propulsion the formula for burnout velocity attained by a multistage rocket has been derived in terms of payload ratio and structural factor of each stage. These two parameters in the kth stage are given by

$$\pi_k = \frac{m_{i(k+1)}}{m_{ik}}, \ \epsilon_k = \frac{m_{sk}}{m_{sk} + m_{pk}},$$

where m_{ik}, m_{pk} and m_{sk} are the initial mass, the propellant mass and the structural mass in the kth stage. The overall burning time and the overall burnout velocity, as obtained by A. Miele (1962) are :

$$t_* = \frac{1}{g} \sum_{k=1}^{n} \frac{V_{Ek}}{\tau_{ik}} (1 - \epsilon_k)(1 - \pi_k)$$

and

$$V_* = \sum_{k=1}^{n} V_{Ek} \log \frac{1}{\epsilon_k + (1 - \epsilon_k)\pi_k},$$

where gravity is neglected, and τ_{ik} and V_{Ek} respectively denote the initial thrust-to-weight ratio and the equivalent exhaust velocity of the kth stage.

Barrére's (1960) and Ruppe's (1966) optimisation problem consists in determining the optimum distribution of partial payload ratios (or mass ratios) between successive stages so as to maximize the overall payload ratio with given structural factor/ratio and equivalent exhaust velocity of each stage and the overall burnout velocity in gravitationless space. Their investigations reveal that equal structural factors/ratios and equal equivalent exhaust velocities for all stages lead to equal optimum partial payload ratios. Furthermore, the study of the same problem is extended to gravitational space by Barrére (1960), consequently accounting for an additional term—final thrust-to-weight ratio in each stage.

Various types of optimisation pertaining to multistage rocket performance have been dealt with in the text books on this subject. Srivastava (1962) found optimum stage mass distribution leading to the minimum propellant mass for given payload mass and burnout velocity; while the textbook analyses and Srivastava's analysis involve the terms like "partial payload ratio," "structural factor"/"construction parameter," the present one is straight forward, without these terms being involved.

The burning time and burnout velocity are obtained in terms of the propellant mass and the structural mass of each stage without accounting for partial payload ratio or structural factor and the optimisation criteria are to determine (*i*) the maximum payload mass, i.e., the minimum propellant mass with given burnout velocity, gross initial mass and each stage equivalent exhaust velocity and structural mass in gravity free space; (*ii*) the maximum payload mass, i.e., the minimum propellant mass with given equivalent exhaust velocity, structural mass, propellant mass flow of each stage, burnout velocity and gross initial mass in the gravity-included case.

Vertical flight in vaccum with the thrust tangent to the flight path is considered under the assumption that rocket is initially at rest and the final velocity of each stage is equal to that of the next.

Burning Time and Burnout Velocity

If m_{pk}, m_{sk} and m_i be the propellant mass, the structural mass in the kth stage and the gross initial mass respectively, the payload mass of n-stage rocket is expressed as

$$M_L = m_i - \sum_{k=1}^{n} m_{pk} - \sum_{k=1}^{n} m_{sk}. \qquad \text{...(1)}$$

If V_{Ek}, m_{ik} and m_{fk} be the equivalent exhaust velocity, initial mass, and final mass respectively in the kth stage, the overall burnout velocity as obtained by Miele neglecting gravity, is

$$V = \sum_{k=1}^{n} V_{EK} \log \frac{m_{ik}}{m_{fk}},$$

where $\quad m_{ik} - m_{fk} = m_{pk}$

and $$m_{i(k-1)} - m_{ik} = m_{p(k-1)} + m_{s(k-1)}.$$

Putting $k = 2, 3, \ldots k$, $m_{i1} = m_i$ and then adding all the columns, one gets

$$m_{ik} = m_i - \sum_{r=1}^{k-1} m_{pr} - \sum_{r=1}^{k-1} m_{sr}$$

and

$$m_{fk} = m_i - \sum_{r=1}^{k} m_{pr} - \sum_{r=1}^{k} m_{sr} + m_{sk}.$$

As a consequence of these relationships, the overall burnout velocity becomes

$$V = -\sum_{k=1}^{n} V_{Ek} \log \frac{m_i - \sum_{r=1}^{k} m_{pr} - \sum_{r=1}^{k} m_{sr} + m_{sk}}{m_i - \sum_{r=1}^{k} m_{pr} - \sum_{r=1}^{k} m_{sr} + m_{pk} + m_{sk}}. \qquad \ldots(2)$$

