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OVERREFLECTION AND OVERTRANSMISSION OF ROSSBY WAVES BY ZONAL SHEAR

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The reflection and transmission of Rossby waves by zonal shear for the cases $U(y) = e^y$ sech y and $\tanh^2 y$ is investigated by examining the eigen function of the disturbance equation. The coefficients of reflection and transmission are evaluated asymptotically. For $U(y) = e^y$ sech y the condition for overreflection is obtained, and for $U(y) = \tanh^2 y$, it is found that overtransmission occurs as well as overreflection, and condition for overtransmission is obtained.

Key Words : Overreflection; Overtransmission; Rossby Waves; Zonal Shear

INTRODUCTION

IN atmospheric studies there has been considerable emphasis on the vertical propagation of Rossby waves, a possibility that was not recognized during Rossby's time and on the interactions between Rossby waves and the zonal shear. Theoretical studies on the overreflection of Rossby waves have been reported by several authors.[1] Linearized equations of a disturbance superimposed upon a barotropic zonal flow allow two kinds of solutions. One of these is called the normal mode which rapidly converges to zero at infinity and the other which does not vanish but oscillates at infinity are related to the scattering problem of Rossby waves and it was shown that the interactions between the Rossby wave and the shear flow in an inherent structure of the eigen functions of the continuous mode of disturbance superimposed upon the shear flow. In far fields these solutions are expressed as a superposition of incoming and outgoing Rossby waves, allowing us to determine the transmission and reflection coefficients of Rossby waves.

For the case $U(y) = \tanh y$, Yamadar and Gotoh[2] calculated the reflection coefficient of Rossby waves and found that strong overreflection occurs in some narrow range of parameters. Yamada and Okamura[3] investigated the reflection and transmission of Rossby waves by zonal shear for the cases $U(y) = \tanh y$ and $\text{sech}^2 y$. They evaluated the coefficient of reflection and transmission numerically and obtained some asymptotic properties analytically. They also found that for the case $U(y) = -\text{sech}^2 y$ overtransmission occurs as well as overrefletion.

In this paper, we investigate the coefficients of reflection and transmission of Rossby waves for the cases $U(y) = e^y$ sech y and $\tanh^2 y$. For $U(y) = \tanh^2 y$, it is found that overtransmission occurs as well as overreflection, and the condition for overreflection and/or overtransmission is obtained.

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Mathematical Formulations

We consider two-dimensional motion of incompressible inviscid fluid in an unbounded domain under the β-plane approximations taking the x-axis of the cartesian co-ordinates eastward and the y-axis northward. Consider a steady parallel zonal flow $[U(y), 0]$ which is uniform at infinity as

$$U(y) \to \begin{cases} U_+ \; (y \to \infty) \\ U_- \; (y \to -\infty) \end{cases} \qquad \ldots(1)$$

Assuming amplitude disturbance in the form of stream functions

$$\Phi = \phi(y)\, e^{i\alpha(x-ct)},$$

where α is the wave number in x-direction and c the complex phase speed, we obtain

$$(U - c)\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi + (\beta - U_{yy})\,\phi = 0, \qquad \ldots(2)$$

where β is the gradient of vorticity assumed positive throughout.

Here we investigate the solutions of the above equation which do not vanish at $y = \pm\infty$ but oscillate there. We call such solutions as Rossby modes and when $U(y) - c$ vanishes they are associated with critical latitudes. In this work we are concerned only with the viscous critical latitudes, and so the matching conditions of the Rossby modes at the critical latitudes are the same as those established by Lin[4] for a non-rotating fluid.

According to their asymptotic behaviour, the Rossby modes are divided into two groups as follows :

Group A :

$$\phi(y) \approx \begin{cases} e^{-i\gamma_+ y} + R\, e^{i\gamma_+ y}, & (y \to \infty) \\ T\, e^{-i\gamma_- y}, & (y \to -\infty) \end{cases} \qquad \ldots(3)$$

Group B :

$$\phi(y) \approx \begin{cases} T\, e^{i\gamma_+ y}, & (y \to \infty) \\ e^{i\gamma_- y} + R\, e^{i\gamma_- y}, & (y \to -\infty) \end{cases} \qquad \ldots(3)$$

where γ_+, γ_-, R and T are constant with respects to y, and γ_+ and γ_- are related to the phase speed c as

$$c = U_+ - \frac{\beta}{\alpha^2 + \gamma_+^2} = U_- - \frac{\beta}{\alpha^2 + \gamma_-^2}. \qquad \ldots(4)$$

We can give the Rossby mode in Group A, the physical interpretations that one part of the incident wave from $y = \infty$ is reflected by the shear with the reflection coefficient R, while the other part dissipates if $I_m(\gamma_-) > 0$ or penetrates the shear with transmission coefficients T if $I_m(\gamma_-) = 0$. A similar interpretation is possible for the Rossby mode in Group B.

