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SOME COVARIANCE IDENTITIES, INEQUALITIES AND THEIR APPLICATIONS: A REVIEW*

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A reivew of covariance identities characterizing specific distributions and covariance identities and inequalities for general distributions are given. Some applications to limit theorems in probability and admissibility theory in statistical inference are mentioned. New concepts of multivariance and multicorrelation for an even dimensional random vector are introduced extending the usual concept of variance and correlation. The concept of r-th order strong mixing is defined and an inequality for the joint cumulant of an r-dimensional random vector is indicated.

Key Words: Hoeffding Identity; Characterization of Distributions; Joint Cumulant; r-th Order Strong Mixing; Covariance Identity; Covariance Inequality; Two-Part Dependence Assumption; Strong Mixing; Quantile Function

1 Introduction

Suppose X is a continuous random variable with the exponential density

$$f(x,\theta) = \exp\{\theta x - \psi(\theta)\}k(x) \qquad \dots (1)$$

with support $R=(-\infty, \infty)$ where $k(.)$ is differentiable. Let

$$t(x) = -\frac{k'(x)}{k(x)} \qquad \dots (2)$$

where $k'(x)$ denotes the derivative of $k(x)$. Then, for any absolutely continuous function g on R such that $E|g'(X)| < \infty$,

$$E[(t(X)-\theta)g(X)] = E[g'(X)]. \qquad \dots (3)$$

It is easy to see that the random variable $t(X)$ is unbiased for the parameter θ by choosing $g(x) \equiv 1$ in the above identity. In particular, the eq. (3) can be written in the form

$$Cov(t(X)-\theta, g(X)) = E[g'(X)]. \qquad \dots (4)$$

We call such an identity a *covariance identity*. The proof of the identity is easy and we include it for completeness. Note that

$$\begin{aligned} Eg'(X) &= \int_R g'(x)e^{\theta x-\psi(\theta)}k(x)dx \\ &= -\int_R g(x)e^{\theta x-\psi(\theta)}[\theta k(x)+k'(x)]\,dx \qquad \dots (5) \\ &= E[(t(X)-\theta)g(X)]. \end{aligned}$$

The second equality is a consequence of the assumption that $E|g'(X)| < \infty$ since $g(x)e^{\theta x-\psi(\theta)}k(x) \to 0 \text{ as } x \to \pm\infty$. This result is due to Hudson[1].

Consider a subclass of densities $f(x, \theta)$ for which

$$E_\theta\{(X-\mu)g(X)\} = E_\theta\{a(X)g'(X)\} \qquad \dots (6)$$

for some function $a(x)$ and for all absolutely continuous functions $g(.)$ such that $E|a(X)g'(X)| < \infty$ where X is a random variable with density $f(x, \theta)$ and $E_\theta(X) = \mu$. It can be shown that if the above identity holds, then the density $f(x, \theta)$ has to be of the form

$$f(x,\theta) = \exp\left\{\mu\int\frac{1}{a(x)}dx - \chi(\theta)\right\} \frac{1}{a(x)}\exp\left\{-\int\frac{x}{a(x)}dx\right\} \qquad \dots (7)$$

where the integrals are interpreted as indefinite integrals under some conditions. If X has the above

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density, then the identity eq. (6) follows from eq. (3)

Example 1: If X is $N(\mu, 1)$, then

$$E\{(X-\mu)g(X)\} = E\{g'(X)\}$$

whenever $E|g'(X)| < \infty$.

Example 2: If X is *Gamma* $(\mu, 1)$, that is,

$$f(x,\mu) = \frac{1}{\gamma(\mu)} x^{\mu-1} e^{-x}, x > 0 \quad \dots (8)$$

$$= 0 \text{ otherwise}$$

then $f(x,\mu)$ is of the form (7) with $a(x)=x$.

The following theorem gives a characterization of the continuous exponential family.

Theorem 1.1: *Let X be a random variable and $k(x)$ be a positive differentiable function. Let $t(x) = -\frac{k'(x)}{k(x)}$. If the relation*

$$E[(t(X)-\theta)g(X)] = E[g'(X)]$$

holds for functions of the form $g(x)=e^{iux}$, $u \in R$, then X has a density with respect to the Lebesgue measure and it is of the form

$$f(x,\theta) = \exp\{\theta x - \psi(\theta)\}k(x)$$

The above theorem is due to Prakasa Rao[2].

In particular, the following special case, due to Stein[3], gives a characterization of the normal distribution.

