Research Paper

Axi-Symmetric Crushing of Thin Walled Frusta and Tubes

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An analytical straight fold model with partly inside and partly outside folding in axi-symmetric crushing of thin metallic frusta has been presented. Existing total outside fold model of a frusta and partly inside and partly outside fold models as well as total outside or total inside fold models of a tube can be derived from the present model. The relations for obtaining the inside and outside fold lengths in the case of tubes are derived. The difference in the values of yield stress of the material in compression and tension has been incorporated in the analysis. Variation of circumferential strain during the formation of a fold has been taken into account. The mean and the variation of crushing load have been computed. The results have been compared with experiments and good agreement has been observed. The results are of help in understanding the phenomenon of actual fold formation.

Key Words: frusta, tubes, crushing, folding, straight-fold

Introduction

For their application to the design for crashworthiness of road and air vehicles, mechanics of axial crushing of thin metallic tubes and frusta has been extensively studied in the past experimentally[1-3]. The aim is to absorb the kinetic energy dissipated in a crash or an accident of these vehicles in plastic deformation that occurs during crushing of these thin walled structures and consequently save human beings from injury and costly equipment from excessive damage. The mechanics of crushing phenomenon is quite complex and not amenable to complete analytical solution. Experiments on tubes have shown that under axial loads when these deform in axi-symmetric mode, the fold formed is partly inside and partly outside the initial tube diameter[4-6]. Analytical solutions available are not many and those available have made several simplifying assumptions which include the deformation to be inextensional and the fold to be only outside[8] or inside[9,10] the initial diameter of the tube. These analyses only determined the mean collapse load for which mean circumferential strain was enough. There is hardly any analysis available which determines the inside/outside folding of the frusta or the tubes. Available analysis[4,11], which considered inside/outside folding in tubes assumed that both parts are equal in length and those available for the load-deformation computation assume the fold shape. Experiments have shown that this is not true and the inner fold is smaller than the outer fold[5,6,11-13]. Such analysis for conical frusta does not exist, and those available[7] consider the folding to be only outside.

In the present paper, a mathematical formulation is presented for axi-symmetric crushing of frusta with partly inside and partly outside folding based on energy considerations. The model developed considers the variation of circumferential strain during the formation of a fold and the difference in yield strength of material in tension and compression. The existing total outside fold model of frusta[7] and the partly inside and partly outside fold models[6] have been derived from the proposed model. The mean and the variation of crushing load for frusta and tubes have been computed. The results have been compared with experiments and good agreement has been observed.

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Analysis of Frusta

For the analysis, we consider a thin frustum of thickness t, smaller end radius R_1, and angle of taper α, as shown in figure 1. The axi-symmetric crushing of the frustum is also shown in this figure, by the formation of partly inside and partly outside straight folding. The undeformed portion a'-b'-c'-d'-e' of the frustum takes the shape a-b-c-d-e after axi-symmetric crushing. The length of first and second limbs of first fold is denoted by h_1 and h_2 respectively out of which mh_1 and mh_2 are inside the initial line of frustum for first and second limb respectively, and m is the folding parameter defined as the ratio of the inside portion to the total length of the fold. The angle of inclination of the limbs of the fold, θ_1 and θ_2 have been measured from the initial line of the frustum; their initial value in the undeformed state is zero and their maximum value after complete crushing is $(\pi/2-\alpha)$ and $(\pi/2+\alpha)$ respectively. In the present analysis, complete crushing of the fold has been assumed because the consideration of effective crushing distance[10] would lead to the lower values of the energy dissipation. The yield strength of the material of the frustum in compression and tension has been taken as f_{yc} and f_{yt} respectively, and $r = f_{yc}/ f_{yt}$.

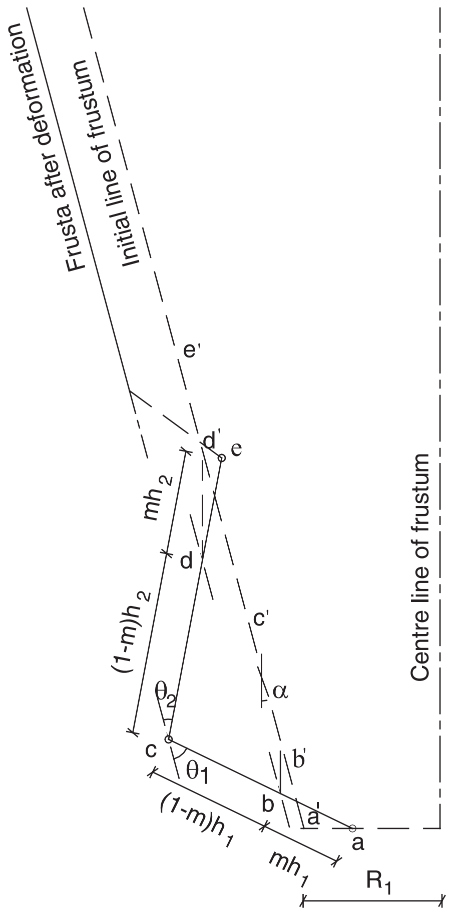

Figure 1: *Axial crushing model of a frustum*

The plastic moment of resistance of the material of the frustum has been taken as