Making use of the invariance of $I_m(\phi_y, \phi^*)$ (the wave invariant), except across the critical latitude, we can obtain a relation between the reflection and transmission coefficients of the Rossby modes in Group A as follows :

$$Re(\gamma_+)(1 - |R|^2) = Re(\gamma_-)|T|^2 + \sum_{i=1}^{N} \pi \left[(\beta - U_{yy}) \frac{|\phi|^2}{|U_y|} \right]_{y = y_{c_i}} \quad ...(5)$$

where we have assumed that there are critical latitudes at $y = y_{c_i}$ $(1 \leqslant i \leqslant N)$.[1,5] A similar relation can also be obtained for the Rossby modes in Group B. Here we evaluate the effect of the critical latitudes on the coefficients R and T in the cases $U(y) = e^y \operatorname{sech} y$ and $\tanh^2 y$.

Results and Discussion

(*a*) $U(y) = e^y \operatorname{sech} y$

In this case there is only one critical latitude for Rossby modes with

$$\left(\frac{\beta}{2} - \alpha^2\right)^{1/2} < \gamma_+ \quad ...(6)$$

in Group A. Since we are interested in phenomena arising at the critical latitudes we focus our attention on the Rossby modes in Group A.

It is difficult in general to solve (2) analytically in this case, but the asymptotic form of the reflection coefficient R for large values of γ_+ can be obtained by the method of matched asymptotic expansion as follows.

Transforming the independent variable y to $z = \exp(-2y)/k$ where $k = \exp(-2y_c)$ equation (2) takes the form

$$z^2 \frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} - (\alpha^2/4)\phi - \frac{(1+k)(1+kz)}{8k(1-z)} \times \left[\beta + \frac{8kz(1-kz)}{(1+kz)^3}\right]\phi = 0 \quad ...(7)$$

and the points $y = (-\infty, y_c, \infty)$ to $z = (\infty, 1, 0)$. The asymptotic form of the solution is obtained by making use of the WKB approximation.

Transforming ϕ into g by

$$\phi = \exp\left[k^{-1/2} \int g dz\right] \quad ...(8)$$

expanding g into power series of $k^{1/2}$ as $g = g_0 + k^{1/2} g_1 + kg_2 + \ldots$ the equation (7) becomes

$$\frac{z^2 g^1}{\sqrt{k}} + \frac{z^2 g^2}{k} + \frac{zg}{\sqrt{k}} - \frac{\alpha^2}{4} - \frac{(1+k)(1+kz)}{8k(z-1)}\left[\beta + \frac{8kz(1-kz)}{(1+kz)^3}\right] = 0.$$

Equating the coefficients of the lowest power of k to zero, we have

$$\int g_o\, dz = \pm (\beta/2)^{1/2} \tan^{-1} (z-1)^{1/2}$$

and $$\int g_1 dz = \ln (z-1)^{1/4}.$$

Hence, for large values of γ_+

$$\phi = (z-1)^{1/4} \{A \exp [\gamma_+ \tan^{-1} (z-1)^{1/2}] + B \exp[-\gamma_+ \tan^{-1} (z-1)^{1/2}]\} \quad \ldots(9)$$

The branch of the root should be chosen consistent with the contour of integration around the critical latitude, so that $\tan^{-1} (z-1)^{1/2} \sim (-i/2) \ln z$, for small $|z|$.

Thus we have the result

$$\phi(y) \propto \exp(-i\gamma_+ y) + c \exp(-\pi\gamma_+ + i\gamma_+ y)$$

for $y \geqslant \ln(\gamma_+/\beta^{1/2})$. $|c| = 1$ which gives the asymptotic form of the reflection coefficient as

$$|R| \sim \exp(-\pi\gamma_+),\ \gamma_+ \rightarrow \infty \quad \ldots(10)$$

which is independent of the values of α and β. In the other limit ($\gamma_+ \rightarrow 0$) the perfect reflection is found, that is, $|R| \rightarrow 1,\ \gamma_+ \rightarrow 0$.

