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WAVES GENERATED BY DISTURBANCES AT AN INERTIAL SURFACE IN AN OCEAN OF FINITE DEPTH

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This paper is concerned with the two-dimensional unsteady motion in an ocean of finite depth bounded by an inertial surface, generated by initial surface disturbances. The depression of the inertial surface is obtained, and for the particular case of an initial surface disturbance concentrated at a point, its asymptotic form is deduced by the method of stationary phase for large time and distance from this point.

Key Words : Unsteady Motion; Ocean of Finite Depth; Surface Disturbance; Inertial Surface; Velocity Potential; Method of Stationary Phase

INTRODUCTION

WAVES are generated when an explosion occurs above or within an ocean. Within the framework of linearised theory of water waves, problems associated with the generation of these waves can be formulated as an initial value problem. The initial condition on the surface of the ocean is taken as prescribed initial displacement or impulsive pressure distributed over a certain region of the ocean surface according as the explosion occurs within or above the ocean. For the two-dimensional unsteady motion due to an initial surface disturbance in the form of displacement or impulsive pressure concentrated at a point (chosen as the origin), the velocity potential and the free surface elevation were given in the treaties of Lamb[1] and Stoker.[2] For the three-dimensional unsteady motion, Kranzer and Keller[3] considered the case of an axially symmetric initial surface disturbance and compared the theory with experiments. Chaudhuri[4] and Wen[5] considered the case when the initial surface disturbance consists of both surface impulse and surface elevation distributed over any arbitrary region of the surface of an ocean of uniform finite depth.

In all these problems, the ocean was assumed to be bounded above by a free surface. However, in recent years there has been considerable interest on problems involving generation of waves due to various types of sources of arbitrary time-dependent strength submerged in an ocean covered by an inertial surface composed of a thin layer of uniformly distributed noninteracting particles (e. g., broken ice, floating mat).[6–9] This motivates us to extend the class of problems

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concerning unsteady motion due to prescribed initial surface disturbances in an ocean with a free surface mentioned above to the case when the ocean is covered by an inertial surface. Very recently, Mandal[10] considered a two-dimensional problem of this type in a deep ocean covered by an inertial surface wherein the waves are generated due to prescribed initial surface disturbances. In the present paper, the authors have extended the problem considered[10] for a deep ocean to an ocean of uniform finite depth. The corresponding initial value problem is reduced to a boundary value problem by using Laplace transform in time. This boundary value problem is then solved by using Green's integral theorem in the ocean region to the transformed potential function and a source function obtained earlier.[7] By Laplace inversion, the potential function describing the motion and hence, the depression of the inertial surface at any time is obtained in terms of an infinite integral. This integral is evaluated asymptotically for large time and distance by the method of stationary phase when the initial disturbance at the inertial surface is concentrated at a point taken as the origin. Known results are recovered by making the ocean depth tending to infinity.

Statement and Formulation of the Problem

The authors consider the unsteady two-dimensional potential flow of an inviscid, incompressible, homogeneous liquid of uniform finite depth under the action of gravity only. The liquid is covered by an inertial surface composed of a thin but uniform distribution of disconnected floating particles of area density $\epsilon\rho(\epsilon \geqslant 0)$ where ρ is the volume density of the liquid. $\epsilon = 0$ corresponds to a liquid covered by a free surface. The motion in the liquid is generated by a given displacement of the inertial surface at an instant when the liquid is at rest. We choose a rectangular cartesian co-ordinate system where the y-axis is taken vertically downwards, $y = 0$ corresponds to the undisturbed position of the inertial surface and $y = h$ is the bottom of the liquid. Since the motion is started from rest, it is irrotational and can be described by a velocity potential $\varphi(x, y, t)$ for $t \geqslant 0$. Within the framework of linearised theory of water waves, it can be shown that φ satisfies the following initial value problem

$$\nabla^2\varphi = 0,\ 0 \leqslant y \leqslant h,\ t \geqslant 0, \qquad \ldots(1)$$

