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Mathematical Elasticity

STRESS-CONCENTRATIONS IN A THIN INSULATED METALLIC DISC IN THE FORM OF AN INVERSE OF AN ELLIPSE LYING HORIZONTALLY IN A UNIFORM ELETRIC FIELD

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The paper deals with a plane-strain problem concerning an isotropic, elastic disc in the form of an inverse of an ellipse, subjected to a uniform electric field. In the problem, stresses and displacements have been expressed in terms of two analytic functions $\Omega(z)$ and $\omega(z)$. The condition that the boundary is stress-free has been expressed in the form of a functional equation. Using function theoretic method and splitting up $\Omega(\sigma)$ into even and odd functions of σ, the functional equation has been solved and the solution has been obtained in a closed form.

Keywords : Function Theory; Metallic Disc; Potential Gradient; Body-Stress Equation

INTRODUCTION

THE problems of rotating elastic discs in the form of a circle and of an ellipse had been solved by Stevenson (1943). Stevenson had used Airy's stress-function and had expressed the stresses and displacements in terms of two complex-potential functions. He had not adopted the function-theoretic method of Muskhelishvili (1953). Later, the problem of a rotating elastic disc in the form of a cardioid was solved by Mitra (1955), who adopted the above two complex-potentials and used the function-theory of Muskhelishvili to solve his problem.

In the present problem, an insulated metallic disc in the form of an inverse of an ellipse has been placed in an electric field of uniform potential gradient. Following the method of Mitra, the boundary equations involving the complex-potential functions have been transformed into integral equations by means of mapping function which maps the boundary on the unit circle. It has has possible to solve the above integral equations by using function-theoretic method and then stress-functions have been determined in a closed form.

THEORY

We suppose that a thin, insulated metallic disc in the form of an inverse of an ellipse is placed parallel to a uniform electric field. Consequently induced charges are

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developed on the periphery of the disc which would be influenced by the electric lines of force along the plane of the disc taken as $x - y$ plane. The disc would, therefore, experience plane elastic strains and elastic stresses would be generated on the body. It is evident that the disc remains totally uncharged as a whole. Incidentally, the plate being metallic would have a uniform potential K (say).

If $W(z, \bar{z})$ be an analytic function such that the potential $U(z, \bar{z}) = \frac{d}{dz} W(z, \bar{z})$, then in our problem

$$\frac{d}{dz} W(z, \bar{z}) = K,$$

where $z = x + iy$ and $\bar{z}$ is conjugate of z. If χ be the Airy's biharmonic stress-function, the stresses satisfying the equations of equilibrium can be expressed as

$$\widehat{xx} = \frac{\partial^2 \chi}{\partial y^2} + \rho U, \; \widehat{yy} = \frac{\partial^2 \chi}{\partial x^2} + \rho U, \; \widehat{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} \qquad \text{...(1)}$$

Let D the displacement and Θ, Φ the stress-combinations be defined by

$$D = u + iv \qquad \text{...(2)}$$

$$\Theta = \widehat{xx} + \widehat{yy} \qquad \text{...(3)}$$

$$\Phi = \widehat{xx} - \widehat{yy} + 2i\widehat{xy} \qquad \text{...(4)}$$

From stress-strain relations, we get

$$(1 - 2\eta)\,\Theta = 2\mu \left(\frac{\partial D}{\partial z} + \frac{\partial \bar{D}}{\partial \bar{z}} \right) \qquad \text{...(5)}$$

and $$\Phi = 4\mu \frac{\partial D}{\partial \bar{z}}, \qquad \text{...(6)}$$

where $$\eta = \frac{\lambda}{2(\lambda + \mu)}.$$

In plane strain the body-stress equation is

$$\frac{\partial \Phi}{\partial z} + \frac{\partial}{\partial \bar{z}} (\Theta - 2\rho U) = 0. \qquad \text{...(7)}$$

Substituting from (5) and (6), in (7), we get

$$\frac{\partial}{\partial \bar{z}} \left\{ K \frac{\partial D}{\partial z} + \frac{\partial \bar{D}}{\partial \bar{z}} - \rho \frac{1 - 2\eta}{\mu} U \right\} = 0$$

where $K = 3 - 4\eta$.

Integrating with respect to $\bar{z}$, we get

$$K\frac{\partial D}{\partial z}+\frac{\partial \bar{D}}{\partial \bar{z}}-\rho\frac{1-2\eta}{\mu}U=C\Omega'(z) \qquad \text{...(8)}$$

$\Omega'(z)$ being an arbitrary analytic function of z, an C an orbitrary constant.