In order to find the corresponding burnout velocity in the gravity included case, it is required to find the burning time, which in the present analysis, can be expressed as

$$t = \sum_{k=1}^{n} \frac{m_{pk}}{\beta_k}, \qquad \ldots(3)$$

where β_k is the propellant mass flow in the kth stage. Hence, the overall burnout velocity in this case is given by

$$V = -\sum_{k=1}^{n} V_{Ek} \left\{ \log \frac{m_i - \sum_{r=1}^{k} m_{pr} - \sum_{r=1}^{k} m_{sr} + m_{sk}}{m_i - \sum_{r=1}^{k} m_{pr} - \sum_{r=1}^{k} m_{sr} + m_{pk} + m_{sk}} + \frac{g m_{pk}}{\beta_k V_{Ek}} \right\}$$

$\ldots(4)$

Optimal Stage-Propellant Mass Distribution for Maximum Payload Mass with given Burnout Velocity in Gravity-Free Space

Using eqn. (2), the constraint eqn. can be formed as

$$\phi \equiv V + \sum_{k=1}^{n} V_{Ek} \log \frac{m_i - \sum_{r=1}^{k} m_{pr} - \sum_{r=1}^{k} m_{sr} + m_{sk}}{m_i - \sum_{r=1}^{k} m_{pr} - \sum_{r=1}^{k} m_{sr} + m_{pk} + m_{sk}} = 0$$

$\ldots(5)$

Now the problem of maximising the payload mass given by (1), under the given conditions is the same as maximising the following function :

$$U = M_L + \Lambda\phi \qquad \text{...(6)}$$

where Λ is the Lagrange's Multiplier and as such the optimisation equations are

$$\frac{\partial M_L}{\partial m_{pj}} + \Lambda \frac{\partial \phi}{\partial m_{pj}} = 0 \qquad \text{...(7)}$$

for $j = 1, 2, 3, \ldots n,$

i.e.,

$$1 + \Lambda \left[\frac{V_{Ej}}{m_i - \sum_{r=1}^{j} m_{pr} - \sum_{r=1}^{j} m_{sr} + m_{sj}} + \sum_{k=j}^{n-1} V_{E(k+1)} \left\{ \frac{1}{m_i - \sum_{r=1}^{k+1} m_{pr} - \sum_{r=1}^{k} m_{sk}} - \frac{1}{m_i - \sum_{r=1}^{k} m_{pr} - \sum_{r=1}^{k} m_{sr}} \right\} \right] = 0. \qquad \text{...(8)}$$

which is the jth optimisation equation. Similarly, by replacing j by $(j+1)$ in eqn. (8), the $(j+1)$th optimisation equation can be formed and subtracting the latter from the former, Λ can be eliminated to yield the following relationship :

$$\frac{V_{Ej}}{m_i - \sum_{k=1}^{j} m_{pk} - \sum_{k=1}^{j} m_{sk} + m_{sj}} = \frac{V_{E(j+1)}}{m_i - \sum_{k=1}^{j} m_{pk} - \sum_{k=1}^{j} m_{sk}}$$

or

$$M_j = \frac{m_{sj} V_{E(j+1)}}{V_{Ej} - V_{E(j+1)}} \qquad \text{...(9)}$$

$$(j = 1, 2, \ldots n-1)$$

where

$$M_j = m_i - \sum_{k=1}^{j} m_{pk} - \sum_{k=1}^{j} m_{sk}. \qquad \text{...(10)}$$

