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Non-Newtonian Fluids

FREE CONVECTION FLOW OF A SECOND-ORDER FLUID ALONG A POROUS VERTICAL PLATE WITH PERIODIC SUCTION

R. C. CHAUDHARY *and* P. R. SHARMA

Department of Mathematics, University of Rajasthan, Jaipur-302 004, India

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This paper studies three-dimensional flow and heat transfer for a non-Newtonian fluid i.e., free convection flow of a second-order fluid along a porous vertical plate with transverse sinusoidal suction velocity distribution. The effect of the non-Newtonian fluid parameter on the stress component and heat flux has been discussed numerically. It is found that the results for Newtonian fluid differ from those obtained by Singh, Sharma and Misra (1978).

Keywords : Porous; Temperature; Viscoelasticity; Cross Viscocity

INTRODUCTION

IN modern times, there has been a noticeable increase in interest in free convection problem because the process of heat transfer has many important technological applications. The two-dimensional flow problems have been solved by many research workers but a few three-dimensional problems have been studied. Recently, Gersten and Gross (1974) have studied the three dimensional incompressible laminar boundary layer past a porous flat plate when a slightly sinusoidal transverse suction velocity distribution at the wall is applied. Singh *et al.* (1978) have examined the three-dimensional free convection flow and heat-transfer along a porous vertical plate when the same suction velocity distribution is applied as in Gersten and Gross (1974). Verma and Sharma (*communicated*) have studied the free convection flow of a second order fluid along an infinite, vertical, porous plate with constant suction. In the present paper, we extend the results of Singh *et al.* (1978), to a three-dimensional non-Newtonian flow problem. Here we investigate the effect of non-Newtonian fluid parameters on the wall shear stress and heat transfer in the flow due to a periodic suction velocity distribution. Our results are compared with the results for Newtonian case as obtained by Gersten and Gross (1974) and Singh *et al.* (1978), when $G = 0$ and $G = 0$ and 5 respectively. The results in the Newtonian case differ from those of Singh *et al.* (1978). It seems that the constants in (5.8) of Singh *et al.* (1978) are not evaluated correctly.

Nomenclature

τ_{ij} Stress tensor

p Pressure

δ_{ij} Metric tensor

v_i Velocity vector

a_i Acceleration vector

$v_w(z)$ periodic suction velocity at the wall

ϵ small parameter in periodic suction velocity

v_0 Suction velocity for $\epsilon = 0$

l Wavelength of the periodic suction velocity

(x, y, z) Cartesian coordinates

(u, v, w) Velocity components along x, y, z-direction

U Uniform free stream velocity

ρ Density of the fluid

g Acceleration due to gravity

β Coefficient of expansion

$\bar{k}$ Thermal conductivity

T_w Temperature at the wall

T Temperature of the fluid

T_∞ Temperature of the fluid in the free stream

p_∞ Pressure in the free stream

θ Dimensionless temperature $\left(= \dfrac{T - T_\infty}{Tw - T_\infty} \right)$

R Reynolds number $\left(= - \dfrac{v_0 \rho l}{\phi_1} \right)$

ϕ_1 Coefficiently of viscosity

ϕ_2 Coefficient of visco-elasticity

ϕ_3 Coefficient of Cross-viscosity.

K Dimensionless visco-elasticity $\left(= \dfrac{\phi_2}{\rho l^2} \right)$

S Dimensionless Cross-viscosity $(= \phi_3/\rho l^2)$

P Prandtl number $\left(\dfrac{\phi_1 C_p}{\bar{k}} \right)$

α Thermal diffusivity

C_p Specific heat at constant pressure

$\bar{y}$ Dimensionless coordinate $\left(= \dfrac{y}{l} \right)$

$\bar{z}$ Dimensionless coordinate $\left(= \frac{z}{l}\right)$

G The Grasshof Number $\left(= \frac{\nu g\beta(T_w - T_\infty)}{U v_0^2}\right)$

The model of the second order fluid as suggested by Coleman and Noll (1960) is used in the present analysis. The constitutive equation of an incompressible non-Newtonian fluid is

$$\tau_{ij} = - p\delta_{ij} + \phi_1 A_{ij} + \phi_2 B_{ij} + \phi_3 A_{ik} A^k_j , \qquad \ldots(1)$$

where

$$A_{ij} = V_{i,j} + v_{j,i} \qquad \ldots(2)$$

and

$$B_{ij} = a_{i,j} + a_{j,i} + 2v^m_{,i} v_{m,j}, \qquad \ldots(3)$$

comma denoting covariant differentation. The solution of 6.8 per cent polyisobutylene in cetane at 30 °C behaves as a second order fluid.