$$\frac{\partial^2}{\partial t^2}(\varphi - \epsilon\varphi_y) - g\,\varphi_y = 0,\ y = 0,\ t > 0, \qquad \ldots(2)$$

$$\varphi_y = 0,\ y = h,\ t \geqslant 0, \qquad \ldots(3)$$

$$\varphi - \epsilon\varphi_y = 0,\ y = 0,\ t = 0, \qquad \ldots(4)$$

$$\frac{\partial}{\partial t}(\varphi - \epsilon\varphi_y) = g\,f(x),\ y = 0 \text{ and } t = 0, \qquad \ldots(5)$$

where $f(x)$ is the initial depression of the inertial surface and g is the gravity.

Let $\Phi(x, y, p)$ denote the Laplace transform of $\varphi(x, y, t)$ in time defined as

$$\Phi(x, y\, p) = \int_0^\infty \varphi(x, y, t) \exp(-pt)\, dt,\ p > 0.$$

By the use of this Laplace transform in time, the initial value problem described by (1) to (5) reduces to the following boundary-value problem :—

$$\nabla^2\Phi = 0,\ 0 \leqslant y \leqslant h, \qquad \ldots(6)$$

$$p^2 - (g + \epsilon p^2)\, \Phi_y = g f(x) \text{ on } y = 0, \qquad \ldots(7)$$

$$\Phi_y = 0 \text{ on } y = h. \qquad \ldots(8)$$

This BVP is solved by a suitable use of Green's integral theorem in the next section.

Solution of the BVP

The Green's function $G\,(x, y, \xi, \eta, p)$ satisfying

$$\nabla^2 G = 0,\ 0 \leqslant y \leqslant h \text{ except at } (\xi, \eta),$$

$$p^2 G - (g + \epsilon p^2)\, G_y = 0 \text{ on } y = 0,$$

$$G \sim \ln r \text{ as } r = \{(x - y)^2 + (\xi - \eta)^2\}^{1/2} \to 0,$$

$$G_y = 0 \text{ on } y = h,$$

has the form given by[7]

$$G = -\ln \frac{r}{r'} + 2 \int_0^\infty \left\{ \frac{\epsilon \cosh k(h-y) \cosh k(h-\eta)}{D(k)} + \frac{\exp(-kh) \sinh ky \sinh k\eta}{k} \right\} \frac{\cos k(x-\xi)}{\cosh kh}\, dk$$

$$+ 2 \int_0^\infty \frac{\cosh k(h-y) \cosh k(h-\eta)}{k\, D(k)} \frac{\cos k(x-\xi)}{\cosh kh} \frac{\mu^2}{\mu^2 + p^2}\, dk, \qquad \ldots(9)$$

$$\left.\begin{aligned} \text{where}\quad r' &= \{(x-\xi)^2 + (y+\eta)^2\}^{1/2}, \\ D(k) &= \cosh kh + \epsilon k \sinh kh \\ \text{and}\quad \mu^2 &= gk \sinh kh / D(k). \end{aligned}\right\} \qquad \ldots(10)$$

Applying Green's integral theorem to Φ and G in the liquid region, we obtain

$$\Phi(\xi, \eta, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (G - \epsilon G_y)\,(x, 0, \xi, \eta, p) f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^\infty \frac{\mu^2}{\mu^2 + p^2} \frac{\cosh k(h-\eta)}{k \sinh kh} \int_{-\infty}^{\infty} \cos k(x-\xi) f(x)\, dx\, dk.$$

Laplace inversion gives

$$\varphi(\xi, \eta, t) = \frac{1}{\pi} \int_0^\infty \mu \sin \mu t \, \frac{\cosh k(h-\eta)}{k \sinh kh} \int_{-\infty}^{\infty} \cos k(x-\xi) \, f(x) \, \mathrm{d}x \, \mathrm{d}k. \quad \ldots(11)$$