The optimum propellant mass in the jth stage can be given by

$$m_{pj} = M_{(j-1)} - M_j - m_{sj}$$

$$= V_{Ej} \left[\frac{m_{s(j-1)}}{V_{E(j-1)} - V_{Ej}} - \frac{m_{sj}}{V_{Ej} - V_{E(j+1)}} \right], \qquad \text{...(11)}$$

where $j = 2, 3, \ldots n-1,$ and

$$m_{p1} = m_i - \frac{V_{E1}}{V_{E1} - V_{E2}} m_{s1} \qquad \text{...(12)}$$

Eqns. (11) and (12) reveal that such an optimisation of all the stages is possible if

$$V_{E_1} > V_{E_2} > V_{E_3} \ldots V_{E(n-1)} > V_{En} \quad \ldots(13)$$

i.e., the equivalent exit velocity decreases with the number of stages. The stages having equal equivalent exit velocities will not contribute to the maximum payload mass under given restrictive conditions.

In the light of the optimum conditions (9) to (13), the expression for burnout velocity can be rewritten as

$$V = -\sum_{k=1}^{n-1} V_{Ek} \log \frac{M_k + m_{sk}}{M_k + m_{pk} + m_{sk}} - V_{En} \log \frac{M_{(n-1)} - m_{pn}}{M_{(n-1)}}$$

and with given burnout velocity V, the optimum propellant mass in the last stage can be obtained as (using $M_k + m_{pk} + m_{sk} = M_{k-1}$ from (9))

$$V = - V_{E_1} \log \frac{V_{E_1} m_{s1}}{(V_{E_1} - V_{E_2})\, m_i} - \sum_{k=2}^{n-1} V_{Ek} \log \frac{(V_{E(k-1)} - V_{Ek})\, m_{sk}}{(V_{Ek} - V_{E(k+1)})\, m_{s(k-1)}}$$

$$- V_{En} \log \left\{1 - \frac{m_{pn}(V_{E(n-1)} - V_{En})}{m_{s(n-1)} V_{En}}\right\}.$$

$$m_{pn} = \left[1 - e^{-(V/V_{En})} \left\{\frac{m_i}{m_{s1}}\left(1 - \frac{V_{E2}}{V_{E1}}\right)\right\}^{(V_{E_1}/V_{En})}\right.$$

$$\left. \times \prod_{k=2}^{n-1} \left\{\frac{m_{s(k-1)}}{m_{sk}} \cdot \frac{V_{Ek} - V_{E(k+1)}}{V_{E(k-1)} - V_{Ek}}\right\}^{(V_{Ek}/V_{En})}\right] \frac{V_{En} m_{s(n-1)}}{V_{E(n-1)} - V_{En}}. \quad \ldots(14)$$

Thus, with the given equivalent exhaust velocity and the structural mass of each stage, gross initial mass and overall burnout velocity, a propellant mass distribution among the stages exists leading to the minimum total propellant consumption, i.e., to the maximum payload mass.

By means of eqns. (1), (9), (10) and (14), the maximum payload mass can be computed as

$$M_L(\max) = m_i - \sum_{k=1}^{n-1} m_{pk} - \sum_{k=1}^{n-1} m_{sk} - m_{sn} - m_{pn}$$

$$= \frac{V_{En} m_{s(n-1)}}{V_{E(n-1)} - V_{En}} - m_{sn} - m_{pn}.$$

$$M_L(\max.) = \frac{V_{En} m_{s(n-1)}}{V_{E(n-1)} - V_{En}} - \left[1 - e^{-(V/V_{En})} \left\{\frac{m_i}{m_{s1}}\left(1 - \frac{V_{E2}}{V_{E1}}\right)\right\}^{(V_{E_1}/V_{En})}\right.$$