Theorem 1.2 : *A random variable X has a normal distribution with mean and variance unity if and only if for all absolutely continuous functions g with $E|g'(X)| < \infty$, the identity*

$$E[(X-\theta)g(X)] = E[g'(X)]$$

holds.

Another special case of Theorem 1.1. is the following characterization of a gamma distribution.

Theorem 1.3; *A positive random variable X has a Gamma distribution with density $p(.)$ with parameters (α, β), that is,*

$$p(x) = (\alpha^{\beta+1}/\gamma(\beta+1))x^{\beta}e^{-\alpha x}, x > 0$$

=0 otherwise

if and only if for all absolutely continuous functions g with $E|g'(X)| < \infty$, the identity

$$E[(\alpha - \beta X^{-1})g(X)] = E[g'(X)]$$

holds

The above results for continuous exponential families have analogues for the discrete exponential families.

A random variable X is said to have a distribution belonging to the *discrete exponential family* if it has a discrete distribution with the probability mass function of the form

$$P(X=x) = c(\theta)\theta^x p(x), x = 0,1,2,\dots \quad \dots (9)$$

with $p(x) > 0$, x=0, 1, 2,... Hudson[1] proved the following theorem.

Theorem 1.4: *Let X be as stated above and g be a real-valued function with $g(-1)=0$ and $E|g(X)| < \infty$. Then*

$$\theta E[g(X)] = E[t(X)g(X-1)] \quad \dots (10)$$

where

$$t(x) = 0 \text{ if } x = 0$$
$$= p(x-1)/p(x) \text{ if } x \geq 1. \quad \dots (11)$$

The following converse result characterizes the distribution defined above[2].

Theorem 1.5: *Let X be nonnegative integer valued random variable and suppose that $p(x)>0$, $x=0,1,2,\dots$ Further suppose that the identity eq. (10) holds for $g(x)=s^x$ for all s in an interval containing the origin where $t(x)$ is as defined by (11). Then X has the distribution defined by function (9).*

As special cases, it follows that the identity

$$E[(X-\lambda)s^X] = E[X(s^X - s^{X-1})], \lambda > 0$$

characterizes the Poisson distribution, the identity

$$E[(t(X)-p)s^X] = E[t(X)(s^X - s^{X-1})],$$

$$t(x) = \frac{x}{r+x-1}$$

characterizes the Negative binomial distribution for fixed $0<p<1$ and a positive integer $r>1$ and the identity

$$E[(X-np)s^X] = E[X(s^X - s^{X-1})]$$

characterizes the Binomial distribution for fixed integer $n \geq 1$ and $0<p<1$ among the random variables X with support $\{0, 1, 2, ..., n\}$[2].

As applications of the above results, one can obtain limit theorems in probability. For instance, the following result gives a necessary and sufficient condition for the limiting distribution of a sequence of probability measures Q_n to be the standard normal distribution.

Theorem 1.6: *A sequence of probability measures Q_n converges weakly to the standard normal distribution if and only if*

$$\int_R [xf(x) - f'(x)]Q_n(dx) \to 0 \text{ as } n \to \infty$$

for every f for which f', f'' exist and $x\,f(x)$, $x f'(x)$ and $f''(x)$ are bounded and continuous.

The above theorem is due to Hazendock[4]. Chen[5] gave a proof of this result as an application of the characterization of the normal distribution given above. A general result of this nature for limiting distributions belonging to the exponential families is given in Prakasa Rao[2].

Consider an exponential family in R^p with the density function

$$f(z) = e^{\mu.z - M(\mu) - K(z)} I_E(z)$$

with respect to the Lebesgue measure on R^p where E is a finite union of open connected sets in R^p and $K(z)$ has continuous partial derivatives $D_iK(z)$, $1 \leq i \leq p$ and $\mu = (\mu_1 .... \mu_p)$. Suppose that f approaches zero monotonically as z approaches the boundary of E along the coordinate axes.

An example of such a distribution is a p-variate normal distribution with mean μ and covariance matrix Σ. Chou[6] proved the following result.

