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WAVES PRODUCED BY DISTURBANCES AT THE INTERFACE BETWEEN TWO SUPERPOSED FLUIDS

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The generation of waves at the interface between two superposed fluids under the action of gravity due to initial disturbances at the interface is considered in this paper assuming linear theory. Laplace transform technique and Green's integral theorem are employed in the mathematical analysis. For the case of an initial interface depression or impulse concentrated at the origin, the method of stationary phase provides the interface depression asymptotically. The asymptotic forms of the interface are presented graphically. It is observed that the presence of the upper fluid, however small be its density has a significant effect on the wave motion.

Key Words: Superposed Fluids; Interface; Initial Disturbances; Velocity Potentials; Laplace Transform; Green's Theorem; Method of Stationary Phase

Introduction

The problem of two-dimensional unsteady motion in deep water produced by disturbances in the form of initial surface elevation or impulse concentrated at a point on the free surface is discussed in the treatise of Lamb[1] and Stoker[2] assuming linear theory. Fourier transform technique was used in the mathematical analysis and the free surface elevation is given in the form of infinite integrals which were then evaluated asymptotically for large $gt^2/4x$ $(x \neq 0)$ by the method of stationary phase. Kranzer and Keller[3] considered axially symmetrical initial surface disturbance in water of finite depth and compared the theory with experimental results. They also gave a brief account of the various works related to this problem. Wen[4] considered initial disturbance over an arbitrary region of the free surface and obtained the free surface depression approximately by using the method of stationary phase for double integral. There has been a considerable interest in the study of the problems of generation of surface water waves in water covered by an inertial surface which is composed of a thin but uniform distribution of noninteracting floating particles. Mandal[5] considered the two-dimensional unsteady motion in deep ocean covered by an inertial surface due to initial disturbances at the inertial surface. The corresponding problem for an ocean of uniform finite depth was treated by Mandal and Ghosh[6]. Again, Mandal and Mukherjee[7] studied three dimensional unsteady motion in deep ocean covered by an inertial surface due to a prescribed axi-symmetric initial disturbance at the inertial surface while the corresponding problem for an ocean of uniform finite depth was considered by Mandal and Ghosh[8]. Also, Mandal and Ghosh[9] considered the problem of generation of water waves in an ocean of finite depth covered by an inertial surface due to an arbitrary periodic pressure distribution on the inertial surface as well as an initial displacement of the inertial surface.

Extension of this initial value problem to two superposed fluids is briefly mentioned in the survey article of Wehausen and Laitone[10] although no details were given. Here we consider interface waves in two superposed fluids due to initial disturbances at the interface where the upper fluid extends infinitely upwards and the lower fluid extends infinitely downwards. This serves as a model for air-water and our purpose here is to study the effect of air (a fluid of very low density) on the wave motion produced by initial disturbances on an ocean surface. To solve the corresponding initial value problem for two superposed fluids, we use a new potential function inthe lower region which is a linear combination of the velocity potential for the lower fluid and another potential defined in the lower region by reflection of the velocity potential for the upper fluid about the common interface. This new potential function satisfies an initial value problem whose solution is obtained by using Laplace transform in time and Green's integral theorem. Fortunately, the interface depression can be expressed in terms of this new potential function and is obtained in terms of a double integral involving the function describing the initial disturbance of the interface. For the special case of an initial interface depression or impulse concentrated at the origin, the interface depression reduces to a single integral involving oscillatory functions. Using the principle of stationary phase, asymptotic form of the interface depression is obtained for large $gt^2/4x$ where g is the gravity, t is time and $x(\neq 0)$ is the distance from the concentrated initial disturbance. Known results for a single fluid medium are recovered by making the upper fluid vacuo.

To determine the effect of the presence of upper fluid on the wave motion, the asymptotic forms of the interface depression are presented graphically. The figures exhibit variations of the interface depression at a fixed point x for large time and at a large fixed time for x. When the upper fluid is vacuo, forms of the free surface depression are also presented graphically side by side. It is observed that the presence of an upper fluid, however small may be its density, has significant effect on the wave motion.

