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A HYDRODYNAMICAL PROBLEM OF UNIFORM STREAMING

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Solution of the hydrodynamical problem of uniform streaming of an inviscid incompressible homogeneous fluid past a fixed infinitely long rigid strip whose cross section is an arc of a circle which subtends an angle 2α at the centre is discussed. The solution of this problem is first reduced to that of a governing Fredholm integral equation of the first kind with the Helbert kernel. This integral equation is solved by using some known inversion formulas. Finally, the authors present the expressions for some physical quantities of interest and readily derive the known limiting results for a semi-circular infinite rigid strip, an infinite rigid circular cylinder and an infinite rigid strip.

Key Words : Hilbert Kernel; Infinite Circular Rigid Strip; Kinetic Energy; Fredholm Integral Equation

INTRODUCTION

THE hydrodynamical problems of uniform streaming of an inviscid incompressible homogeneous fluid past two-dimensional and three-dimensional rigid abstracts have been discussed by various authors—Lamb,[1] Stakgold,[2] Schiffer and Szego,[3] Jain and Kanwal,[4] Vaid and Jain[5] and Noble.[6] Recently, Lal and Jain[7] presented the solution of the two-dimensional hydrodynamic problem of a uniform stream of an inviscid incompressible homogeneous fluid flowing past a fixed semi-circular infinite rigid strip. They formulated this boundary value problem by the usual Green's function approach and readily inverted its governing Fredholm's integral equation with the Hilbert kernel $\cot[\frac{1}{2}(\theta - \theta_1)]$, $0 < \theta, \theta_1 < \pi$, by using formula of Lewin.[8] Using the approach of Lal and Jain,[7] the authors present here the solution of the two-dimensional hydrodynamic problem of a uniform stream of an inviscid homogeneous liquid flowing past a fixed infinitely long rigid strip whose cross section is an arc of a circle which subtends an angle 2α at its centre. The governing Fredholm integral equation of this problem possesses the Hilbert kernel $\cot[\frac{1}{2}(\theta - \theta_1)]$, $-\alpha < \theta, \theta_1 < \alpha$ which is solved by using formulas of Shail.[9] Finally, expressions for the physical quantities of interest are derived which readily yield the corresponding known limiting results for a semi-circular infinite rigid strip given by Lal and Jain[7] and are infinite rigid circular cylinder presented by Lamb[1] and an infinite rigid strip.

Mathematical Formulation and the Solution

In a uniform stream of an inviscid homogeneous liquid of velocity $U\hat{n}$, $\hat{n} = \hat{i} \cos\gamma + \hat{j} \sin\gamma$, we place an infinite circular cylindrical rigid strip subtending an angle 2α at its axis. In cylindrical polar co-ordinates (r, θ, z), the strip is defined by $r = a$, $-\alpha < \theta < \alpha$, $-\infty < z < \infty$. The velocity potential function $\phi_i(r, \theta)$ of the undisturbed uniform stream is given by

$$\phi_i(r, \theta) = -Ur\cos(\theta - \gamma) \qquad \ldots(1)$$

and the secondary and total velocity potential functions $\phi_s(r, \theta)$ and satisfy $\phi(r, \theta)$ the Laplace equation and the relation

$$\phi(r, \theta) = -Ur\cos(\theta - \gamma) + \phi_s(r, \theta). \qquad \ldots(2)$$

Thus, it becomes necessary to solve the following two-dimensional boundary value for the velocity potential function $\phi_s(r, \theta)$ of the secondary field :

$$\nabla\phi^2{}_s(r, \theta) = 0, \text{ in } D, \qquad \ldots(3)$$

and
$$\left(\frac{\partial\phi_s}{\partial r}\right)_{r=a} = U\cos(\theta - \gamma), \ -\alpha < \theta < \alpha, \qquad \ldots(4)$$

ϕ_s, $\dfrac{\partial\phi_s}{\partial r}$ are continuous across the arcs $r = a$, $\alpha < \theta < \pi$,

$$-\pi < \theta < -\alpha, \qquad \ldots(5)$$

where D is the region of the x, y plane exterior to the circular arc $C : r = a$, $-\alpha < \theta < \alpha$. Furthermore, $\nabla\phi_s \to 0$ as $r \to \infty$, and the jump in the value of potential function $\phi_s(r, \theta)$ across the arc C vanishes at the edges $\theta = \pm\alpha$.