Taking conjugate of (8) and solving for $\frac{\partial D}{\partial z}$ and taking

$$C=\frac{1-k^2}{8\mu},$$

we get

$$8\mu\frac{\partial D}{\partial z}=k\Omega'(z)-\overline{\Omega'(z)}+\gamma\rho\frac{\partial W}{\partial z}, \qquad \text{...(9)}$$

where $\quad U=\frac{\partial}{\partial z}W(z,\bar{z}).$

Integrating with respect to z and introducing an arbitrary analytic function $\overline{\omega'(z)}$, we get

$$8\mu D=k\Omega(z)-z\overline{\Omega'(z)}-\overline{\omega'(z)}+\gamma\rho W. \qquad \text{...(10)}$$

From (5), (6) and (10) we find that Θ, Φ can be expressed as

$$2\Theta=\Omega'(z)+\overline{\Omega'(z)}+(4-\gamma)\rho\frac{\partial W}{\partial z} \qquad \text{...(11)}$$

and

$$-2\Phi=z\overline{\Omega''(z)}+\overline{\omega''(z)}-\gamma\rho\frac{\partial W}{\partial z}, \qquad \text{...(12)}$$

where $\quad \gamma=\frac{2(1-2\eta)}{1-\eta}.$

Boundary Conditions in Complex Potentials

Let the direction of tangent and normal to the bounding curve be denoted by s and n, which are obtained by rotating the axes of x and y through an angle α, so that

$$n+is=ze^{-i\alpha}. \qquad \text{...(13)}$$

The stress-combination Θ', Φ', referred to new axes are given by

$$\left.\begin{aligned}\Theta'&=\widehat{nn}+\widehat{ss}\\ \Phi'&=\widehat{nn}-\widehat{ss}+2i\widehat{sn}\end{aligned}\right\} \qquad \text{...(14)}$$

where $\quad \Theta'=\Theta,\ \Phi'=\Phi e^{-2i\alpha}$

Therefore,

$$4(\widehat{nn}+i\widehat{ns})=2(\Theta'+\Phi')=2(\Theta+\Phi e^{-2i\alpha}) \qquad \text{...(15)}$$

$$4(\hat{nn} + i\hat{ns})\frac{\partial z}{\partial s} = \left\{\Omega'(z) + \overline{\Omega'(z)} + (4-\gamma)\,\rho\frac{\partial W}{\partial z}\right\}\frac{\partial z}{\partial s} + \left\{z\,\overline{\Omega''(z)} + \overline{\omega''(z)} - \gamma\rho\frac{\partial W}{\partial \bar{z}}\right\}\frac{\partial \bar{z}}{\partial s}, \quad \ldots(16)$$

whence we get

$$4\left(\hat{nn} + i\hat{ns} - \rho\frac{\partial W}{\partial z}\right)\frac{\partial z}{\partial s} = \frac{\partial}{\partial s}\{\Omega(z) + z\overline{\Omega'(z)} + \overline{\omega'(z)} - \gamma\rho W\} \quad \ldots(17)$$

Since on the boundary $\hat{nn} = \hat{ns} = 0$, the boundary condition (16) reduces to

$$\Omega(z) + z\,\overline{\Omega'(z)} + \overline{\omega'(z)} - \gamma\rho W - 4\rho\int\frac{\partial W}{\partial z}\,dz = -\,4\rho Kz \quad \ldots(18)$$

Since the potential $\frac{dW}{dz} = K$, we have $W = Kz$ and eqn. (18) becomes

$$\Omega(z) + z\,\overline{\Omega'(z)} + \overline{\omega'(z)} = (\gamma - 4)\,\rho Kz$$

The mapping function which represents conformally the region of the disc on the unit-circle C is given by

$$z = m(\zeta) = \frac{l\zeta}{a^2 + \zeta^2},\ a > 1,\ l > 0 \quad \ldots(19)$$

Then $$m'(\zeta) = \frac{l(a^2 - \zeta^2)}{(a^2 + \zeta^2)^2}. \quad \ldots(20)$$

Let us put $\Omega(z) = \Omega\,\{m(\zeta)\} = \Omega(\zeta)$. Since $\bar{\zeta} = \frac{1}{\zeta}$ on the unit circle we have on the boundary of the unit circle

$$\Omega(\zeta) + m(\zeta)\frac{\overline{\Omega'(\zeta)}}{\overline{m'(\zeta)}} + \frac{\overline{\omega'(\zeta)}}{\overline{m'(\zeta)}}$$
$$= \gamma\rho W(\zeta, \bar{\zeta}) - 4\rho Km(\zeta)$$
$$= K\rho(\gamma - 4)\,m(\zeta) = F(\zeta),\ \text{say} \quad \ldots(21)$$

We multiply both sides of the above equation by $\frac{1}{2\pi i}\frac{d\zeta}{\zeta - \sigma}$ and integrate over the unit circle c.