Overreflection ($|R| > 1$) occurs in some cases sharply depending on the parameters especially γ_+. The values of γ_+ at which overreflection occurs can be determined analytically. Noticing that only one critical latitude can appear in this case (say at $y = y_c$) and that $Re(\gamma_-)$ vanishes for the Rossby modes satisfying the condition (6), we can easily show from (5) that, overreflection occurs if and only if,

$$\beta - U_{yy}(y_c) < 0 \quad \ldots(11)$$

that is,

$$(\alpha^2 + \gamma_+^2)^3 + 4(\alpha^2 + \gamma_+^2)^2 - 6\beta(\alpha^2 + \gamma_+^2) + 2\beta^2 < 0 \quad \ldots(12)$$

that is,

$$\left[-\alpha^2 + \frac{\beta}{(1-c_1)}\right]^{1/2} < \gamma_+ < \left[-\alpha^2 + \frac{\beta}{(1-c_2)}\right]^{1/2},$$

$$\text{if } -\alpha^2 + \frac{\beta}{(1-c_1)} > 0, \quad \ldots(13)$$

$$0 < \gamma_+ < \left[-\alpha^2 + \frac{\beta}{(1-c_2)}\right]^{1/2}, \text{ if } -\alpha^2 + \beta/(1-c_1) < 0$$

and $$-\alpha^2 + \beta/(1-c_2) > 0, \quad \ldots(14)$$

where

$$C_1 = (\tfrac{4}{3})^{1/2} \cos \tfrac{1}{3} (\theta + 2\pi),\ C_2 = (\tfrac{4}{3})^{1/2} \cos \tfrac{1}{3} (\theta + 4\pi)$$

for $$\theta = \cos^{-1}\left(27^{1/2}\,\frac{\beta}{4}\right),\ 0 < \theta < \frac{\pi}{2},\ C_1 < C_2 < 0.$$

(b) $U(y) = \tanh^2 y$

In this case, the value of γ_+ is equal to that of γ_- and so every Rossby mode expresses the propagation wave on both sides of the shear. Here we consider only the Rossby modes in Group A. Two critical latitudes $y = y_c$ and $-y_c$ appear for the Rossby mode if

$$\beta - \alpha^2 < \gamma_+^2 \qquad \text{...(15)}$$

while only one critical latitude appears if

$$\beta - \alpha^2 = \gamma_+^2 \qquad \text{...(16)}$$

does not occur otherwise.

The asymptotic form of $|R|$ can be obtained analytically as follows :

Transforming the independent variable y of the equation (2) by

$$z = k \sinh^2 y \text{ where } k = (\sinh y_c)^{-2}$$

we get

$$z(k+z)\frac{d^2\phi}{dz^2} + [z + (k/2)]\frac{d\phi}{dz} - (\alpha^2/4)\,\phi - \frac{(1+k)(k+z)}{k(1-z)}\left[\beta - \frac{2k(k-2z)}{(k+z)^2}\right]\phi = 0. \qquad \text{...(17)}$$

Now we find WKB solution :

Transform ϕ as in (8) we get

$$z(k+z)\left[\frac{g'}{\sqrt{k}} + \frac{g^2}{k}\right] + [z + (k/2)](g/\sqrt{k}) - \alpha^2/4 - \frac{(1+k)(k+z)}{4k(1-z)}\left[\beta - \frac{2k(k-2z)}{(k+z)^2}\right] = 0$$

Equating the lowest power of k to zero we have

$$\int g_0\, dz = \pm\, \beta^{1/2} \tan^{-1}\left[\frac{(1-z)^{1/2}}{z^{1/2}}\right] \qquad \text{...(18)}$$

and

$$\int g_1\, dz = (1/4) \ln\left[\frac{(1-z)}{z}\right] \qquad \text{...(19)}$$

Hence, for large values of γ_+ we have

$$\phi = \exp\left\{(1/4)\ln\left(\frac{1-z}{z}\right) \pm \gamma_+ \tan^{-1}\left[\frac{(1-z)^{1/2}}{z^{1/2}}\right] + 0(k)\right\} \qquad \text{...(20)}$$

Taking care of the integration contour around the critical latitude, we obtain the asymptotic forms of the above solutions as

$$\phi \sim \exp\{(\pi/4)\, i \pm \gamma_+ [(i/2)\ln(\beta\gamma_+^{-2}) + iy]\},\; y \to \infty$$

$$\phi \sim y^{-1/2} \exp\{(\pi/4)\, i \pm [((i/2)\,\gamma_+)\ln(\beta\gamma_+^{-2}) + (\pi/2)\,\gamma_+ - \beta^{1/2} y]\} \quad (y \to 0) \qquad \text{...(21)}$$

Since $U(y) = \tanh^2 y$ is an even function of y we can consider two special solutions, an even solution ϕ_e and odd solution ϕ_0; it is consider one critical latitude $y = y_c$.