Equations of Motion

Let the wall be lying vertically on xz-plane such that x-axis is oriented in the direction of the buoyancy force and y-axis is perpendicular to the plane of the wall and directed into the fluid which is flowing linearly with free stream velocity U. (u, v, w) are the velocity component in the directions of x, y, z respectively.

Consider the sinusoidal suction velocity distribution of the form

$$v_W(z) = v_0\left(1 + \epsilon \cos\frac{\pi z}{l}\right), \ \epsilon \ll 1, \qquad \ldots(4)$$

which consists of basic steady distribution $v_0 < 0$ with superimposed weak distribution $\epsilon v_0 \left(\cos\frac{\pi z}{l}\right)$. The velocity and temperature fields are independent of x due to an asymptotic flow, but the flow itself will be three-dimensional in the presence of a cross-flow. The purpose of the present work is to investigate the effect of buoyancy on the flow and temperature field.

The equations of continuity, momentum and energy are

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \qquad \ldots(5)$$

$$\rho\left(v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g\beta(T - T_\infty) + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}, \qquad \ldots(6)$$

$$\rho\left(v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}, \qquad \ldots(7)$$

$$\rho\left(v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}, \qquad ...(8)$$

and

$$\rho C_p\left(v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=\bar{k}\left(\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right). \qquad ...(9)$$

In eqn. (6), we have used Boussinesq approximation. The terms due to viscous dissipation and the terms arising out of the elastic energy have been assumed negligible in comparison to the heat transfer across the plate and the dimensionless viscoelasticity parameter is taken to be small.

The boundary conditions are

$$y=0: u=0,\ v=v_0\left(1+\epsilon\cos\frac{\pi z}{l}\right),\ w=0,\ T=T_w;$$

$$y\to\infty: u=U,\ v=v_0,\ p=p_\infty,\ w=0,\ T=T_\infty \qquad ...(10)$$

All the physical variables are defined in nomenclature.

METHOD OF SOLUTION

Since ϵ is very small, we assume the solutions of the following form

$$f=f_0+\epsilon f_1+\epsilon^2 f_2+\ldots, \qquad ...(11)$$

where f stands for any of u, v, w, p and θ.

When $\epsilon=0$, the problem becomes two-dimensional and hence eqns. (5) to (9) reduce to

$$\frac{dv_0}{dy}=0, \qquad ...(12)$$

$$Rkl^2\frac{d^3u_0}{dy^3}-l\frac{d^2u_0}{dy^2}-R\frac{du_0}{dy}+\frac{GURv_0}{\nu}\theta=0 \qquad ...(13)$$

$$\frac{d}{dy}\left[p_0-(2K+S)\left(\frac{du_0}{dy}\right)^2\right]=0, \qquad ...(14)$$

$$\frac{d^2\theta_0}{dy}-\frac{v_0}{\alpha}\left(\frac{d\theta_0}{dy}\right)=0. \qquad ...(15)$$

Here, $w_0=0$ and eqn. (14) gives $2K+S=0$ for $p=p_\infty$.

The boundary conditions are

$$y=0:\ u=u_0=0,\ v=v_0,\ \theta_0=1$$

$$y\to\infty: u=u_0=U,\ \theta_0=0. \qquad ...(16)$$

Equation (13) is a third order differential equation when $K\neq 0$ and for $K=0$, it reduces to the equation governing the Newtonian fluid. Hence, the presence of the elasticity of the fluid increases, the order of the governing equation from two to three and threfore we need three boundary conditions for the unique solution of (13). But from physical considerations, only two boundary conditions are $u_0 = 0$ at $y = 0$ and $u_0 = U$ at $y \to \infty$; to overcome this difficulty, we assume the solution in the following form