Thus : $$\frac{1}{2\pi i}\int_c \Omega(\zeta)\,\bar{m}'(\bar{\zeta})\frac{d\zeta}{\zeta - \sigma} + \frac{1}{2\pi i}\int_c \bar{\Omega}'(\bar{\zeta})\,m(\zeta)\frac{d\zeta}{\zeta - \sigma} + \frac{1}{2\pi i}\int_c \bar{\omega}'(\bar{\zeta})\frac{d\zeta}{\zeta - \sigma} = \frac{1}{2\pi i}\int_c F(\zeta)\,\bar{m}'(\bar{\zeta})\frac{d\zeta}{\zeta - \sigma} \quad \ldots(22)$$

where

$$F(\zeta) = \frac{K\rho(\gamma-4)\, l\zeta}{a^2+\zeta^2}. \qquad \ldots(23)$$

The integral on the right hand side of (22) with (23) and (20) becomes

$$\frac{1}{2\pi i}\int_c F(\zeta)\,\overline{m'}\left(\frac{1}{\zeta}\right)\frac{d\zeta}{\zeta-\sigma}$$

$$= \frac{l^2(\gamma-4)K\rho}{2\pi i}\int_C \frac{\zeta^3(a^2\zeta^2-1)}{(a^2+\zeta^2)(a^2\zeta^2+1)^2}\,\frac{d\zeta}{\zeta-\sigma}$$

$$= \frac{l^2(\gamma-4)\,K\rho a^2(1+a^4)\,\sigma}{(1-a^4)^2\,(a^2+\sigma^2)}. \qquad \ldots(24)$$

We now integrate separately the integrals on the left hand side of (22).

We assume

$$\Omega(\zeta) = P(\zeta) + Q(\zeta), \qquad \ldots(25)$$

where $P(\zeta)$ is a polynomial in even powers of ζ, and $Q(\zeta)$ that in odd powers of ζ.

The first integral on the left hand side of (22)

$$= \frac{1}{2\pi i}\int_C \Omega(\zeta)\,\overline{m'}\left(\frac{1}{\zeta}\right)\frac{d\zeta}{\zeta-\sigma}$$

$$= \frac{1}{2\pi i}\int_C \Omega(\zeta)\,\frac{l\zeta^2(a^2\zeta^2-1)}{(a^2\zeta^2+1)^2}\,\frac{d\zeta}{\zeta-\sigma}$$

$$= l\left[\frac{\Omega(\sigma)\,\sigma^2(a^2\sigma^2-1)}{(a^2\sigma^2+1)^2} + \frac{\frac{i}{a}P'\left(\frac{i}{a}\right)+\sigma Q'\left(\frac{i}{a}\right)}{a^2\sigma^2+1}\right.$$

$$\left. + 2\,\frac{P\left(\frac{i}{a}\right) - ia\sigma Q\left(\frac{i}{a}\right)}{a^2\sigma^2+1} + \frac{P\left(\frac{i}{a}\right)(a^2\sigma^2-1) + 2ia\sigma\,.\,Q\left(\frac{i}{a}\right)}{(a^2\sigma^2+1)^2}\right] \qquad \ldots(26)$$

Also, the integral

$$\frac{1}{2\pi i}\int_C \bar{\Omega}'\left(\frac{1}{\zeta}\right)m(\zeta)\,\frac{d\zeta}{\zeta-\sigma}$$

$$= \frac{1}{2\pi i}\int_C \bar{\Omega}'\left(\frac{1}{\zeta}\right)\frac{l\zeta}{a^2+\zeta^2}\,\frac{d\zeta}{\zeta-\sigma}$$

$$= -\,l\left\{P'\left(\frac{i}{a}\right)\frac{ia}{a^2+\sigma^2} - \frac{Q'\left(\frac{i}{a}\right).\,\sigma}{a^2+\sigma^2}\right\}; \qquad \ldots(27)$$

and $$\frac{1}{2\pi i}\int_C \overline{\omega'}\left(\frac{1}{\zeta}\right)\frac{d\zeta}{\zeta-\sigma}=0, \qquad \ldots(28)$$

neglecting the constant term in $\omega'(\zeta)$ which is associated with rigid body displacement.