Formulation of the Problem

We consider two-dimensional motion at the interface between two inviscid, incompressible and homogeneous superposed fluids wherein the upper fluid extends infinitely upwards and the lower fluid extends infinitely downwards. The motion in the fluids is generated due to an initial disturbance in the form of interface depression or an impulsive pressure at the interface. A rectangular cartesian coordinate system is chosen in which the origin is taken at the interface, y-axis vertically downwards in the lower fluid so that $y=0$ is the mean position of the interface. As the motion in the two fluids starts from rest, it is irrotational and can be described by the velocity potentials $\phi_1(x,y,t)$ and $\phi_2(x,y,t)$ in the lower and upper fluids respectively. They satisfy

$$\left.\begin{aligned}\nabla^2\phi_1=0 \quad \text{in} \quad y>0,\\ \nabla^2\phi_2=0 \quad \text{in} \quad y<0,\end{aligned}\right\} t\geq 0, \qquad \dots (1)$$

the linearised dynamic and kinematic conditions at the interface

$$\left.\begin{array}{l}\phi_{1y}=\phi_{2y}=\eta_t,\\ \phi_{1tt}-s\phi_{2tt}=g(\phi_{1y}-s\phi_{2y})\end{array}\right\} \text{on} \quad y=0, \quad t>0, \qquad \ldots (2)$$

where $\eta(x,t)$ is the interface depression, g is the gravity, $s=\rho_2/\rho_1$ $(0\leq s<)$, ρ_1 and ρ_2 are the densities of the lower and upper fluids respectively, the condition of no motion at infinite depth and height

$$\left.\begin{array}{l}\nabla\phi_1\to 0 \quad \text{as} \quad y\to\infty,\\ \nabla\phi_2\to 0 \quad \text{as} \quad y\to-\infty,\end{array}\right\} \qquad \ldots (3)$$

the initial conditions

$$\phi_1=\phi_2=0 \text{ at } t=0 \quad \text{on} \quad y=0 \qquad \ldots (4)$$

and

$$\frac{\partial}{\partial t}(\phi_1-s\phi_2)=g(1-s)f(x) \text{ at } t=0 \text{ on } y=0, \qquad \ldots (5)$$

where $f(x)$ is the initial interface depression.

Let us define a new potential function $\Phi(x,y,t)$ in $y\geq 0$ as

$$\Phi(x,y,t)=\phi_1(x,y,t)-s\phi_2(x,-y,t). \qquad \ldots (6)$$

From (1) to (5) we find that Φ satisfies

$$\begin{array}{l}\nabla^2\Phi=0 \text{ in } y>0,\ t>0,\\ \Phi_{tt}=\dfrac{1-s}{1+s}g\Phi_y \text{ on } y=0,\ t>0,\\ \nabla\phi\to 0 \text{ as } y\to\infty,\\ \Phi=0 \text{ at } t=0 \text{ on } y=0,\\ \Phi_t=(1-s)\,gf(x) \text{ at } t=0 \text{ on } y=0.\end{array} \qquad \ldots (7)$$

Let $\bar{\Phi}(x,y,p)$ denote the Laplace transform of $\Phi(x,y,t)$ denoted by

$$\bar{\Phi}(x,y,p)=\int_0^\infty \Phi(x,y,t)\exp(-pt)\,\mathrm{d}t,\ p>0.$$

By the application of Laplace transform, equations in (7) reduce to

$$\nabla^2\bar{\Phi}=0 \text{ in } y>0,$$

$$p^2\bar{\Phi}=\frac{1-s}{1+s}g\bar{\Phi}_y+(1-s)\,gf(x) \text{ on } y=0, \qquad \ldots (8)$$

$|\nabla \bar{\Phi}| \to 0$ as $y \to \infty$.

(8) is a BVP in the region $y \geq 0$ and we obtain its solution in the next section.

Solution

Let $G(x,y;X,Y;p)$ denote the Green's function satisfying

$\nabla^2 G = 0$ in $y > 0$ except at (X, Y),

$$p^2 G - \frac{1-s}{1+s} g G_y = 0 \text{ on } y = 0,$$

$G \to -\ln\{(x-X)^2 + (y-Y)^2\}^{1/2}$ as $(x,y) \to (x, Y)$,

$|\nabla G| \to 0$ as $Y \to \infty$.

