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CALCULATION OF SKEWNESS FACTOR WITH THE HELP OF FOURIER INVERSION OF HEISENBERG'S EQUATION RELATED TO ISOTROPIC TURBULENCE

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In homogeneous and isotropic turbulence, the velocity-correlation theory leads to derivation of Karman-Howarth[1] equation, while a heuristic approach as suggested by W Heisenberg[2,3] gives the corresponding equation in the energy space.

Starting from Heisenberg's equation so envisaged and taking Fourier inverse of the differential form of that equation, one can readily obtain an equation in the velocity-correlation space which can be compared termwise to the aforesaid Karman-Howarth equation.[1] This evidently helps us to calculate the skewness factor S as

$$S = \frac{B_{ddd}(r)}{[B_{dd}(r)]^{2/3}}, \qquad \ldots(1)$$

where in case of derivation of local isotropy[4–7] from ordinary isotropy, we need introduce

$$\left.\begin{aligned} &\overline{(u'_d - u_d)^2} = B_{dd}(r) \;\; = 2\overline{u^2}(t)\,[1 - f(r,t)] \\ \text{and}\quad &\overline{(u'_d - u_d)^3} = B_{ddd}(r) = 6\,\{\overline{u^2}(t)\}^{3/2}\, k(r,t) \end{aligned}\right\} \qquad \ldots(2)$$

$f(r,t)$ and $k(r,t)$ representing standard second- and third-order velocity-correlations explicity written as

$$\left.\begin{aligned} &\overline{u_1 u_1'} = \overline{u^2(t)}\, f(r,t) \\ \text{and}\quad &\overline{u_1 u_1 u_1'} = \overline{\{u^2(t)\}^{3/2}}\, k(r,t). \end{aligned}\right\} \qquad \ldots(3)$$

In the aforesaid context, the calculated value of S as obtained from two numerical tables provided by K M Ghosh[8] turns out to be $= 0.25$ in either of the cases. This result compares favourably to the value of S obtained in wind-tunnel experiments made by R W Stewart,[9] namely $S = 0.30$.

Key Words : Homogeneous and Isotropic Turbulence; Velocity-correlations; Skewness Factor; Ordinary Isotropy and Local Isostropy; Kolmogoroff's Quasi-equilibrium Hypotheses; Turbulent Energy

Introduction

It is well known that in the case of homogeneous and isotropic turbulence, the equation of translation for second-order velocity-correlation can be written in the form[1]

$$\frac{\partial}{\partial t}\overline{u^2(t)}f(r,t)-\left(\frac{\partial}{\partial r}+\frac{4}{r}\right)\{\overline{u^2(t)}\}^{3/2}k(r,t)$$
$$=2\nu\left(\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}\right)\overline{u^2(t)}\,f(r,t), \quad \ldots(4)$$

When we make use of rule of conversion given in (2), we can convert the equation (4) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{u^2(t)}-\frac{1}{6}\left[\frac{\mathrm{d}}{\mathrm{d}r}B_{ddd}(r)+\frac{4}{r}B_{ddd}(r)\right]$$
$$=-2\nu\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{4}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right]\frac{1}{2}B_{dd}(r) \quad \ldots(5)$$

Since the time-rate of decay of turbulent energy per unit volume per unit density in case of homogeneous and isotropic turbulence is given by

$$\epsilon=-\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{3}{2}\overline{u^2(t)}\right], \quad \ldots(6)$$

so multiplying the equation (5) by $6r^4$ throughout, we recast the equation (5) in the form[10]

$$\frac{\mathrm{d}}{\mathrm{d}r}[r^4B_{ddd}(r)]-6\nu\left[\frac{\mathrm{d}}{\mathrm{d}r}r^4B_{dd}(r)\right]$$
$$=-4\,\epsilon\,r^4 \quad \ldots(7)$$

Calculation of S in Terms of $B_{dd}(r)$ with the Help of Kolmogoroff's Quasi-Equilibrium Hypotheses

According to the modified form of the quasi-equilibrium hypotheses due to Kolmogoroff[4–7] and Oboukhov,[11] $\bar{\epsilon}_r$, the time-rate of decay of turbulent energy per unit volume per unit density can be obtained as its average over the volume of a sphere of suitably small radius r as

$$\bar{\epsilon}_r=\frac{3}{4\pi r^3}\iiint_{|\mathbf{h}|<r}\epsilon(\mathbf{x}+\mathbf{h})\,\mathrm{d}\mathbf{h} \quad \ldots(8)$$

at a location defined by the position vector $\mathbf{x}$ of a vast span of locally isotropic turbulent fluid medium. When $\bar{\epsilon}_r$ is calculated in respect of a sphere of sufficiently small radius r, then following Oboukhov,[11] we can readily obtain