The depression $\zeta(\xi, t)$ of the inertial surface at time t is given by

$$\zeta(\xi, t) = \frac{1}{g} \frac{\partial}{\partial t} (\varphi - \epsilon \varphi_\eta)(\xi, 0, t)$$

$$= \frac{1}{\pi} \int_0^\infty \cos \mu t \int_{-\infty}^{\infty} \cos k(k-\xi) \, f(x) \, \mathrm{d}x \, \mathrm{d}k. \quad \ldots(12)$$

If the initial displacement of the inertial surface is concentrated at the origin, then

$$\zeta(\xi, t) = \frac{\mathrm{l}}{\pi} \int_0^\infty \cos \mu t \cos k\xi \, \mathrm{d}k. \quad \ldots(13)$$

Making $h \to \infty$, these results reduce to the corresponding results given.[10]

Asymptotic Expansions

The method of stationary phase is now applied to produce an approximate value of ζ for the single integral in equation (13), when ξ and t are large but ξ/t is finite. We write (13) as

$$\zeta(\xi, t) = \frac{1}{2\pi} \operatorname{Re} \int_0^\infty \left[\exp\{it\, \Psi(-k)\} + \exp\{it\, \Psi(k)\} \right] \mathrm{d}k, \quad \ldots(14)$$

where $$\Psi(k) = \mu - k\frac{\xi}{t}. \quad \ldots(15)$$

The first integral in (14) has no stationary point in the range of integration while the second integral has stationary points where $\Psi'(k) = 0$.

Now

$$\Psi'(k) = \frac{(gh)^{1/2}}{2} \left[\frac{1}{\{D(k)\}^{3/2}} \left\{ \left(\frac{kh}{\sinh kh}\right)^{1/2} + \left(\frac{kh}{\sinh kh}\right)^{-1/2} \cosh kh \right\} \right] - \frac{\xi}{t}. \quad \ldots(16)$$

The term in the square bracket of (16) decreases monotonically from 2 to 0 as k increases from 0 to ∞. Also $\Psi'(0) = (gh)^{1/2} - \frac{\xi}{t}$ and $\Psi'(\infty)$ is negative. Hence, for $\xi/t > (gh)^{1/2}$, both $\Psi'(0)$ and $\Psi'(\infty)$ are negative. Since $\Psi'(k) < 0$ and is a monotonic decreasing function for $k > 0$, there is no zero of $\Psi'(k)$, so that there is no stationary point. However, for $\xi/t < (gh)^{1/2}$, $\Psi'(0)$ is positive while $\Psi'(\infty)$ is negative and since $\Psi'(k)$ is a monotonic decreasing function for $k > 0, \Psi'(k)$ has a unique zero in the range of integration, so that there exists only one stationary point at $k = k_0$ say. Finally, when $\xi/t = (gh)^{1/2}$, $\Psi'(0) = 0$ so that there is a stationary point at $k = 0$. But as $\Psi''(0)$ is infinite it gives a smaller contribution than $\xi/t < (gh)^{1/2}$ case, so that its contribution can be discarded. Now applying the method of stationary phase to the second integral in (14), we obtain

$$\zeta(\xi, t) \sim \frac{1}{2\pi}\left(\frac{2\pi}{t\,|\,\Psi''(k_0)\,|}\right)^{1/2} \cos\left(t\Psi(k_0) - \frac{\pi}{4}\right). \qquad \ldots(17)$$

It is rather involved to solve for k_0 in the equation $\Psi'(k) = 0$. Instead, we consider the following two special cases.

Case I : *Shallow Ocean* (*h small*)

For small values of h, $\mu \sim \left[gk^2h(1 - \frac{1}{3}k^2h^2 - \epsilon k^2h)\right]^{1/2}$.