Then $G(x,y;X,Y;p)$ can be obtained as (cf. Rhodes-Robinson[11])

$$G = -\ln \frac{r}{r'} + 2\int_0^\infty \frac{\exp\{-k(y+Y)\}}{k} \frac{\omega^2}{\omega^2 + p^2} \cos k(x-X)\, dk, \qquad \ldots (9)$$

where

$r, r' = \{(x-X)^2 + (y \mp Y)^2\}^{1/2}$

and

$$\omega^2 = \frac{1-s}{1+s} gk. \qquad \ldots (10)$$

By using Green's integral theorem to $\bar{\Phi}(x,y,p)$ and $G(x,y;X,Y;p)$ in the region $y \geq 0$, $-\infty < x < \infty$ we find that

$$\bar{\Phi}(X,Y,p) = \frac{1+s}{2\pi} \int_{-\infty}^{\infty} G(x,0;X,Y;p) f(x)\, dx$$

$$= \frac{1+s}{\pi} \int_0^\infty \frac{\omega^2}{\omega^2 + p^2} \frac{\exp(-kY)}{k} \left\{ \int_{-\infty}^{\infty} \cos k(x-X) f(x)\, dx \right\} dk.$$

Using Laplace inversion this gives

$$\Phi(x,y,t) = \frac{1+s}{\pi} \int_0^\infty \omega \sin \omega t \frac{\exp(-kY)}{k} \left\{ \int_{-\infty}^{\infty} \cos k(x-X) f(x)\, dx \right\} dk. \qquad \ldots (11)$$

The interface depression at any time t is given by

$$\eta(X,t) = \frac{1}{g(1-s)} \frac{\partial}{\partial t} \Phi(X,0,t)$$

$$= \frac{1}{\pi}\int_0^\infty \cos\omega t\left\{\int_{-\infty}^{\infty}\cos k(x-X)\,f(x)\,\mathrm{d}x\right\}\mathrm{d}k. \qquad \dots (12)$$

If the initial depression of the interface is concentrated at the origin, then $f(x) = \delta(x)$, so that

$$\eta(x,t) = \frac{1}{\pi}\int_0^\infty \cos\omega t \cos kx\,\mathrm{d}k \qquad \dots (13)$$

by changing X to x.

For $s = 0$ (i.e., in the absence of the upper fluid), (13) reduces to the result given in Stoker[2] obtained by a different method.

Asymptotic Form of the Interface Depression

To obtain the asymptotic form of the interface depression we use the method of stationary phase to (13) for large t. Now, (13) can be written as

$$\eta(x,t) = \frac{1}{4\pi}\int_0^\infty \left[\exp\left\{it\left(\omega + k\frac{x}{t}\right)\right\} + \exp\left\{-it\left(\omega + k\frac{x}{t}\right)\right\}\right.$$

$$\left. + \exp\left\{it\left(\omega - k\frac{x}{t}\right)\right\} + \exp\left\{-it\left(\omega - k\frac{x}{t}\right)\right\}\right]\mathrm{d}k. \qquad \dots (14)$$

Within the range of integration, the first two integrals of (14) have no stationary points. The stationary point for the third and fourth integrals is given by

$$k = \frac{1-s}{1+s}\frac{gt^2}{4x^2} \qquad \dots (15)$$

By the application of stationary phase method to the third and fourth integrals in (14), we find for large $gt^2/4x\ (x \neq 0)$

$$\eta(x,t) \to \frac{1}{\pi^{1/2}x}\left\{\frac{1-s}{1+s}\frac{gt^2}{4x}\right\}^{1/2}\cos\left(\frac{1-s}{1+s}\frac{gt^2}{4x} - \frac{\pi}{4}\right). \qquad \dots (16)$$

It may be noted that if we put $s = 0$, (16) reduces to the free surface depression given in Stoker[2].

If the disturbance is in the form of an impulsive pressure $I(x)$ per unit area applied to the interface, then the initial conditions (4) and (5) are changed to

$$\phi_1 - s\phi_2 = -\frac{I(x)}{\rho_1} \text{ at } t = 0 \text{ on } y = 0$$

and

$$\frac{\partial}{\partial t}(\phi_1 - s\phi_2) = 0 \text{ at } t = 0 \text{ on } y = 0.$$

Using a similar technique, we obtain in this case

$$\Phi(X,Y,t) = -\frac{1}{\pi\rho_1}\int_0^\infty \cos\omega t \exp(-kY)\left\{\int_{-\infty}^{\infty} \cos k(x-X)\, I(x)\, \mathrm{d}x\right\}\mathrm{d}k.$$