Following the analysis given by Lal and Jain,[7] the governing Fredholm integral equation of the first kind for this boundary value problem is given by

$$\int_{-\alpha}^{\alpha} I(\theta_1)\,\mathrm{cosec}^2\left(\frac{\theta - \theta_1}{2}\right) d\,\theta_1 = 8\pi aU\cos(\theta - \gamma), \ -\alpha < \theta < \alpha, \quad \ldots(6)$$

where

$$I(\theta) = \phi_s(a + 0, \theta) - \phi_s(a - 0, \theta), \qquad \ldots(7)$$

and $I(\theta)$ satisfies the edge conditions

$$I(-\alpha) = I(\alpha) = 0. \qquad \ldots(8)$$

when the left member of the integral equation (6) is integrated by parts and the edge conditions (8) are used, the integral equation (6) reduces to the simpler form

$$\int_{-\alpha}^{\alpha} I'(\theta_1)\cot[\tfrac{1}{2}(\theta - \theta_1)]\,d\theta_1 = -4\pi aU\cos(\theta - \gamma), \ -\alpha < \theta < \alpha, \quad \ldots(9)$$

which possesses the Hilbert kernel $\cot[\frac{1}{2}(\theta - \theta_1)]$, $-\alpha < \theta, \theta_1 < \alpha$.

It is now clear how the above integral equation (9) can be solved by using the formula given by Shail[9] for solving the integral equation of the form

$$\int_{-\alpha}^{\alpha} g(\theta_1)\{q + \log |2 \sin \tfrac{1}{2}(\theta - \theta_1)|\} \, d\theta_1 = \pi f(\theta), \; -\alpha < \theta < \alpha, \quad \ldots(10)$$

where $f(\theta)$ is a C^1-function on $[-\alpha, \alpha]$ and q is a constant.

Writing $f(\theta) = f_1(\theta) + f_2(\theta)$, $g(\theta) = g_1(\theta) + g_2(\theta)$, where $f_1(-\theta) = f_1(\theta)$ $f_2(-\theta) = -f_2(\theta)$, with similar definitions of $g_1(\theta)$ and $g_2(\theta)$, the integral equation (10) yields the following two integral equations

$$\int_0^{\alpha} g_1(\theta_1)\{2q + \log 2 |\cos\theta - \cos\theta_1|\} \, d\theta_1 = \pi f_1(\theta), \; 0 < \theta < \alpha, \quad \ldots(11)$$

and

$$\int_0^{\alpha} g_2(\theta) \log \left| \frac{\sin \frac{1}{2}(\theta - \theta_1)}{\sin \frac{1}{2}(\theta + \theta_1)} \right| d\theta_1 = \pi f_2(\theta), \; 0 < \theta < \alpha, \quad \ldots(12)$$

for the determination of the unknown functions $g_1(\theta)$ and $g_2(\theta)$. After making a slight correction, the inversion formulas of Shail[9] for the integral equations (11) and (12) are given by

$$g_1(\theta) = \frac{1}{\pi(\cos\theta - \cos\alpha)^{1/2}} \frac{d}{d\theta} \int_0^{\theta} \frac{\sin t}{(\cos t - \cos\theta)^{1/2}}$$

$$\times \left(\int_t^{\alpha} \left(\frac{\cos\Psi - \cos\alpha}{\cos t - \cos\Psi} \right)^{1/2} f_1'(\Psi) \, d\Psi \right) dt$$

$$+ \frac{2\cos\frac{1}{2}\theta}{\pi\,[2q + \log(\sin^2\alpha/2)]} \int_0^{\alpha} \frac{f_1(\theta_1)\cos\frac{1}{2}\theta_1}{(\cos\theta_1 - \cos\alpha)^{1/2}} \, d\theta_1, \quad \ldots(13)$$

and

$$g_2(\theta) = \frac{1}{\pi} \frac{d}{d\theta} \int_0^{\alpha} \frac{\tan\frac{t}{2}\sec^2\frac{t}{2}}{\left(\tan^2\frac{t}{2} - \tan^2\frac{\theta}{2}\right)^{1/2}}$$

$$\left(\int_0^{t} \frac{f_2'(\Psi)}{\left(\tan^2\frac{t}{2} - \tan^2\frac{\Psi}{2}\right)^{1/2}} \, d\Psi \right) dt. \quad \ldots(14)$$