$$\times \prod_{k=2}^{n-1} \left\{\frac{m_{s(k-1)}}{m_{sk}} \cdot \frac{V_{Ek} - V_{E(k+1)}}{V_{E(k-1)} - V_{Ek}}\right\}^{(V_{Ek}/V_{En})}\Bigg] \frac{V_{En} m_{s(n-1)}}{V_{E(n-1)} - V_{En}} - m_{sn}$$

$$= \left[e^{-(V/V_{En})} \left\{\frac{m_i}{m_{s1}} \left(1 - \frac{V_{E2}}{V_{E1}}\right)\right\}^{(V_{E1}/V_{En})} \right.$$

$$\times \prod_{k=2}^{n-1} \left\{\frac{m_{s(k-1)}}{m_{sk}} \cdot \frac{V_{Ek} - V_{E(k+1)}}{V_{E(k-1)} - V_{Ek}}\right\}^{(V_{Ek}/V_{En})}\Bigg] \frac{V_{En} m_{s(n-1)}}{V_{E(n-1)} - V_{En}} - m_{sn} \quad \ldots(15)$$

where the corresponding minimum propellant mass is

$$M_p(\min) = m_i - \sum_{k=1}^{n} m_{sk} - M_L(\max). \quad \ldots(16)$$

Conversely, with given equivalent exit velocity and structural mass of each stage, gross initial mass and payload mass, there exists an optimum distribution of the given propellant mass amongst the stages, which maximizes the overall burnout velocity.

Optimal Stage Propellant Mass Distribution for Maximum Burnout Velocity including Gravity

Because of eqns. (4) and (10) in this section the constraint equation is

$$\phi \equiv V + \sum_{k=1}^{n} V_{Ek} \log \frac{M_k + m_{sk}}{M_k + m_{pk} + m_{sk}} + g \sum_{k=1}^{n} \frac{m_{pk}}{\beta_k}. \quad \ldots(17)$$

In the same way as in the previous section, the optimisation equations for maximum payload mass are represented by

$$\frac{\partial U}{\partial m_{pj}} = 1 + \Lambda \left[\frac{V_{Ej}}{m_i - \sum_{r=1}^{j} m_{pr} - \sum_{r=1}^{j} m_{sr} + m_{sj}} \right.$$

$$+ \sum_{k=j}^{n-1} V_{E(k+1)} \left\{ \frac{1}{m_i - \sum_{r=1}^{k+1} m_{pr} - \sum_{r=1}^{k} m_{sr}} - \frac{1}{m_i - \sum_{r=1}^{k} m_{pr} - \sum_{r=1}^{k} m_{sr}} \right\}$$

$$\left. + \frac{g}{\beta_j} \right] = 0 \quad (j = 1, 2, 3, \ldots n - 1),$$

where $U = M_L + \Lambda\phi$ being the Lagrange's Multiplier, so that the following optimum conditions are obtained :

$$V_{E1}\left(\frac{1}{m_i - m_{p1}} + \frac{g}{\beta_1 V_{E1}}\right) = V_{E2}\left(\frac{1}{m_i - m_{p1} - m_{s1}} + \frac{g}{\beta_2 V_{E2}}\right)$$

$$V_{Ej}\left(\frac{1}{m_i - \sum_{r=1}^{j} m_{pr} - \sum_{r=1}^{j} m_{sr} + m_{sj}} + \frac{g}{\beta_j V_{Ej}}\right)$$

$$= V_{E(j+1)}\left(\frac{1}{m_i - \sum_{r=1}^{j} m_{pr} - \sum_{r=1}^{j} m_{sr}} + \frac{g}{\beta_{(j+1)} V_{E(j+1)}}\right), \qquad \ldots(18)$$

where $(j = 1, 2, \ldots n - 1)$.