$$u_0 = u_{00} + Ku_{01} + 0\,(K^2), \qquad \ldots(17)$$

where terms of $0(K^2)$ are neglected for very small value of K(i.e. $K \ll 1$), for a second-order fluid like polyisobutylene in cetane. Hence we have the solutions

$$v = v_0 < 0, \qquad \ldots(18)$$

$$u_{00} = U\left[1 + \left(\frac{G}{P^2 - P} - 1\right) e^{-Rv/l} - \frac{G}{P^2 - P}\, e^{-PRv/l}\right], \qquad \ldots(19)$$

$$u_{01} = U\left[\left\{-\frac{GPR^2}{(P-1)^2} - R^3\left(1 - \frac{G}{P^2-P}\right)\frac{y}{l}\right\} e^{-Rv/l} + \frac{GPR^2}{(P-1)^2}\, e^{-PRv/l}\right], \qquad \ldots(20)$$

for $P \neq 1$

and

$$\theta = \theta_0 = e^{Pv_0 y/\nu}. \qquad \ldots(21)$$

When $\epsilon \neq 0$, we use eqn. (11) for u, v, w, p and θ in eqns. (5) to (9) and collect terms of the first order. The perturbation equations of $0(\epsilon)$ are

$$\frac{\partial y_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \qquad \ldots(22)$$

$$R\left(v_0 \frac{\partial u_1}{\partial y} + v_1 \frac{\partial u_0}{\partial y}\right)$$

$$= \frac{GUv_0^2 R}{\nu}\,\theta_1 - v_0 l\left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}\right) + RKl^2\left[v_0 \frac{\partial^3 u_1}{\partial y^3} + \frac{\partial^2}{\partial y^2}\right.$$

$$\left.\times\left(v_1 \frac{\partial u_0}{\partial y}\right) + v_0 \frac{\partial^2}{\partial z^2}\left(\frac{\partial u_1}{\partial y}\right) + \frac{\partial^2}{\partial z^2}\left(v_1 \frac{\partial u_0}{\partial y}\right)\right]$$

$$+ RSl^2\left[2\frac{\partial}{\partial y}\left(\frac{\partial u_0}{\partial y}\frac{\partial v_1}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u_0}{\partial y}\frac{\partial v_1}{\partial z} + \frac{\partial u_0}{\partial y}\frac{\partial w_1}{\partial y}\right)\right], \qquad \ldots(23)$$

$$Rv_0 \frac{\partial v_1}{\partial y} = -\frac{R}{\rho}\frac{\partial p_1}{\partial y} - v_0 l\left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right) + RKl^2 v_0\left(\frac{\partial^3 v_1}{\partial y^3} + \frac{\partial^3 v_1}{\partial z^2 \partial y}\right), \qquad \ldots(24)$$

$$Rv_0 \frac{\partial w_1}{\partial y} = -\frac{R}{\rho}\frac{\partial p_1}{\partial z} - v_0 l\left(\frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2}\right) + RKl^2 v_0\left(\frac{\partial^3 w_1}{\partial y^3} + \frac{\partial^3 w_1}{\partial y \partial z^2}\right), \qquad \ldots(25)$$

$$v_0 \frac{\partial \theta_1}{\partial y} + v_1 \frac{\partial \theta_0}{\partial y} = \alpha \left(\frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2} \right). \quad ...(26)$$

This is a set of linear differential equations which describe the three-dimensional cross-flow. The equations will be solved for u_1, v_1, w_1, p_1 and θ_1.

Cross Flow Solution

The eqns. (22), (24) and (25) are independent of the main flow component $u_1(y, z)$ and the temperature field $\theta_1(y, z)$. The suction velocity consists of a basic uniform distribution v_0 with a superimposed weak sinusoidal distribution $\epsilon v_0 \cos \pi \bar{z}$. Therefore, the velocity components $v_1(y, z)$, $w_1(y, z)$ and $p_1(y, z)$ are also separated into main and small sinusodial components. We assume the following forms for $v_1(y, z)$, $w_1(y, z)$ and $p_1(y, z)$ as

$$v_1(\bar{y}, \bar{z}) = \pi v_0 v_{11}(\bar{y}) \cos \pi \bar{z}, \quad ...(27)$$

$$w_1(\bar{y}, \bar{z}) = - v_0 \bar{v}'_{11}(\bar{y}) \sin \pi \bar{z}, \quad ...(28)$$

$$p_1(\bar{y}, \bar{z}) = \rho v_0^2 \, p_{11}(\bar{y}) \cos \pi \bar{z}. \quad ...(29)$$

Here, a prime denotes differentiation with respect to $\bar{y}$.