Theorem 1.7: *Let Z be a random vector with the density function as defined above. Let g be a function from R^p to R such that g is an indefinite integral of*

$\partial g / \partial z_i$ for all $i = 1, ..., p$ with $E\|\nabla g(Z)\| < \infty$ and

$E\|(\nabla K(Z) - \mu)g(Z)\| < \infty$ if $p > 1$. Then

$$E(\nabla K(Z) - \mu)g(Z)) = E\nabla g(Z).$$

A random variable X is said to have an univariate elliptical distribution with parameters μ and $\gamma > 0$ if it has a density of the form

$$f_h(x \mid \mu, \gamma) = \gamma^{-1/2} h(x-\mu)^2/\gamma)$$

for some function h. If $\mu = 0$ and $\gamma = 1$, then the density of X is called a spherical density corresponding to the radial function h. Examples of such densities include the standard normal, standard Cauchy, Student's t with a given degrees of freedom, mixtures of normal etc.

The following identity holds for spherical densities[7].

Theorem 1.8: *Suppose X has a spherical density corresponding to the radial function h. Let*

$$\eta(x) = \int_x^\infty t h(t^2) dt / h(x^2).$$

If g is absolutely continuous with $E[g(X)]^2 < \infty$ and $E[g'(X)\eta(X)] < \infty$, then

$$E[g'(X)\eta(X)] = E[Xg(X)].$$

In general, if X has an elliptical density with radial function h and parameters μ and γ, then

$$E[g'(X)\beta(X)] = E[(X-\mu)g(X)]$$

where

$$\beta(x) = \int_{-\infty}^x (\mu - t)\gamma^{-1/2} h\left(\frac{(t-\mu)^2}{\gamma}\right)dt \Bigg/ \left[\gamma^{-1/2} h\left(\frac{(x-\mu)^2}{\gamma}\right)\right]$$

It can be shown that the above identities characterize the corresponding densities. As an application of these identities, one can prove the following result on the inadmissability of the usual estimator for elliptical densities under quadratic loss.

Suppose X_i, $1 \leq i \leq p$ are independent univariate elliptical random variables with unknown parameters μ and known $\gamma_i > 0$ and known radial functions h_i for $1 \leq i \leq p$. Define

$$\zeta_i = \zeta_i(X_i), \zeta_i(x) = \int_{-\infty}^x \frac{1}{\beta_i(y)} dy,$$

Let

$$\bar{\zeta} = (1/p)\sum_{i=1}^{p}\zeta_i(X_i)$$

and

$$S = \sum_{i=1}^{p}(\zeta_i - \bar{\zeta})^2.$$

Then, for $p \geq 4$, the estimator

$$X^* = \left(X_i - \frac{p-3}{S}(\zeta_i - \bar{\zeta}), 1 \leq i \leq p \right)$$

dominates

$X = (X_i, 1 \leq i \leq p)$

as an estimator of

$\mu = (\mu_i, 1 \leq i \leq p)$

and the difference in risk is $E_\mu[(p-3)^2/S]$ assuming that $E(1/S) < \infty$[7].

A more recent result giving a covariance identity for the exponential distribution is the following due to Bobkov and Houdre[8].

***Theorem 1.9** : Let ξ, η and ζ be independent standard exponentially distributed random variables. Then, for any two absolutely continuous functions f, g such that $E|f(\xi)|^2 < \infty$ and $E|g(\xi)|^2 < \infty$,*

$$\text{cov}(f(\xi), g(\xi)) = E[f'(\xi+\eta)g'(\xi+\zeta)].$$

It can be shown that if the above identity holds for i.i.d.random variables ξ, η and ζ for all absolutely continuous f, g such that $E|f(\xi)|^2 < \infty$ and $E|g(\xi)|^2 < \infty$, then either ξ, η and ζ are standard exponential random variables or $-\xi$ and hence $-\eta$ and $-\zeta$ are standard exponential random variables[9]. The problem involves solving a functional equation of the type

$$\phi(t+s) - \phi(t)\phi(s) = -ts\phi(t)\phi(s)\phi(t+s)$$

for $-\infty < t, s < \infty$ where $\phi(\cdot)$ is a complex-valued function. More general identities of this nature are proved in Prakasa Rao[9].

Let X be a continuous random variable with density f, mean μ and variance σ^2. Cacoullos and Papathanasiou[10] proved that for an absolutely continuous function g with $E|w(X)g'(X)| < \infty$, the covariance identity

$$\text{cov}(X, g(X)) = \sigma^2 E[w(X)g'(X)] \quad \dots (12)$$

holds with the function $w(x)$ defined by

$$\sigma^2 w(x)f(x) = \int_{-\infty}^{x}(\mu - t)f(t)dt.$$

It was also shown that the function $w(x)$ characterizes the density. For instance, the function $w(x) \equiv 1$ if and only if X is normally distributed.