Eqn. (18) can be simplified to

$$M_j^2 + A_j M_j - B_j = 0, \qquad \ldots(19)$$

where M_j is as denoted by (10)

$$\left.\begin{aligned} A_j &= m_{sj} + \frac{V_{Ej} - V_{E(j+1)}}{g\left(\frac{1}{\beta_j} - \frac{1}{\beta_{(j+1)}}\right)} \\ \text{and} \quad B_j &= \frac{m_{sj} V_{E(j+1)}}{g\left(\frac{1}{\beta_j} - \frac{1}{\beta_{j+1}}\right)} \end{aligned}\right\} \qquad \ldots(20)$$

The solution of the quadratic eqn. (19) is

$$M_j = -A_j \pm \sqrt{A_j^2 + 4B_j}.$$

For $B_j < 0$, (20) implies $\frac{1}{\beta_j} < \frac{1}{\beta_{j+1}}$. $\qquad \ldots(21)$

$$A_j^2 + 4B_j = \left\{m_{sj} + \frac{V_{Ej} + V_{E(j+1)}}{g\left(\frac{1}{\beta_j} - \frac{1}{\beta_{j+1}}\right)}\right\}^2 - \frac{4V_{Ej}V_{E(j+1)}}{g^2\left(\frac{1}{\beta_{j+1}} - \frac{1}{\beta_j}\right)^2}.$$

If $A_j < 0$, then for $m_{sj} > 0$, $m_{sj} < \frac{V_{Ej} - V_{E(j+1)}}{g\left(\frac{1}{\beta_{j+1}} - \frac{1}{\beta_j}\right)} > 0$. $\qquad \ldots(22)$

Because of (21), $A_j < 0$ and $B_j < 0 \Rightarrow V_{Ej} > V_{E(j+1)}$. $\qquad \ldots(23)$

Combining eqns. (21) to (23), one gets

$$A_j^2 + 4B_j < \frac{4V_{E(j+1)}}{g^2\left(\frac{1}{\beta_{j+1}} - \frac{1}{\beta_j}\right)^2}(V_{E(j+1)} - V_{Ej}) < 0. \qquad \ldots(24)$$

$A_j < 0$ and $B_j < 0$ give imaginary roots of eqn. (18).

Since there is only one positive real root of eqn. (19) and that occurs for $B_j > 0$, this optimisation is valid for $B_j > 0$, i.e., by (20) $\beta_{j+1} > \beta_j$ for $j = 1, 2, \ldots n-1$, i.e., for the propellant mass flow which increases with the number of stages. Hence, the optimum value of M_j is given by

$$M_j = -A_j + \sqrt{A_j^2 + 4B_j} \qquad \ldots(26)$$

so that the optimum propellant mass in the jth stage is

$$m_{pj} = -M_j + M_{(j-1)} - m_{sj}$$
$$= A_j - A_{(j-1)} - \sqrt{A_j^2 + 4B_j} + \sqrt{A_{(j-1)}^2 + 4B_{(j-1)}} - m_{sj}, \qquad \ldots(27)$$

where $j = 2, 3, \ldots n-1$; and that in the first stage is due to the first of set (18).

Thus, for given equivalent exit velocity, propellant massflow and structural mass of each stage, gross initial mass and overall burnout velocity, there exists a propellant mass alongwith a definite stage-distribution so that the payload mass becomes maximum, while the optimum propellant mass in the last stage can be expressed in terms of the given burnout velocity by means of eqns. (17) and (26) :

$$V = -\sum_{k=1}^{n-1} V_{Ek}\left\{\log\frac{M_k + m_{sk}}{M_k + m_{pk} + m_{sk}} + \frac{gm_{pk}}{\beta_k V_{Ek}}\right\}$$
$$- V_{En}\left(\log\frac{M_{(n-1)} - m_{pn}}{M_{(n-1)}} + \frac{gm_{pn}}{\beta_n V_{En}}\right). \qquad \ldots(28)$$

Hence, the maximum payload mass is given by

$$M_L(\max) = M_{(n-1)(\text{opt})} - m_{sn} - m_{pn(\text{opt})}, \qquad \ldots(29)$$

where $M_{(n-1)(\text{opt})}$ and $m_{pn(\text{opt})}$ are evaluated from eqns. (26) and (28) respectively.