If the impulse is concentrated at the origin, then

$$\Phi(X,Y,t) = -\frac{1}{\pi\rho_1}\int_0^\infty \cos\omega t \cos kX \exp(-kY)\, \mathrm{d}k.$$

In this case

$$\eta(x,t) = \frac{1}{\{g(1-s^2)\}^{1/2}}\frac{1}{\pi\rho_1}\int_0^\infty k^{1/2}\sin\omega t\cos kx\, \mathrm{d}k$$

by changing X to x, so that for large $gt^2/4x$ $(x \neq 0)$.

$$\eta(x,t) \to \frac{2}{\pi^{1/2}(1-s)\, g\rho_1 xt}\left\{\frac{1-s}{1+s}\frac{gt^2}{4x}\right\}^{3/2}\sin\left(\frac{1-s}{1+s}\frac{gt^2}{4x} - \frac{\pi}{4}\right). \quad \ldots (17)$$

Again, in the absence of the upper fluid (i.e., for $s=0$), (17) coincides with the result given in Stoker[2].

Discussion

To study the effect of the presence of the upper fluid on the wave motion generated by initial interface disturbances at the interface between two superposed fluids, the interface depression $\eta(x,t)$ is graphically presented in figures 1 to 4 for the case of an initial interface depression concentrated at the origin and in figures 5 to 8 for the case of an initial impulse also concentrated at the origin by using the asymptotic forms of $\eta(x,t)$.

These figures show the variation of $\eta(x,t)$ at a fixed point x ($x=5$, 10cm) when the time increases from 10 to 80 secs and at a fixed large time t ($t=10$, 15 secs) for all x between 1 to 8cm. As we have considered an air-water model, the value of s is taken to be 0.0013 (Lamb[1], p. 576). When the upper fluid is made vacuo ($s=0.0$), $\eta(x,t)$ represents the free surface depression. This is also presented graphically side by side in each figure. From these figures we observed that the basic characters of the wave motion for the two-fluid case is almost similar to those for the one-fluid case which were discussed elaborately in Stoker's[2] book. However, due to the presence of upper fluid, it is observed from these figures that the phase of the wave profiles for fixed x changes considerably with progress of time. For example, a particular phase, say zero, of η occurs seven times between $t=15$ to 65 secs for the single-fluid case while it occurs five times for the two-fluid case for $x=5.0$cm. (Fig. 1) and for $x=10.0$cm. (Fig. 2) zero phase of η occurs five times between $t=15$ to

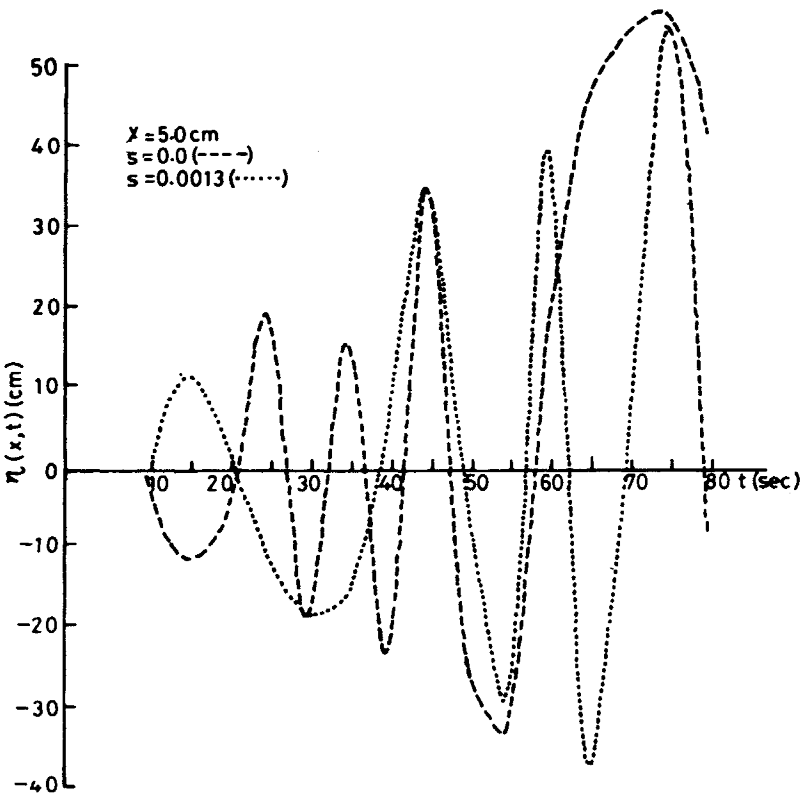

Fig 1 Waves profiles due to an initial disturbance in the form of an initial depression ($x = 5.0$ cm)