Using eqns. (27) to (29) in eqns. (22), (24) and (25) we have

$$RKv_{11}''' - v_{11}'' - R(\pi^2 K + 1) v_{11}' + \pi^2 v_{11} = \frac{R}{\pi} p_{11}', \quad ...(30)$$

and

$$RKv_{11}^{iv} - v_{11}''' - R(\pi^2 K + 1) v_{11}'' + \pi^2 v_{11}' = \pi R p_{11}, \quad ...(31)$$

with the boundary conditions

$$v_{11}(0) = 1/\pi \text{ and } v_{11}'(0) = 0. \quad ...(32)$$

We assume

$$v_{11} = v_{110} + K v_{111} + 0(K^2)$$

and
$$p_{11} = p_{110} + K p_{111} + 0(K^2) \quad ...(33)$$

The boundary conditions are reduced to

$$v_{110}(0) = 1/\pi, \quad v_{111}(0) = v_{111}'(0) = 0 = v_{111}'(0). \quad ...(34)$$

From eqns. (30), (31), (33) with (34), we have

$$v_{110} = \frac{1}{\pi(\pi - \lambda)} (\pi e^{-\lambda \bar{y}} - \lambda e^{-\pi \bar{y}}), \quad ...(35)$$

$$v_{111} = \frac{\lambda(\pi + \lambda)^2 (\lambda + R)}{\pi^2 (R - 2\lambda)} [e^{-\pi \bar{y}} - e^{-\lambda \bar{y}} + (\pi - \lambda) \bar{y} e^{-\lambda \bar{y}}]. \quad ...(36)$$

Using eqns. (35) and (36) in (31), we get p_{110} and p_{111}. As v_{110}, v_{111}, p_{110} and p_{111} are known, we find $v_1(\bar{y}, \bar{z})$, $w_1(\bar{y}, \bar{z})$ and $p_1(\bar{y}, \bar{z})$. The expressions of $v_1(\bar{y}, \bar{z})$, $w_1(\bar{y}, \bar{z})$ and $p_1(\bar{y}, \bar{z})$ are not presented here for the sake of brevity.

SOLUTION FOR FLOW AND TEMPERATURE FIELDS

Now we consider the eqns. (23) and (26). The velocity component u_1 and temperature field θ_1 are also seperated into mean and simusoidal components u_1 and θ_1 : we assume the following forms for u_1 and θ_1 :

$$u_1(\bar{y}, \bar{z}) = U u_{11}(\bar{y}) \cos \pi\bar{z}, \qquad \ldots(37)$$

$$\theta_1(\bar{y}, \bar{z}) = \theta_{11}(y) \cos \pi\bar{z}. \qquad \ldots(38)$$

The boundary conditions reduce to

$$\bar{y} = 0 : u_{11} = 0,\ \theta_{11} = 0;$$

$$\bar{y} \to \infty : u_{11} = 0,\ \theta_{11} = 0. \qquad \ldots(39)$$

Further, assuming

$$u_{11} = u_{110} + K u_{111} + 0(K^2)$$

and $$\theta_{11} = \theta_{110} + K\theta_{111} + 0(K^2). \qquad \ldots(40)$$

The corresponding boundary conditions are

$$\bar{y} = 0 : u_{110} = 0 = u_{111},\ \theta_{110} = 0 = \theta_{111};$$

$$\bar{y} \to \infty : u_{110} = 0 = u_{111},\ \theta_{110} = 0 = \theta_{111}. \qquad \ldots(41)$$

From eqns. (23) and (26) with the boundary conditions (41), we find u_{110}, u_{111}, θ_{110} and θ_{111}; and finally, with eqns. (37) and (38), we have u_1 and θ_1. Here, the expressions of u_1 and θ_1 are not given as these expression are lengthy.