$$\bar{\epsilon}_r=\bar{\epsilon} \quad \ldots(9)$$

i.e. $\bar{\epsilon}_r$ is independent of the radius of the sphere. Hence, in this context, replacing ϵ by $\bar{\epsilon}$, we rewrite the relation (7) as

$$r^4 B_{ddd}(r) - 6\nu \left[r^4 \frac{d}{dr} B_{dd}(r) \right] = -4 \int_0^r \epsilon\, r^4 dr$$

$$= -4\bar{\epsilon}\,\frac{r^5}{5} \qquad \ldots(10)$$

Now, we can make use of the standard definition of the skewness factor as given by

$$S = \frac{B_{ddd}(r)}{[B_{dd}(r)]^{3/2}},$$

[*cf.* relation (1)]

where in the context of derivation of local isotropy from ordinary, we need to use the rule of conversion as

$$B_{dd}(r) = \overline{(u'_d - u_d)^2} = \overline{2u^2(t)}\,[1 - f(r, t)]$$

and $$B_{ddd}(r) = \overline{(u'_d - u_d)^3} = 6\,\{\overline{u^2(t)}\}^{3/2}\, k(r, t)$$

[*cf.* relations in (2)].

Thus, with the help of the relation (1) and first one of the relation in (2), we can revise the equation (10) in the form,

$$r^4 S\, B^{3/2}_{dd}(r) - 6\nu \left[r^4 \frac{d}{dr} B_{dd}(r) \right] = -4\,\bar{\epsilon}\,\frac{r^5}{5}. \qquad \ldots(11)$$

Use of Self-Similar Solutions of Heisenberg's Equation[14] with its Fourier Inversion to Calculate S from Equation (11)

For the purpose of evaluation of S from the relation (11), we shall now make use of the self-similar solutions of Heisenberg's equation for decay of homogeneous and isotropic turbulent energy spectrum function $\mathcal{H}(k, t)$ as

$$-\frac{\partial}{\partial t} \int_0^k \mathcal{H}(k, t)\, dk = 2(\nu + \zeta \int_k^\infty \sqrt{\frac{\mathcal{H}(k',t)}{k'^3}}\, dk' \int_0^k k^2 \mathcal{H}(k, t)\, dk \quad \ldots(12)$$

which has been shown by Karman and Lin[15] as to be Fourier transformed version of Karman-Howarth equation.[1-3]

From the works of Sen,[14] it is known that the most general form of self-similar solutions of Heisenberg's equation (12) is given by

$$\mathcal{H}(k, t) = \frac{1}{\zeta^2 k_0^3 t_0^2} \left(\frac{t}{t_0}\right)^{3c-2} \varphi\left[\frac{k}{k_0}\left(\frac{t}{t_0}\right)^c\right], \qquad \ldots(13)$$

where k_0, t_0 are suitable constants, $\varphi\left[\frac{k}{k_0}\left(\frac{t}{t_0}\right)^c\right]$ is a function of $\frac{k}{k_0}\left(\frac{t}{t_0}\right)^c$ with c defined as a parametric constant so introduced by Sen[14] which lies in between the limits $0 < c < \frac{2}{3}$. Now considering Fourier inverse of $\mathcal{H}(k, t)$ so suggested in (13), we can easily obtain

$$\overline{u^2(t)}\, f(0,t) = [\int_0^\infty \mathcal{H}(k,t)\, e^{-ikr}\, \mathrm{d}k]_{r=0}.$$

$$= \frac{1}{\zeta^2 k_0^3 t_0^2} \left(\frac{t}{t_0}\right)^{3c-2} \int_0^\infty \varphi\left[\frac{k}{k_0}\left(\frac{t}{t_0}\right)^c \mathrm{d}k \quad \ldots(14)$$

Using the expression so obtained in (14), we can calculate from the first one of relations in (2),

$$B_{dd}(r) = 2\,\overline{u^2}\,[1 - f(r,t)] = [\psi(0,t) - \psi(r,t)], \text{ say} \quad \ldots(15)$$

where with the substitution $\frac{k}{k_0}\left(\frac{t}{t_0}\right)^c = y$ one obtains

$$\psi(r,t) = \left[\frac{1}{\zeta^2 k_0^3 t_0^2}\left(\frac{t}{t_0}\right)^{2(c-1)} \int_0^\infty \varphi(y) \exp\left[-ik_0\left(\frac{t}{t_0}\right)^{-c} yr\right] \mathrm{d}y\right] \ldots(16)$$

Now, we are in a position to calculate ϵ defined through the relation (6) by taking time derivative of the expression in (16) for the limiting case $r \to 0$ and multiplying it by $-\frac{3}{2}$. Thereafter, we use this result along with the relations (15) and (16) in our final equation (11) to calculate the skewness factor S of the velocity-correlation functions relating to homogeneous and isotropic turbulence[12] which lead to locally isotropic behaviour of the turbulent flow field.[13]