Finally, using the relation

$$g(\theta) = g_1(\theta) + g_2(\theta), \quad \ldots(15)$$

we obtain the required solution of the integral equation (10). To invert the governing integral equation (9), we rewrite this equation in the form

$$\frac{d}{d\theta}\int_{-\alpha}^{a} I'(\theta_1)\log|\sin\tfrac{1}{2}(\theta-\theta_1)|\,d\theta_1 = -2\pi aU\cos(\theta-\gamma),$$

$$-\alpha<\theta<\alpha, \qquad \ldots(16)$$

which on integrating both sides with respect to θ, yields

$$\int_{-\alpha}^{\alpha} I'(\theta_1)\{-\log 2+\log|2\sin\tfrac{1}{2}(\theta-\theta_1)|\}\,d\theta_1 = -2\pi aU\sin(\theta-\gamma)+A,$$

$$-\alpha<\theta<\alpha,\ldots, \qquad \ldots(17)$$

where A is an unknown constant. This integral equation is of the form (10) whose inversion formula is given by the equation (13), (14) and (15). When we write $I'(\theta)=I_1'(\theta)+I_2'(\theta)$, where $I_1'(-\theta)=I_1'(\theta)$, $I_2'(-\theta)=-I_2'(\theta)$, the above equation (17) gets split up into the following two integral equations :

$$\int_0^{\alpha} I_1'(\theta_1)\{-2\log^2+\log^2|\cos\theta-\cos\theta_1|\}\,d\theta_1$$

$$=\pi\left[2\,aU\cos\theta\sin\gamma+\frac{A}{\pi}\right],\ 0<\theta<\alpha; \qquad \ldots(18)$$

and

$$\int_0^{\alpha} I_2'(\theta_1)\log\left|\frac{\sin\frac{1}{2}(\theta-\theta_1)}{\sin\frac{1}{2}(\theta+\theta_1)}\right|d\theta_1=\pi[-2aU\sin\theta\cos\gamma].$$

$$0<\theta<\alpha. \qquad \ldots(19)$$

Using formulae (13) and (14), we obtain the required values of the unknown functions $I_1'(\theta)$ and $I_2'(\theta)$ from equations (18) and (19), These values are

$$I_1'(\theta)=\frac{\sqrt{2}\cos\frac{1}{2}\theta}{(\cos\theta-\cos\alpha)^{1/2}}\left\{\frac{A}{\pi\delta}+2\,aU\sin\gamma\left[-\cos\theta\right.\right.$$

$$\left.\left.+\left(\frac{1}{\delta}+1\right)\cos^2\frac{\alpha}{2}\right]\right\}, \qquad \ldots(20)$$

and

$$I_2'(\theta)=\frac{(2\sqrt{2}\,aU\cos\gamma)}{(\cos\theta-\cos\alpha)^{1/2}}\sin\frac{\theta}{2}\left[\cos\theta+\sin^2\frac{\alpha}{2}\right], \qquad \ldots(21)$$

where

$$\delta=2\log\left(\frac{1}{2}\sin\frac{\alpha}{2}\right). \qquad \ldots(22)$$

Since $I_1'(\theta)$ and $I_2'(\theta)$ are even degree and odd degree functions of θ respectively, we obtain by using the edge conditions (8)

$$\int_0^{\alpha} I_1'(\theta)\,d\theta = \tfrac{1}{2}\int_{-\alpha}^{\alpha} I_1'(\theta)\,d\theta = \tfrac{1}{2}\int_{-\alpha}^{\alpha}[\,I_1'(\theta) + I_2'(\theta)\,]\,d\theta$$

$$= \tfrac{1}{2}\int_{-\alpha}^{\alpha} I'(\theta)\,d\theta = 0. \qquad \ldots(23)$$