The shear stress component along the direction of x-axis is

$$C_{fx} = -\frac{\tau x}{\rho v_0 U} = \frac{1}{UR}\,[u'_{00} + ku'_{01} - KRu'_{00} + \epsilon\,\{Uu'_{110} + UKu'_{111}$$

$$- KRUu''_{110} - \pi KRv_{110}u''_{00} + 3\pi KRv'_{110}\,u'_{00}\}\cos \pi\bar{z}]_{\bar{y}=0}, \qquad \ldots(42)$$

$$= 1 + \frac{G}{P} + \epsilon F_1(G, P, R, K)\cos \pi\bar{z}, \text{ for } p \neq 1 \qquad \ldots(43)$$

The heat flux at the wall in terms of Nusselt number is

$$Nu = \frac{\bar{k}}{\rho v_0 c_p}\left(\frac{\partial\theta}{\partial y}\right) y = 0,$$

$$= 1 + \epsilon\,[1 - F_2(P, R, K)] \cos \pi\bar{z},$$

$$= 1 + \epsilon\,[1 - \{F_{20}(P, R) + KF_{21}(P, R)\}] \cos \pi\bar{z}, \qquad \ldots(44)$$

where

$$F_{20}(P, R) = \frac{1}{\lambda - \pi}\left[\frac{\lambda^2 + P(\lambda^2 - \pi^2)}{\pi\lambda(P + 1)}\,\bar{\lambda} + \frac{\pi P(PR + \lambda)}{\lambda(P + 1)} - \frac{\lambda PR + \pi^2}{\pi}\right], \qquad \ldots(45)$$

$$F_{21}(P, R) = \frac{(\pi + \lambda)^2\,(\lambda + R)}{\pi^2 R\lambda(2\lambda - R)\,(P + 1)^2}\,[\{\pi PR\lambda(P + 1) - \lambda^2 R(p + 1)^2$$

$$- \pi P(\pi - \lambda)\,(PR + 2\lambda)\}\,\bar{\lambda} + \lambda^2 R(PR + \pi)\,(P + 1)^2$$

$$- \pi PR\lambda(P + 1)\,(PR + \lambda) + \pi P(\pi - \lambda)\,(PR + \lambda)$$

$$\times\,(PR + 2\lambda) - \pi PR\lambda(\pi - \lambda)\,(P + 1)], \qquad \ldots(46)$$

$$\lambda = \frac{R}{2} + \sqrt{\left(\frac{R^2}{4} + \pi^2\right)}, \qquad \ldots(47)$$

and

$$\bar{\lambda} = \frac{PR}{2} + \sqrt{\left(\frac{P^2R^2}{4} + \pi^2\right)}. \qquad \ldots(48)$$

If we put $K = 0$ in the eqns. (37), (38) and (40), the expressions for F_1 and F_2 differ from those obtained by Singh *et al.* (1978) due to their incorrect derivation.

Also, when $G = 0$ and $K = 0$, the expressions for F_1 and F_2 differ from those of Gersten and Gross (1974) and Singh *et al.* (1978). It seems the constants of (5.8) are not derived correctly by Singh *et al.* (1978).

Conclusions

The discussion of results is as follows :

(*i*) C_{fx} (*Wall Shear Stress in Main Flow Direction*)

The effect of cross-flow in the wall shear stress is given by the function F_1 which is shown against R in Fig. 1 for $K = 0$ and F_1 goes on increasing indefinitely. For $K = -0.1$. It shows that for a Newtonian fluid the function F_1 increases from 0 to 1 with the increasing value of R. But in a non-Newtonian case this function decreases with the increasing value of R(i.e., $R \leqslant 5$) and increases for the higher values of R due to the presence of viscoelastic parameter K. The magnitude of F_1 is less for $G = 0$ in comparison to $G = 5$ in the case of a Newtonian fluid, it is also true for a non-Newtonian fluid for increasing value of R(i.e., $R \leqslant 5$), but greater in the case of a non-Newtonian fluid for $G = 5$ in comparison to $G = 0$ for higher values of R.