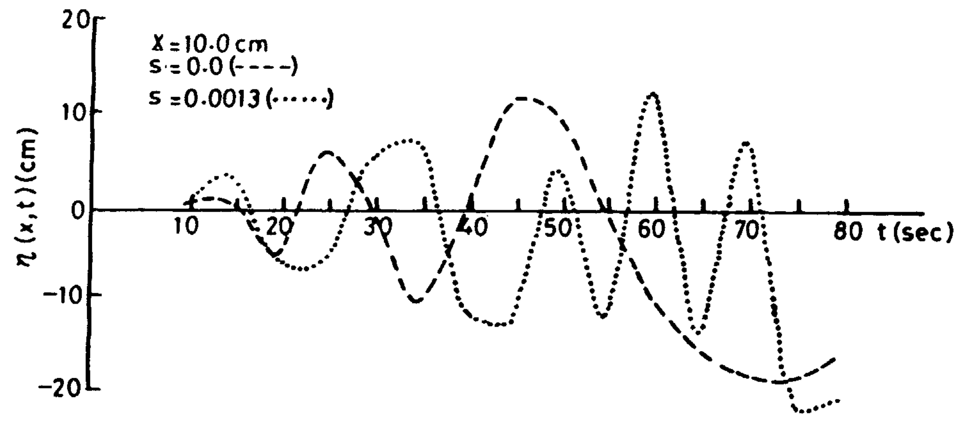

Fig 2 Waves profiles due to an initial disturbance in the form of an initial depression ($x = 10.0$ cm)

65 secs for the single-fluid case while it occurs seven times for the two-fluid case for an initial interface or free surface depression concentrated at the origin. Figs 5 and 6 also show somewhat similar type of behaviour when the wave motion is caused by an initial impulse concentrated at the origin of the interface or the free surface. In Figs 3, 4 and 7, 8 the interface as well as free surface depression are presented for fixed large time ($t = 10$, 15 secs) against x. As one moves away from the origin, the amplitude diminish, which is quite expected, but here also the phases changes considerably.

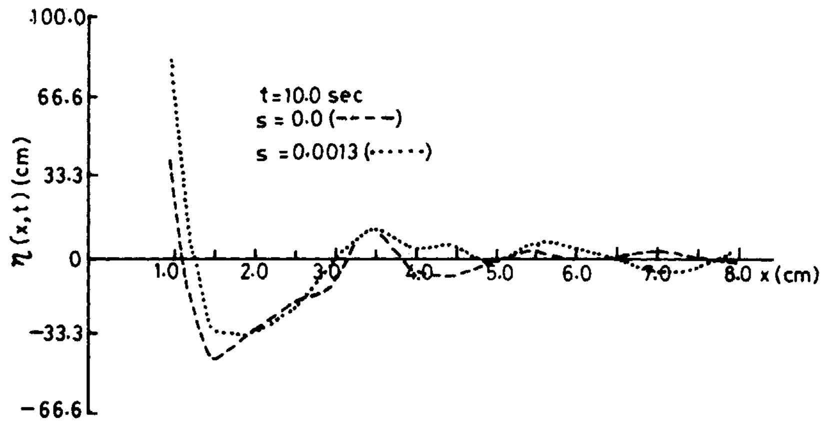

Fig 3 Waves profiles due to an initial disturbance in the form of an initial depression ($t = 10.0$ sec)