Cacoullos and Papathanasiou[11] gave a multivariate extension of the above result. Papathanasiou[12] extended the result of Chou[6] to a wider class of multivariate distributions by obtaining identity of the form

$$E|(-\nabla \log f(X))g(X)] = E[\nabla g(X)]$$

where ∇ denotes the gradient operator.

Cacoullos and Papathanasiou [13] obtained the following general covariance identity.

***Theorem 1.10** : Suppose $h(x)$ and $g(x)$ are functions such that g is absolutely continuous, and $E|k(X)g'(X)| < \infty$ where*

$$k(x) = \frac{1}{f(x)}\int_{-\infty}^{x}(E[h(X)] - h(t))f(t)\,dt. \quad \dots (13)$$

Then

$$\text{cov}(h(X), g(X)) = E[k(X)g'(X)]. \quad \dots (14)$$

Houdre and Perez-Abreu[14] discuss covariance identities for functionals of the Wiener and Poisson processes. They obtain covariance identity for functions of a multivariate normal random vector as a special case of their general results.

Let $X = (X_1, \dots, X_p)$ be a p-dimensional normal random vector with mean μ and covariance matrix Σ. Let $\Phi(x) = (\phi_1(x), \dots, \phi_k(x))'$ and $\Psi(x) = (\psi_1(x), \dots, \psi_k(x))'$ be mappings from R^p to R^k such that $\text{cov}(\phi_i(X), \psi_j(X))$ exists for $1 \leq i, j \leq p$. Define $\text{cov}(\Phi(X), \Psi(X))$ to be the matrix with $\text{cov}(\phi_i(X), \psi_j(X))$ as its (i, j)-th element. Let ∇ be the gradient operator and ∇^r be the iterated gradient operator. Houdre and Perez-Abrou[14] proved that

$$\text{cov}(\Phi(X), \Psi(X)) = \sum_{r=1}^{N}\frac{(-1)^{r+1}}{r!}E\{(\nabla^r\Phi(X))\Sigma^{\otimes r}(\nabla^r\Psi(X))'\} + R_N \quad \dots (15)$$

where E is the expectation, $\Sigma^{\otimes r}$ is the r-th Kronecker product of Σ with itself and R_N includes terms involving gradients of order $N+1$. For applications of these results. (see Houdre[15]).

We have so far discussed covariance identities for specific distributions which characterize those distributions.

We now consider general results for covariance of functions of a random variable.

The following result is due to Hoeffding[16] (cf. Lehmann[17]).

Theorem 1.11 *: Let F denote the joint and* F_X *and* F_Y *denote the marginal distributions of X and Y respectively. Then*

$$\text{cov}(X,Y) = \int_{R^2} [F(x,y) - F_X(x)F_Y(y)]dxdy \quad \ldots (16)$$

provided the cov (X, Y) *exists.*

Proof: Let (X_1, Y_1) and (X_2, Y_2) be independent random vectors with the joint distribution function F. Then

$$2\text{cov}(X < Y) = 2\{E(XY) - E(X)E(Y)]$$
$$= E[(X_1 - X_2)(Y_1 - Y_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [I(u,X_1) - I(u,X_2)]$$
$$[I(v,Y_1) - I(v,Y_2)]dudv$$

where $I(u, x)=1$ if $u \le x$ and $=0$ otherwise. Since $E|XY|$, $E|X|$ and $E|Y|$ exist, the expectation can be taken inside the integral sign and the Fubini's theorem is applicable. It is easy to see that the expectation of the integrand is

$$2[F(x,y) - F_X(x)F_Y(y)]$$

completing the proof.

Note that the identity (16) can be written in the form

$$\text{cov}(X,Y) = \int_{R^2} \text{cov}(I(X \le x), I(Y \le y))dxdy \quad \ldots (17)$$

here after called the *Hoeffding identity.* Block and Fang[18] obtained a multivariate version of this identity using the notion of a joint cumulant for a random vector. They proved that

$$\text{cum}(X_1,\ldots,X_k) = \int_{R^k} \text{cum}\,(\chi(x_1,X_1),\ldots,\chi(x_k,X_k))dx_1\ldots dx_k \quad \ldots (18)$$

where

$$\chi(x,X) = 1 \text{ if } x < X$$

$= 0$ otherwise

and

$$\text{cum}\,(X_1,\ldots,X_k) = \sum(-1)^{p-1}(p-1)!\; E\left(\prod_{j\in v_1} X_j\right)\ldots E\left(\prod_{j\in v_p} X_j\right), \quad \ldots (19)$$

Here the summation extends over all the partitions $(v_1, \ldots v_p)$, $p=1, 2, \ldots, k$ of $\{1, 2, \ldots, k\}$. It is easy to check that the cum(X, Y) is the same as the cov (X, Y) (cf. Block and Fang[18]).