In the context of the aforesaid analysis, for the parametric value $c = \frac{1}{2}$, we may make use of the numerical tables of $[y, \varphi(y)]$* as obtained by Chandrasekhar,[16] and similarly for $c = \frac{2}{7}$, we need use the numerical tables prepared by Ghosh.[8]

In the aforesaid derivation, for different value of the Reynolds number of turbulent flows as mentioned at the top of respective numerical table of $(y, \varphi(y))$ prepared by Chandrasekhar,[16] we can readily obtain the numerical value of S from the equation (11) when the relations (6), (14), (15) and (16) are numerically calculated for the purpose of substitution in (11) in case of each such table of values. Similarly, for the table of values for $(y, \varphi(y))$ as prepared by Ghosh,[8] we can obtain S from the equation (11) when the relations (6), (14), (15) and (16) are numerically calculated for each of his two tables and substituted in the equation (11). It is to be noted that the numerical tables of Chandrasekhar[16] represent the decay phenomena of homogeneous and isotropic turbulence when both energy cascading viz., transfer of energy from one wave-band to all other wave-bands to its right and viscous dissipation are equally important, while the numerical tables provided by Ghosh[8] represent the process of decay of homogeneous and isotropic turbulence when the energy cascading is the dominating phenomena and viscous

*In the current paper what have been mentioned as tables $(y, \varphi(y)$, need be read in the works of Chandrasekhar[16] or those of Ghosh[8] as $(x, f(x))$ tables.

dissipation is negligible. This is known to be true for the early-period decay process when the Reynolds number of turbulent flows is infinitely large.

Concluding Remarks

The numerical tables of $(y, \varphi(y))$ prepared by Chandrasekhar[16] will lead to the evaluation of S when both energy transfer and viscous damping are equally important features during the process of decay of homogeneous and isotropic turbulence.

The numerical tables of $(y, \varphi(y))$ provided by Ghosh[8] will help us calculate S when the energy transfer is the dominating feature as compared to the viscous damping during the process of decay of homogeneous and isotropic turbulence. This is known to be a situation that develops during the early-period of the said decay process when Reynolds number of turbulent flows so considered is infinitely large.

The evaluation of S from the tables provided by Ghosh[8] is expected to be nearer to the value of S which is reasonably true in the context of Kolmogoroff's Quasi-equilibrium Hypotheses as compared to the value of S calculated from any tables prepared by Chandrasekhar,[16] since in Kolmogoroff's hypotheses, an *a priori* assumption is always made to the effect that $R \gg 1$.

Incidentally, we make use of two numerical tables of Ghosh[8] which lead to evaluation of S after suitable numerical quadrature and evident substitution of relevant terms in (1) leading to evaluation of S as,

$$\left.\begin{aligned} S &= 0.25, \; -0.251 \\ \text{and} \quad S &= 0.251, \; -0.251 \end{aligned}\right\} \qquad \ldots(17)$$

for the aforesaid two tables when $R \to \infty$.

In fact, the results in eq (12) favourably compare to wind-tunnel measurements made by R W Stewart[9] as

$$S \approx -0.30 \qquad \ldots(18)$$

for $R \to \infty$.

[*cf.* p. 172, *The theory of homogeneous Turbulence* of G K Batchelor, Cambridge University Press (1953)].

References

1 T V̈ Karman and L Howarth *Proc R Soc (London)* **164** (1938) p 192
2 W Heisenberg *ZPhys* **124** (1948*a*) p 628
3 W Heisenberg *Proc R Soc (London)* A **195** (1948*b*) p 402
4 A N Kolmogoroff *Dokl Akad Nauk* SSSR **30** (1941*a*) No 4, p 299
5 ——— *Dokl Akad Nauk SSSR* **31** (1941*b*) No 6, p 538
6 ——— *Dokl Akad Nauk SSSR* **32** (1941*c*) No. 1, p 19
7 ——— *J fluid Mech* **13** (1962) p 82
8 K M Ghosh *Bull Calcutta math Soc* **47** (1955) No 2, p 71

9 R W Stewart *Proc Camb phil Soc* **47** (1951) p 146
10 S Panchev *Random Functions and Turbulence* First Edition Pergamon Press Oxford, New York (1971) p 168
11 A M Oboukhov *J fluid Mech* **13** (1962) p 77
12 G I Taylor *Proc R Soc (London)* **151A** (1935) 421–478.
13 G K Batchelor *Proc Camb phil Soc* **43** (1947) No 4, p 533
14 N R Sen *Bull Calcutta math Soc* **43** (1951) p 1
15 T V̈ Karman and C C Lin *Adv appl Mech* **2** (1951) p 1
16 S Chandrasekhar *Proc R Soc* (*London*) A **200** (1949) p 20