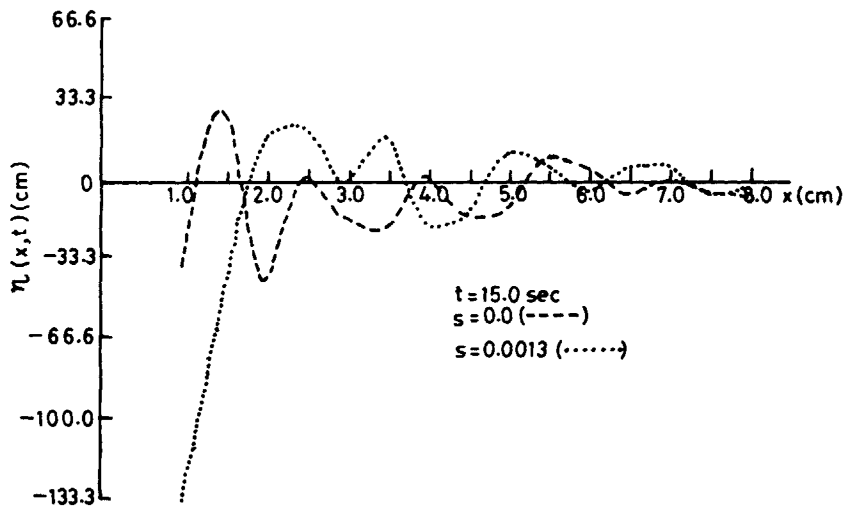

Fig 4 Waves profiles due to an initial disturbance in the form of an initial depression ($t = 15.0$ sec)

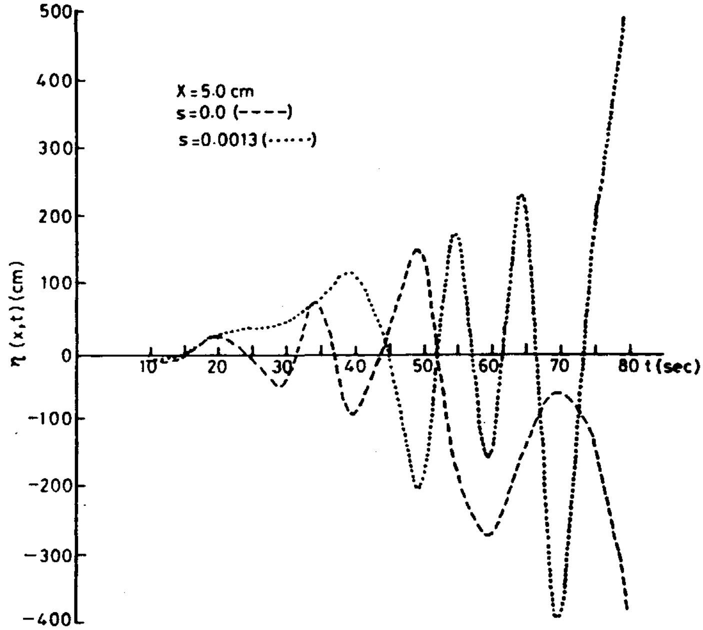

Fig 5 Waves profiles due to an initial disturbance in the form of an impulse ($x = 5.0$ cm)

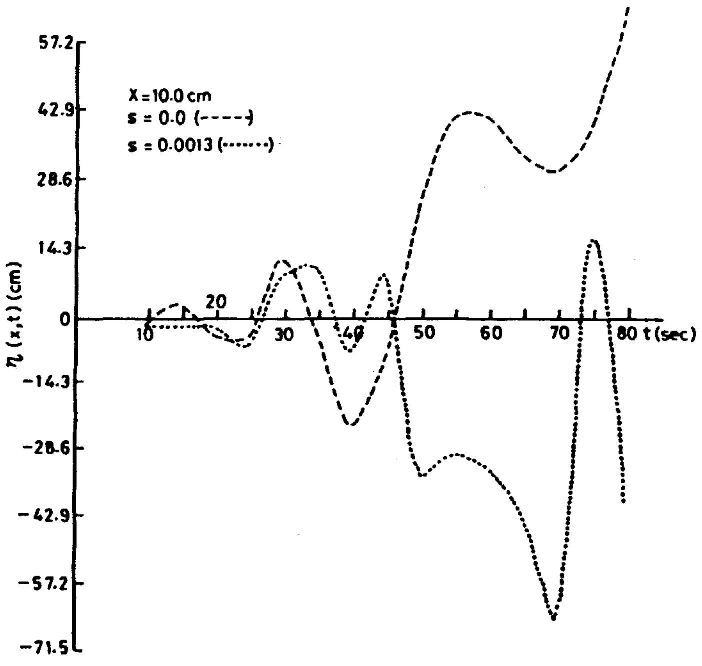

Fig 6 Waves profiles due to an initial disturbance in the form of an impulse ($x = 10.0$ cm)