Eqn. (24) together with (25), (26), (27), and (28) leads to the determination of $\Omega(\sigma)$, involving unknown constants $P\left(\frac{i}{a}\right)$, $P'\left(\frac{i}{a}\right)$, $Q\left(\frac{i}{a}\right)$, $Q'\left(\frac{i}{a}\right)$ as given by

$$l\left[\frac{\Omega(\sigma)\,\sigma^2(a^2\sigma^2-1)}{(a^2\sigma^2+1)^2}+\frac{\frac{i}{a}P'\left(\frac{i}{a}\right)+\sigma Q'\left(\frac{i}{a}\right)}{a^2\sigma^2+1}\right.$$
$$\left.+\,2.\,\frac{P\left(\frac{i}{a}\right)-ia\sigma Q\left(\frac{i}{a}\right)}{a^2\sigma^2+1}+\frac{P\left(\frac{i}{a}\right)(a^2\sigma^2-1)+2ia\sigma\times Q\left(\frac{i}{a}\right)}{(a^2\sigma^2+1)^2}\right]$$
$$+\,l\left\{-P'\left(\frac{i}{a}\right)\times\frac{ia}{a^2+\sigma^2}+\frac{\sigma Q'\left(\frac{i}{a}\right)}{a^2+\sigma^2}\right\}$$
$$=l^2(\gamma-4)\,K\rho\times\frac{a^2(1+a^4)\,\sigma}{(1-a^4)^2\,(a^2+\sigma^2)} \qquad \ldots(29)$$

Replacing σ by $-\sigma$ in the above eqn. (29), we get an equation for $\Omega(-\sigma)$. From these two equations giving $\Omega(\sigma)$ and $\Omega(-\sigma)$ we get on addition and substraction (remembering that $\Omega(-\sigma)=P(\sigma)-Q(\sigma)$) the following equations for $P(\sigma)$ and $Q(\sigma)$:

$$\frac{P(\sigma)\,\sigma^2(a^2\sigma^2-1)}{(a^2\sigma^2+1)^2}+\frac{iP'\left(\frac{i}{a}\right)\sigma^2(1-a^4)}{a(a^2+\sigma^2)(a^2\sigma^2+1)}$$
$$+\,P\left(\frac{i}{a}\right).\,\frac{3a^2\sigma^2+1}{(a^2\sigma^2+1)^2}=0 \qquad \ldots(30)$$

$$\frac{Q(\sigma)\,\sigma(a^2\sigma^2-1)}{(a^2\sigma^2+1)^2}+Q'\left(\frac{i}{a}\right)\frac{(1+a^2)(1+\sigma^2)}{(a^2+\sigma^2)(a^2\sigma^2+1)^2}$$
$$+\,2iaQ\left(\frac{i}{a}\right)\frac{-a^2\sigma^2}{(a^2\sigma^2+1)^2}=\frac{(\gamma-4)\,K\rho l a^2(1+a^4)}{(1-a^4)^2\,(a^2+\sigma^2)} \qquad \ldots(31)$$

To determine the constants $P\left(\frac{i}{a}\right)$, $P'\left(\frac{i}{a}\right)$, $Q\left(\frac{i}{a}\right)$, $Q'\left(\frac{i}{a}\right)$, we put $\sigma=0$, $\sigma=\frac{1}{a}$ in (30) and (31) and we get

$$P\left(\frac{i}{a}\right)=0,\quad P'\left(\frac{i}{a}\right)=0 \qquad \ldots(32)$$

so that from (30)

$$P(\sigma) = 0 \qquad \ldots(33)$$

Also,

$$\left.\begin{aligned} Q\left(\frac{i}{a}\right) &= -\frac{i(\gamma - 4)\, K\rho l a(1 - a^2)}{(1 - a^4)^2} \\ Q'\left(\frac{i}{a}\right) &= \frac{(\gamma - 4)\, K\rho l a^2(1 + a^4)}{(1 + a^2)\,(1 - a^4)^2} \end{aligned}\right\} \qquad \ldots(34)$$

so that from (31)

$$Q(\sigma) = \frac{(\gamma - 4)\, K\rho l a^2 \sigma}{(1 + a^2)\,(a^2 + \sigma^2)}. \qquad \ldots(35)$$

$\Omega(\sigma)$ is thus found to be an odd function of σ given by (35). Substituting for $\Omega(\sigma)$ in (22) we can evaluate $\omega(\sigma)$.

The variation of Hoop's stress on the boundary at different points is given by

$$\tfrac{1}{2}\left[\frac{\Omega'(\sigma)}{m'(\sigma)} + \frac{\bar{\Omega}'(\bar{\sigma})}{\bar{m}'(\bar{\sigma})}\right]$$

$$= \frac{(\gamma - 4)\, k\rho a^2}{1 + a^2}.$$

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