The solutions are obtained as follows :

$$\left.\begin{aligned} \phi_e &\approx e^{-i\gamma_+ y} + A\, e^{i\gamma_+ y} \\ \phi_0 &\approx e^{-i\gamma_+ y} + B\, e^{i\gamma_+ y} \\ A \approx B &\approx -\, ep\,[-\,\pi\gamma_+ - i\gamma_+ \ln(\beta\gamma_+^{-2})] \end{aligned}\right\} \quad ...(22)$$

From these solutions we get the solution which has the asymptotic forms

$$\phi = (\phi_e + \phi_0)/2 \approx \begin{cases} e^{-i\gamma_+ y} + [(A+B)/2]\, e^{i\gamma_+ y}, & y \to \infty \\ \dfrac{A-B}{2}\, e^{-i\gamma_+ y} & , y \to -\infty \end{cases} \quad ...(23)$$

which give

$$|R| = e^{-\pi\gamma_+}, \quad |T| = 0 \quad ...(24)$$

The left hand expression of (24) agrees with (10) while the right shows that the order of the leading term of $|T|$ is higher than that of $|R|$ and therefore, the higher order calculation would be necessary to properly obtain the asymptotic form of $|T|$.

The overtransmission ($|T| > 1$) in this case is found to occur together with the overreflection in some cases. In the overtransmission the waves absorb energy during their penetration into the shear zone. Since γ_- coincides with γ_+ we may interpret the quantity.

$$E = |R|^2 + |T|^2 \quad ...(25)$$

as an outward energy flux of unit magnitude. The wave taken energy from the shear if $E > 1$, while the shear absorbs the energy of the wave if $E < 1$.

Relation (5) reduced in this case to

$$1 - E = P(\beta - 2 - 6c^2 + 8c)/(2c^{1/2}\,\mathrm{sech}^2\, y_c) \quad ...(26)$$

and

$$c = 1 - (\beta/\alpha^2 + \gamma_+^2),$$

where P is a positive constant and we have taken into account the contributions from two critical latitudes $y = y_c$ (< 0) and $-y_c$. Since the wave takes energy from the shear if $E > 1$, the overtransmission occurs if $E > 1$. Therefore, the condition (26) becomes

$$8c - 6c^2 + \beta - 2 < 0 \quad ...(27)$$

Since P and $2c^{1/2}\,\mathrm{sech}^2\, y_c$ are positive. That is,

$$(\beta - \alpha^2)^{1/2} < \gamma_+ < [-\,\alpha^2 - 2 + (4 + 6\beta)^{1/2}]^{1/2} \quad ...(28)$$

if $(\beta - \alpha^2) > 0$.

$$0 < \gamma_+ < [-\alpha^2 - 2 + (4 + 6\beta)^{1/2}]^{1/2} \qquad \ldots(29)$$

if $(\beta - \alpha^2) < 0$,

where in both cases $\qquad -\alpha^2 - 2 + (4 + 6\beta)^{1/2} > 0$.

Conclusion

Under the conventional β-plane approximation of revolution we have obtained a formal solution of the linearized inviscid equations of disturbances superimposed upon a general unbounded and unstratified plane parallel shear flow $U(y)$ which is uniform at $|y| = \infty$.

For $U(y) = e^y \operatorname{sech} y$ we found overreflection of Rossby waves to take place in some cases and we obtained the conditions for overreflection. In an easterly jet with $U(y) = \tanh^2 y$, it is found that overtransmission occurs as well overreflection. These phenomena are due to energy transfer from the shear flow to the Rossby waves. And the condition for overreflection and/or overtransmission is obtained. Hence, overtransmission occurs in a very narrow range of wave numbers. The amplitude of an incident Rossby wave with a particular wave number increases as the wave penetrates the shear zone. This suggests that a Rossby wave with a particular wave number should be dominant in this part of the flow field.

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