Yu[19] obtained a generalization of the Hoeffiding identity to absolutely continuous functions of the components of a random vector. She showed that if f_i, $1 \le i \le 2k$ are absolutely continuous functions on R, then for any random vector $x = (X_1, \ldots, X_{2k})$,

$$L(f_i(X_i), 1 \le i \le 2k) = \int_{R^{2k}} \left\{\prod_{i=1}^{2k} f_i'(x_i)\right\} \times L(I(X_i \le x_i), 1 \le i \le 2k)dx_1\ldots dx_{2k} \quad \ldots (20)$$

where

$$L(X_1,\ldots,X_{2k}) = \sum_{\zeta}(-1)^{card(\zeta)} \times E\left(\prod_{j\in\zeta} X_j\right)E\left(\prod_{j\in\zeta^c} X_j\right) \quad \ldots (21)$$

the sum being taken over all subsets ζ of $\{1, 2, \ldots, 2k\}$ including the empty set.

Here $card(\zeta)$ denotes the cardinality of the set ζ and ζ^c denotes the complement of the set ζ in $\{1, 2, \ldots, 2k\}$.

Quesada-Molina[20] generalized the Hoeffding identity to quasi-monotone functions $K(x, y)$ in the sense that

$$K(x,y) - K(x',y) - K(x,y') + K(x',y') \ge 0 \quad \ldots (22)$$

whenever $x \le x'$, $y \le y'$. He proved that

$$EK(X,Y) - EK(X^*,Y^*) = \int_{R^2} \text{cov}(I(X \le x), I(Y \le y))K(dx,dy) \quad \ldots (23)$$

where X^*, Y^* are independent random variables independent of $(X,.Y)$ with X^* and Y^* having the same marginal distributions as X and Y respectively.

An example of a quasi-monotone function is $K(x, y)=xy$ and the above result reduces to the Hoeffding identity in this case. Another example of a quasi-monotone function is $K(x, y)=(x-a)^r(y-b)^s$ where r, s are positive odd integers and a, b are real numbers.

As an application of the above result, one can prove that if two random variables are positively quadrant dependent (PQD), that is,

$$P(X \le x, Y \le y) \ge P(X \le x)P(Y \le y), x, y \in R,$$

then

$$E[(X-a)^r(Y-b)^s] \ge E[(X-a)^r]E[(Y-b)^s] \quad \text{... (24)}$$

whenever r and s are positive odd integers assuming of course that the expectations involved exist.

The result of Quesada-Molina[20] has been generalized to quasi-monotone functions of a $2k$-dimensional random vector in Prakasa Rao[21] and a new concept of multivariance and multicorrelation has been introduced.

For any random vector $(X_1, ..., X_{2k})$, define

$$R(X_1,...,X_{2k}) = (1/2)L(X_1,...,X_{2k})$$
$$= (1/2)\sum_{\zeta}(-1)^{card(\zeta)} E\left(\prod_{j\in\zeta} X_j\right) \times E\left(\prod_{j\in\zeta^c} X_j\right). \quad \text{... (25)}$$

The quantity $R(X_1,..., X_{2k})$ can be interpreted as multivariance of $(X_1, ..., X_{2k})$. It is easy to check that $R(X, Y)=\text{cov}(X, Y)$ and that $R(X_1, ..., X_{2k})=0$ whenever X_i, $1, \le i \le 2k$ are independent. An alternate way of representing $R(X_1, ..., X_{2k})$ is

$$R(X_1,...,X_2) = (1/2)E\left[\prod_{i=1}^{2k}(X_i - X_i')\right] \quad \text{... (26)}$$

where $(X_1, ..., X_{2k})$ and $(X_1',...,X_{2k}')$ are i.i.d. random vectors. Using this representation, it can be shown that

$$|R(X_1,...,X_{2k})| \le (1/2)2^{\frac{(2^k-1)k}{2^{k-1}}} \left[\mu_{X_1}^{(2^k)},...,\mu_{X_{2k}}^{(2^k)}\right]^{\frac{1}{2^k}} \quad \text{... (27)}$$