We then obtain

$$k_0 = \left[\frac{3\left\{gh - \left(\frac{\xi}{t}\right)^2\right\}}{h^2\ 4gh + 9\epsilon - \left(\frac{\xi}{t}\right)^2\Big\}}\right]^{1/2}. \qquad \ldots(18)$$

In the absence of the inertial surface, i.e., for $\epsilon = 0$, eq. (18) reduces to the result given by Wen[5] after taking into consideration a printing mistake there. Thus for this case

$$\Psi''(k) = (gh)^{1/2}\frac{1}{\{k_0^2(1 - \frac{1}{3}k_0^2h^2)\}^{3/2}}\Big[k_0^2\left(1 - \frac{1}{3}k_0^2h^2\right)\left\{1 - \frac{1}{2}k_0^2h(g\epsilon + 4h)\right\}$$
$$- k_0^2\left(1 - \frac{2}{3}k_0h\right)^2\left\{1 - \frac{1}{6}k_0^2h(9\epsilon + 4h)\right\}\Big], \qquad \ldots(19)$$

where k_0 is given by (18). Using (18), (19) in (17), the asymptotic form of $\zeta(\xi, t)$ is easily obtained.

Case II : *Deep Ocean* (*h Large*)

For large values of h, $\mu \sim \left(\frac{gk}{1 + \epsilon k}\right)^{1/2}$ and (17) gives the result given.[10]

If the motion is set up by the action of an initial surface impulse $I(x)$ per unit area applied to the inertial surface, then only the conditions (4) and (5) will be changed to

$$\varphi - \epsilon\varphi_y = -\frac{I(x)}{\rho} \text{ at } t = 0,\ y = 0,$$

$$\frac{\partial}{\partial t}(\varphi - \epsilon\varphi_y) = 0 \text{ at } t = 0 \text{ on } y = 0.$$

By the same technique, we obtain

$$\varphi(\xi, \eta, t) = -\frac{1}{\pi\rho}\int_0^\infty \frac{\cos \mu t \cosh k(h-\eta)}{D(k)} \int_{-\infty}^{\infty} \cos k(x-\xi)\, I(x)\, dx\, dk.$$

If the impulse is concentrated at the origin, then we get

$$\varphi(\xi, \eta, t) = -\frac{1}{\pi\rho}\int_0^\infty \frac{\cosh k(h-\eta)}{D(k)} \cos \mu t \cos k\xi\, dk. \qquad \ldots(20)$$

In this case the form of the inertial surface is given by .

$$\zeta(\xi, t) = \frac{1}{\pi\rho g^{1/2}}\int_0^\infty \left(\frac{k}{\coth k_0 h + \epsilon k}\right)^{1/2} \sin \mu t \cos k\xi\, dk. \qquad \ldots(21)$$

By the use of the method of stationary phase, the asymptotic form of ζ becomes

$$\zeta(\xi, t) \sim \frac{1}{2\pi\rho g^{1/2}}\left(\frac{k_0}{\coth\ k_0 + \epsilon k_0}\right)^{1/2}\left(\frac{2\pi}{t|\Psi''(k_0)|}\right)^{1/2} \sin\left(t\,\Psi(k_0) - \frac{\pi}{4}\right),$$

where $\Psi(k)$ is given in (15).

For shallow as well as deep oceans approximations to $\zeta(\xi, t)$ can be made as before.

References

1 H Lamb *Hydrodynamics* Dover New York (1945)
2 J J Stoker *Water Waves* Interscience New York (1957)
3 H C Kranzer and J B Keller *J appl Phys* **30** (1959) 398–407
4 K Chaudhuri *Appl Sci Res* **19** (1968) 274–284
5 S L Wen *Int J Math Edu Sci Tech* **13** (1982) 55–58
6 P F Rhodes-Robinson *J Aust math Soc Ser B* **25** (1984) 366–383
7 B N Mandal and K Kundu *J Aust Math Soc Ser B* **28** (1986) 271–278
8 B N Mandal *Mech Res Commun* **13** (1986) 335–339
9 B N Mandal and K Kundu *Int J Engng Sci* **25** (1987) 1383–1386
10 B N Mandal *Appl Sci Res* **45** (1988) 67–73