The present analysis also admits of equal propellant mass flows for all stages and in that case the required optimum conditions are the same as these given by eqns. (9) to (16) in the previous section where V is to be replaced by V' and such that

$$V' = V - \frac{M_p g}{\beta}, \qquad \ldots(30)$$

where M_p is the total propellant mass and β the propellant mass flow of each stage.

Conversely, for given equivalent exit velocity, propellant mass flow and structural mass of each stage, gross initial mass and payload mass, there exists an optimum stage-distribution of the given propellant mass, which maximizes the overall burnout velocity.

Numerical Example 1

In the light of the foregoing analysis, let us consider a three-stage-rocket flown vertically in a vacuum in gravitationless space having

$$\left.\begin{array}{lll} V_{E_1} = 10{,}000\text{ft/sec}, & V_{E_2} = 8{,}000\text{ft/sec}, & V_{E_3} = 6{,}000\text{ft/sec}, \\ V = 15{,}000\text{ft/sec}, & m_i = 100{,}000\text{lb}, & \\ m_{s_1} = 10{,}000\text{lb}, & m_{s_2} = 5{,}000\text{lb}, & m_{s_3} = 2{,}000\text{lb}. \end{array}\right\} \quad \ldots(31)$$

Here the optimum propellant mass in each of three stages, which can maximize the payload mass is given by

$$m_{p_1} = m_i - \frac{V_{E_1} m_{s_1}}{V_{E_1} - V_{E_2}}$$

$$m_{p_2} = \frac{V_{E_2} m_{s_1}}{V_{E_1} - V_{E_2}} - \frac{V_{E_3} m_{s_2}}{V_{E_2} - V_{E_3}}$$

$$m_{p_3} = \left[1 - e^{-(V/V_{E_3})} \left\{\frac{m_i}{m_{s_1}}\left(1 - \frac{V_{E_2}}{V_{E_1}}\right)\right\}^{(V_{E_1}/V_{E_3})} \times \left\{\frac{m_{s_1}}{m_{s_2}} \cdot \frac{V_{E_2} - V_{E_3}}{V_{E_1} - V_{E_2}}\right\}^{(V_{E_2}/V_{E_3})}\right] \frac{V_{E_3} m_{s_2}}{V_{E_2} - V_{E_3}} \quad \ldots(32)$$

Substituting (31) in (32), the following optimum results are obtained :

$$m_{p_1} = 50{,}000\text{lb}, \ m_{p_2} = 20{,}000\text{lb}, \ m_{p_3} = 5{,}000\text{lb (approx.)} \quad \ldots(33)$$

The maximum payload mass is

$$M_{L(\max)} = m_i - \sum_{k=1}^{3} (m_{pk} + m_{sk}) = 8{,}000\text{lb}. \quad \ldots(34)$$

Numerical Example 2

In this example, a three-stage rocket is flown vertically in a vacuum in gravitational field such that

$$\begin{array}{l} m_i = 100{,}000\text{lb}, \ m_{s_1} = 5{,}000\text{lb}, \ m_{s_2} = 4{,}000\text{lb}, \ m_{s_3} = 3{,}000\text{lb} \\ \beta_1 = 10g\text{lb/sec}, \quad \beta_2 = 12g\text{lb/sec}, \quad \beta_3 = 18g\text{lb/sec} \\ V_{E_1} = 10{,}000\text{ft/sec}, V_{E_2} = 9{,}000\text{ft/sec}, V_{E_3} = 8{,}000\text{ft/sec} \\ M_L = 12{,}400\text{lb}, \quad g = \text{acceleration due to gravity} = 32\text{ft/sec}^2 \end{array} \quad \ldots(35)$$

Now, for these given conditions, the optimum stage-distribution of the given propellant mass $M_p(= 62{,}600\text{lb})$ is to be determined and also the corresponding maximum burnout velocity.