When the value of $I_1'(\theta)$ from equation (20) is substituted in the above relation, obtained required value of the unknown constant A is

$$A = -2\pi Ua \sin\gamma \cos^2\frac{\alpha}{2}\,. \qquad \ldots(24)$$

Finally, using the relations (20), (21), (24) and $I'(\theta) = I_1'(\theta) + I_2'(\theta)$, the required solution of the equation (17) is obtained as

$$I'(\theta) = \frac{(2\sqrt{2}\,aU)}{(\cos\theta - \cos\alpha)^{1/2}}\left\{\cos\frac{\theta}{2} - \sin\gamma\left[\cos^2\frac{\alpha}{2} - \cos\theta\right] + \sin\frac{\theta}{2}\cos\gamma\left[\cos\theta + \sin^2\frac{\alpha}{2}\right]\right\} \quad -\alpha < \theta < \alpha. \qquad \ldots(25)$$

The following value of the unknown function $I(\theta)$ is obtained by integrating both sides of the equation (25) with respect to θ and using the edge conditions (8) :

$$I(\theta) = -2\sqrt{2}\,aU\cos\left(\gamma - \frac{\theta}{2}\right)(\cos\theta - \cos\alpha)^{1/2}, \quad -\alpha < \theta < \alpha, \qquad \ldots(26)$$

which agrees with the corresponding limiting result given by Lal and Jain[7] for a semi-circular strip, when $\alpha \to \pi/2$.

Physical Quantities of Interest

Expression for some physical quantities of interest is obtained from relations (25) and (26). Initially, the integral representation formula given by Lal and Jain[7] for the velocity potential $\phi_s(r, \theta)$ is

$$\phi_s(r, \theta) = -\frac{a}{2\pi}\int_{-\alpha}^{\alpha} I(\theta_1)\frac{\partial}{\partial r_1}\log\left[r^2 + r_1^2 - 2rr_1\cos(\theta - \theta_1)\right]^{1/2}\Big]_{r_1 = a}\,d\theta_1 \qquad \ldots(27)$$

where $I(\theta)$ is defined by relation (7) and its exact value is given by the equation (26). Equation (27) yields the values

$$\phi_s(r, \theta) = \begin{cases} -\dfrac{a_0}{2\pi} - \dfrac{1}{2\pi}\displaystyle\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n[a_n\cos n\theta + b_n\sin n\theta], & r < a, \\[2ex] \dfrac{1}{2\pi}\displaystyle\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n[a_n\cos n\theta + b_n\sin n\theta], & r > a, \end{cases} \qquad \ldots(28)$$

where

$$a_0 = \int_{-\alpha}^{\alpha} I(\theta)\, d\theta = -4\pi\, aU \cos\gamma \sin^2 \frac{\alpha}{2} \quad \text{...(29)}$$

$$a_n = \int_{-\alpha}^{\alpha} I(\theta) \cos n\theta d\theta = -\frac{1}{n} \int_{-\alpha}^{\alpha} I'(\theta) \sin n\theta\, d\theta = -\frac{2}{n} \times \int_0^{\alpha} I_2'(\theta) \sin n\theta d\theta$$

$$= \frac{\pi\, aU \cos\gamma}{n} [P_{n+1}(\cos\alpha) - P_{|n-3/2|1/2}(\cos\alpha)] - \cos\alpha [P_n(\cos\alpha) - P_{n-1}(\cos\alpha)],\ n \geqslant 1, \quad \text{...(30)}$$

$$b_n = \int_{-\alpha}^{\alpha} I(\theta) \sin n\theta\, d\theta = \frac{1}{n} \int_{-\alpha}^{\alpha} I'(\theta) \cos n\theta\, d\theta = \frac{2}{n} \times \int_0^{\alpha} I_1'(\theta) \cos n\theta\, d\theta$$