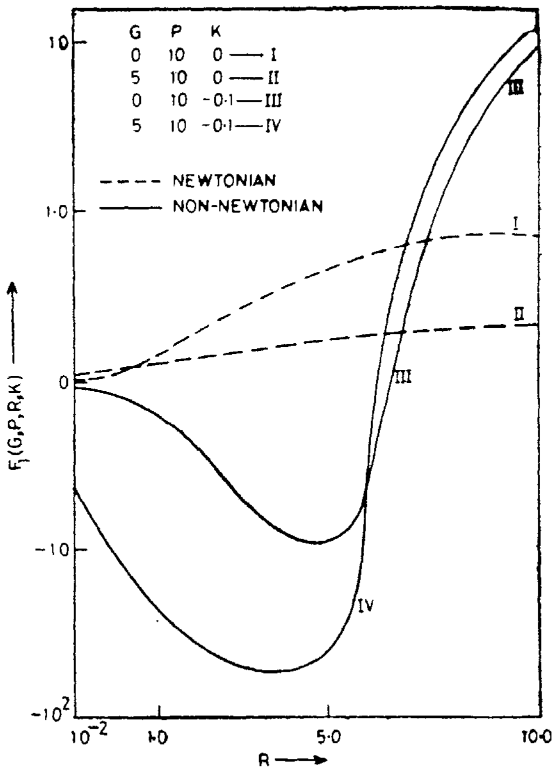
G P K
0 10 0 — I
5 10 0 — II
0 10 -0·1 — III
5 10 -0·1 — IV
NEWTONIAN
NON-NEWTONIAN
10
1·0
0
-10
-10²
F₁(G,P,R,K)
10⁻²
1·0
5·0
10·0
R
I
II
III
III
IV

FIG. 1.

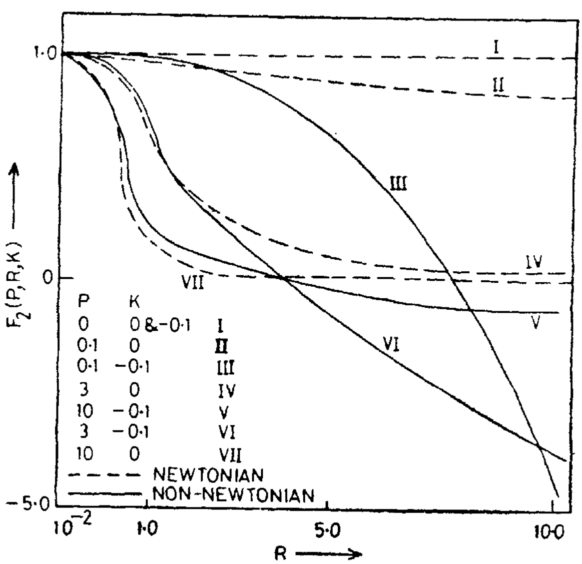
1·0
0
-5·0
F₂(P,R,K)
P K
0 0 &-0·1 I
0·1 0 II
0·1 -0·1 III
3 0 IV
10 -0·1 V
3 -0·1 VI
10 0 VII
NEWTONIAN
NON-NEWTONIAN
10⁻²
1·0
5·0
10·0
R
I
II
III
IV
V
VI
VII

FIG. 2.

(*ii*) *Nu* (*Nusselt-Number*)

The heat flux at the wall in terms of Nusselt number Nu is shown in Fig. 2. When $R \to 0$, $F_2 \simeq 1$ for $K = 0$ and $K = -0.1$. When $R \to \infty$, $F_2 \simeq 0$ for $K = 0$ and F_2 goes on decreasing indefinitely for $K = -0.1$.

When $P = 1$, the Reynolds analogy holds.

$$Nu = C_{fx} \text{ for } P = 1 \text{ and } K = 0. \qquad ...(49)$$

In the case of $P = 0$, the Nusselt number is independent of the Reynolds number and the periodic suction velocity;

$$Nu = 1 \text{ for } P = 0; \ K = 0 \text{ and } K = -0.1 \qquad ...(50)$$

Finally, when $P \to 0$, $F \simeq 1$ for $K = 0$ and $K = -0.1$.

When $P \to \infty$, $F_2 \simeq 0$ for $K = 0$ and $F_2 \to -\infty$ for $K = -0.1$. It means, F_2 for an ordinary viscous liquid decreases from 1 to 0 for large values of R and in the case of a non-Newtonian fluid it decreases rapidly. The difference of magnitude of F_2 for these two fluids goes on increasing with increasing values of P and R and for $P = 0$ the behaviour of F_2 remains unchanged for Newtonian and non-Newtonian fluids.

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