In view of eqns. (20), (26) and (27), the propellant masses in the first and second stage are given by

$$\left.\begin{array}{l} m_{p_1} = 37{,}550\text{lb}, \quad m_{p_2} = 15{,}050\text{lb}, \text{ and consequently} \\ m_{p_3} = M_p - (m_{p_1} + m_{p_2}) = 10{,}000\text{lb} \end{array}\right\} \quad \ldots(36)$$

The maximum burnout velocity is due to eqns. (28), (43) and (44) :

$$V_{\max} = -\sum_{k=1}^{2} V_{Ek}\left\{\log\frac{M_k + m_{sk}}{M_k + m_{pk} + m_{sk}} + \frac{gm_{pk}}{\beta_k V_{Ek}}\right\}$$

$$- V_{E_3}\left(\log\frac{M_2 - m_{p3}}{M_2} + \frac{gm_{p3}}{\beta_3 V_{E_3}}\right) = 6{,}170\text{ft/sec} \qquad \ldots(37)$$

Discussion

In textbook analysis including those of Miele (1962), Barrére (1960) and Ruppe (1966), propellant mass of each stage is in proportion to the structural mass of the same stage by virtue of a predetermined term like "Structural factor/ratio" and it is "Construction parameter" in Srivastava's analysis. Obviously, whatever may be the optimisation criteria in a multistage rocket, the ratio of optimum propellant mass of each stage to the structural mass of that stage remains unaltered. At the end of each stage the empty propellant tank, pipes, engines and other unnecessary mass realised after exhaustion of that stage propellant mass, are discarded and as such propellant mass of each stage needs to be consistent with the structural mass of that stage. However, in the present analysis, owing to eqns. (11), (12), (14), (18), (26), (27), etc., each stage-propellant mass can be made compatible with the same stage-structural mass by suitably designing the given characteristic parameters (Structural masses, equivalent exhaust velocities, propellant mass flows/thrusts of the stages and initial gross mass). For instance, in gravity-free space (*vide* eqns. (11)) $\frac{m_{s(j-1)}}{V_{E(j-1)} - V_{Ej}} > \frac{m_{sj}}{V_{Ej} - V_{E(j+1)}}$; otherwise $m_{pj} \leqslant 0$ which is absurd. In this context, an interesting aspect can be noticed because of eqn. (11) : if structural mass and equivalent exhaust velocity decrease separately in arithmetic progression from the second to the second last stage, then optimum stage-propellant mass also decreases being in an A.P. series during those stages; whether the optimum propellant masses of the remaining two stages (first and last) can be included in this A.P. series mainly depends on the structural masses and equivalent exhaust velocities of these two stages and gross initial mass.

At length, the author envisages that the present optimisation may be well suited to a single stage rocket, initially fitted with a number of subrockets (air-to-air and/or air-to-ground missiles) for being released in succession during the flight, in as much as such a configuration of single stage rocket is analogous to that of a multistage rocket : m_{sj} is the mass of the jth subrocket instead of that stage-structural mass and m_{pj} the propellant mass to be consumed during the interval between the expulsions of the $(J - 1)$th and Jth subrockets, instead of the Jth stage-propellant mass.

Conclusion

Under given appropriate conditions, various types of optimisation related to multistage performance can be attained. The present analysis, which is developed mainly in terms of propellant mass and structural mass of each stage, is not identical with that carried out by Srivastava (1962) in terms of propellant mass and construction parameter or that given in textbooks on the relevant literature in terms of payload ratios and structural factors of the stages. In the present analysis, the choice of structural mass to be discarded at the end of each stage is arbitrary—not in any definite proportion to the corresponding propellant mass by means of any pre-assigned term like "structural factor" or "construction parameter." Nevertheless, two numerical examples indicate that as the given structural mass of each stage decreases with the number of stages, the optimum propellant mass of each stage also decreases with the number of stages.

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