$$= -\frac{\pi aU \sin\gamma}{n} [P_{n+1}(\cos\alpha) + P_{|n-3/2|1/2}(\cos\alpha)] - \cos\alpha [P_n(\cos\alpha) + P_{n-1}(\cos\alpha)],\ n \geqslant 1, \quad \text{...(31)}$$

where the edge conditions (18) the values of $I(\theta)$, $I_1'(\theta)$ and $I_2'(\theta)$ given by the equations (20)–(26) and the well-known formula

$$P_n(\cos\alpha) = \frac{\sqrt{2}}{\pi} \int_0^{\alpha} \frac{\cos(n + \frac{1}{2})\theta}{(\cos\theta - \cos\alpha)^{1/2}}\, d\theta\,. \quad \text{...(32)}$$

have been used. Thus the values of the velocity potential function $\phi_s(r, \theta)$ are completely determined in the regions $r < a$ and $r > a$. Using the formulae $\mathbf{b}_s = -\nabla\phi_s(r, \theta)$ and $\mathbf{q} = U\hat{n} - \nabla\phi_s(r, \theta)$, the expressions for velocity vector of the secondary flow as well as the total flow are obtained.

Finally as explained by Lal and Jain,[7] an expression is devised for the kinetic energy (KE) of the equivalent problem of the flow of inviscid homogeneous liquid at rest at infinite due to the motion of the infinite cylindrical rigid strip moving with constant velocity $-U\hat{n}$ at the instant when the axis of the strip coincides with the Z-axis. The expression for the KE of this equivalent problem is given by

$$\mathrm{KE} = -\tfrac{1}{2}\, a\rho \int_{-\alpha}^{\alpha} I(\theta)\left(\frac{\partial \phi_s}{\partial r}\right)_{r=\alpha} d\theta = -\tfrac{1}{2}\, Ua\rho \int_{-\alpha}^{\alpha} I(\theta)\cos(\theta-\gamma)\, d\theta$$

$$= -\tfrac{1}{2}\, Ua\rho\, [a_1 \cos\gamma + b_1 \sin\gamma]. \qquad \ldots(33)$$

where ρ is the density of the homogeneous liquid and the values of the constant co-efficients a_1, b_1 are given by the relation (30) and (31). Substituting the values

$$a_1 = -\,2\pi a U \cos\gamma \sin^2\frac{\alpha}{2}\left(1+\cos^2\frac{\alpha}{2}\right);$$

$$b_1 = -\,2\pi a U \sin\gamma \sin^4\frac{\alpha}{2}, \qquad \ldots(34)$$

obtained from the relations (30) and (31) in the formula (33), we obtain

$$\mathrm{KE} = \pi\rho\, U^2 a^2 \sin^2\frac{\alpha}{2}\left[1+\cos^2\frac{\alpha}{2}\cos 2\gamma\right]. \qquad \ldots(35)$$

When we let $\alpha \to \frac{\pi}{2}$, we obtain the corresponding known result for an infinite semi-circular cylindrical rigid strip of radius a given by Lal and Jain[7]

$$\mathrm{KE} = \frac{1}{2}\pi\rho\, U^2 a^2\left[1+\frac{1}{2}\cos 2\gamma\right]. \qquad \ldots(36)$$

Similarly, when $\alpha \to \pi$ in the formulae (28), (29) (30), (31) and (35), we obtain the following corresponding known expressions for an infinite circular rigid cylinder of radius a given by Lamb[1] :

$$\phi_s(r,\theta) = -\,\frac{Ua^2}{r}\,[\cos(\theta-\gamma)\,],\ r > a, \qquad \ldots(37)$$

$$\mathrm{KE} = \pi\rho\, U^2 a^2. \qquad \ldots(38)$$

Finally, when in the formula (35), $\alpha \to 0$, $a \to \infty$, such that $a\alpha \to a_1$ we get the following corresponding known expression for the infinite rigid strip

$$x = 0,\ |\,y\,| < a_1,\ -\infty < z < \infty :$$

$$\mathrm{KE} = \frac{\pi\rho\, U^2 a_1^2 \cos^2\gamma}{2}. \qquad \ldots(39)$$

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