whenever $E|X_i - EX_i|^{2^k} = \mu_{X_i}^{(2^k)} < \infty, 1 \le i \le 2k.$ For $k=1$, this inequality reduces to

$$|R(X,Y) \le [\mu_X^{(2)}\mu_Y^{(2)}]^{1/2} \quad \text{... (28)}$$

which is the standard correlation inequality. The quantity

$$\rho(X_1,...,X_{2k}) = \frac{R(X_1,...,X_{2k})}{\alpha_k[\mu_{X_1}^{(2^k)}...\mu_{X_{2k}}^{(2^k)}]^{\frac{1}{2^k}}} \quad \text{... (29)}$$

where

$$\alpha_k = (1/2)2^{\frac{(2^k-1)k}{2^{k-1}}} \quad \text{... (30)}$$

is called the multicorrelation of $(X_1, ..., X_{2k})$. The above inequality proves that

$$|\rho(X_1,...,X_{2k})| \le 1. \quad \text{... (31)}$$

Note that $\rho(X_1,...X_{2k}) = 0$ if $X_i, 1 \le i \le 2k$ are independent.

We now obtain some inequalities for covariance of two random variables X and Y.

Let $((\Omega, F, P)$ be a probability space and $\boldsymbol{A}$ and $\boldsymbol{B}$ be two sub σ-algebras contained in $\boldsymbol{F}$. Define

$$\alpha(\boldsymbol{A},\boldsymbol{B}) = \sup_{A\in \boldsymbol{A}, B\in \boldsymbol{B}} |P(A\cap B) - P(A)P(B)| \quad \text{... (32)}$$

The coefficient $\alpha(\boldsymbol{A},\boldsymbol{B})$ is known as the strong mixing coefficient and it measures the dependence between the σ-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$. Let $\sigma(X)$ denote the σ-algebra generated by a random variable X. Davydov[22] proved the following inequality: for any positive reals p, q and r such that $p^{-1}+q^{-1}+r^{-1}=1$,

$$|\text{cov}(X,Y)| \le C[\alpha(\sigma(X),\sigma(Y))]^{(1/p)} [E|X|^q]^{(1/q)}][E|Y|^r]^{(1/r)}] \quad \text{... (33)}$$

where C is a constant. He proved further that $C \le 12$. Rio[23] obtained a different upper bound for $|\text{cov}(X, Y)|$ using the quantile functions of X and Y and the strong mixing coefficient $\alpha \equiv \alpha(\sigma(X), \sigma(Y))$. The quantile function of X is defined by

$$Q_x(u) = \inf\{t : P(|X| > t) \le u\}.$$

He proved that

$$|\operatorname{cov}(X,Y)| \le 2\int_0^{\alpha} Q_X(u)Q_Y(u)\,du. \qquad \text{... (34)}$$

Bradley[24] extended the inequality under a "two-part" dependence assumption. His result is the following.

Theorem 1.12 *: Let (Ω, F, P) be a probability space and $\mathcal{A}$ and $\mathcal{B}$ be two sub σ-algebras contained in $\mathcal{F}$. Suppose that there exist $p \ge 1$, $q \ge 1$ such that $p^{-1}+q^{-1}=1$, α and λ in [0, 1] such that for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$*

$$|P(A\cap B)-P(A)P(B)| \le \alpha + \lambda[P(A)]^{(1/p)}[P(B)]^{(1/q)}. \qquad \text{... (35)}$$

Then, for any $\mathcal{A}$-measurable random variable X such that $||X||_p < \infty$ and for any $\mathcal{B}$-measurable random variable Y such that $||Y||_q < \infty$,

$$|\operatorname{cov}(X,Y)| \le 4\int_0^{\alpha} Q_{|X|}(u)Q_{|Y|}(u)\,du + C\lambda(1-\log\lambda)\,\|X\|_p\|Y\|_q \qquad \text{... (36)}$$

where

$$C = 8[3\max(p,q) + (1-2^{-(1/p)})^{-1} + (1-2^{-(1/q)})^{-1}] \qquad \text{... (37)}$$

Bradley[24] indicates that the constant C is bounded by C' *max* (p, q) where C' is a universal constant that does not depend on p and q.

In a recent paper (Prakasa Rao[25]), we have introduced the concept of r-th order strong-mixing and extended the results of Bradley[24] to *cum* $(X_1, X_2, ..., X_r)$.

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