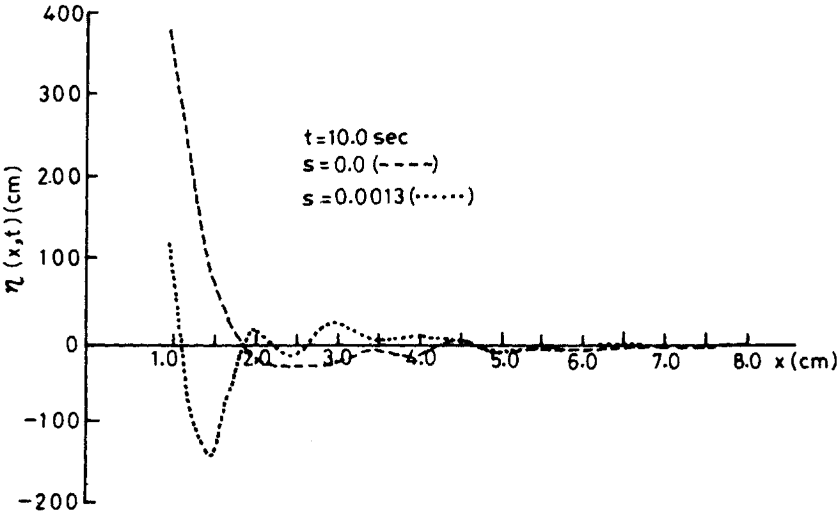

Fig 7 Waves profiles due to an initial disturbance in the form of an impulse ($t = 10.0$ sec)

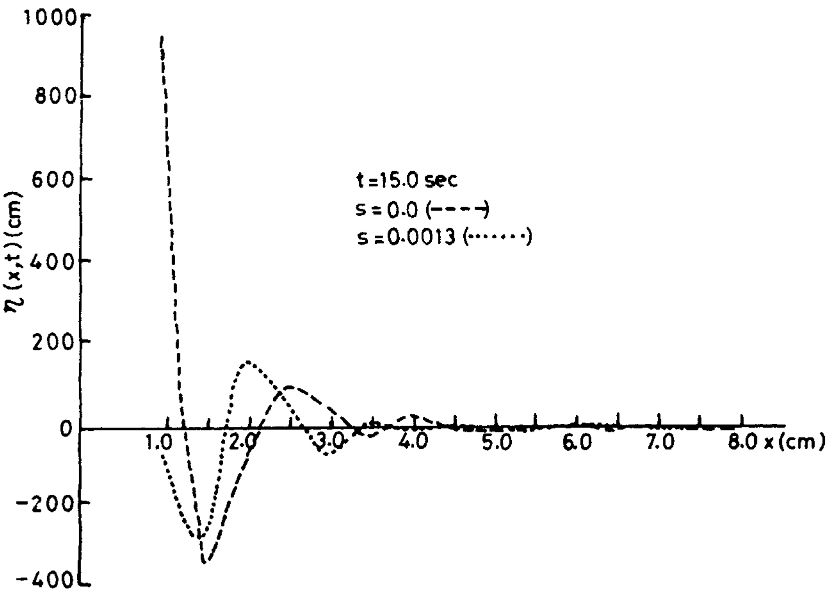

Fig 8 Waves profiles due to an initial disturbance in the form of an impulse ($t = 15.0$ sec)

It may be noted that if the effect of interfacial tension is considered, then the second eq. in (2) is to be replaced by

$$\phi_{1tt} - s\phi_{2tt} = g(\phi_{1y} - s\phi_{2y}) + \frac{T}{\rho_1}\begin{cases}\phi_{1yyy}\\ \phi_{2yyy}\end{cases}$$

so that the second equation of (7) to be modified as

$$\Phi_{tt} = \frac{1-s}{1+s} g \left[\Phi_y + \frac{1}{1-s} \frac{T}{\rho_1 g} \Phi_{yyy} \right],$$

where T is the coefficient of interfacial tension.

For air-water model, $s = 0.0013$ and $T/\rho_1 g = 0.075$ (cf. Lamb[1], p. 455), so that the second term in the square bracket is small in comparison with the first term. This has led us to neglect the surface tension at the interface.

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