$$M_p = \frac{1}{2\sqrt{3}} f_{yt} t^2$$

Energy Absorption in Crushing

The radii of the points a,b,c,d and e in the deformed state of the frustum (Fig. 1) are given by,

$$R_a = R_1 + mh_1\{\sin\alpha - \sin(\theta_1+\alpha)\} \quad \ldots (1)$$

$$R_b = R_1 + mh_1 \sin\alpha \quad \ldots (2)$$

$$R_c = R_1 + h_1\{m\sin\alpha + (1-m)\sin(\theta_1+\alpha)\} \quad \ldots (3)$$

$$R_d = R_1 + \{h_1 + (1-m)h_2\}\sin\alpha \quad \ldots (4)$$

$$R_e = R_1 + \{h_1 + (1-m)h_2\}\sin\alpha - mh_2\sin(\theta_2-\alpha) \quad \ldots (5)$$

Also,

$$R_d = R_1 + h_1\{m\sin\alpha + (1-m)\sin(\theta_1+\alpha)\} - (1-m)h_2\sin(\theta_2-\alpha) \quad \ldots (6)$$

which gives relation between the two angles:

$$\theta_2 = \sin^{-1}[K\sin(\theta_1+\alpha) - (K+1)\sin\alpha] - \alpha \quad \ldots (7)$$

The relationship between the length of the two limbs can be obtained from the geometry in complete crushed state as,

$$h_1 = Kh_2 \quad \ldots (8)$$

where,

$$K = \frac{1+\sin\alpha}{1-\sin\alpha} \quad \ldots (9)$$

For the n^{th} fold (n = 1, for the first fold considered in Fig. 1), the above radii will be such that the new value of R_1 is $R_1 + (n-1)(h_1+h_2)\sin\alpha$.

The energy dissipation in flexure has been assumed to be localized at the hinges which is in the form of rotation at lower, upper and intermediate plastic hinges. The energy dissipated in plastic bending $W_{b\theta}$, in the rotation of the lower limb upto angle θ_1 and upper limb upto angle θ_2 is given by:

$$W_{b\theta} = \underbrace{\int_0^{\theta_2} 2\pi M_p R_e\, d\theta_2}_{\text{(Upper Hinge)}} + \underbrace{\int_0^{\theta_2} 2\pi M_p R_c\, d\theta_2 + \int_0^{\theta_1} 2\pi M_p R_c\, d\theta_1}_{\text{(Intermediate Hinge)}}$$

$$+ a\underbrace{\int_0^{\theta_1} 2\pi M_p R_a\, d\theta_1}_{\text{(Lower Hinge)}} \quad \ldots (10)$$

$$= 2\pi M_p\left[\int_0^{\theta_2}(R_c+R_e)\,d\theta_2 + \int_0^{\theta_1}(aR_a+R_c)\,d\theta_1\right] \quad \ldots (11)$$

During the formation of first fold, lower hinge does not exist, whereas it exists in subsequent folds. It has been incorporated in the above expression by introducing a constant, a. Its value will be zero (i.e. a=0) for first and unity (i.e. a=1) for rest of the folds. Evaluating the integrals, $W_{b\theta}$ is obtained as

$$W_{b\theta} = 2\pi M_p \left[\begin{aligned}&(a+1)(R_1+mh_1\sin\alpha)\theta_1 + h_1\{m(a+1)-1\}\{\cos(\theta_1+\alpha)-\cos\alpha\}\\ &+2\{R_1+h_2(1+K-m)\sin\alpha\}\theta_2 + h_2(2m-1)\{\cos(\theta_2-\alpha)-\cos\alpha\}\end{aligned}\right] \quad \ldots (12)$$

putting, $\theta_1 = \frac{\pi}{2} - \alpha$ and $\theta_2 = \frac{\pi}{2} + \alpha$ in the above equation gives the total energy absorbed in flexure during complete crushing of the fold as,

$$W_b = 2\pi M_p \left[\begin{aligned}&(a+1)(R_1+mh_1\sin\alpha)\left(\frac{\pi}{2}-\alpha\right) - h_1\{m(a+1)-1\}\cos\alpha\\ &+2\{R_1+h_2(1+K-m)\sin\alpha\}\left(\frac{\pi}{2}+\alpha\right) - h_2(2m-1)\cos\alpha\end{aligned}\right] \quad \ldots (13)$$

which for m=0 is reduced to the total outside fold model of frusta[7], in which case the total energy dissipated in flexure is,

$$W_b = 2\pi M_p \left[\left\{ (3+a)\frac{\pi}{2} + (1-a)\alpha \right\} R_1 - \frac{2h_2}{(1-\sin\alpha)} \{ (\pi + 2\alpha)\sin\alpha + \cos\alpha \} \right] \quad \dots (14)$$

For $\alpha = 0$, $a = 1$ and $h_1 = h_2 = h$ (say), equation (13) describes the partly inside and partly outside model of a cylinder of radius R_1 and the energy dissipated in flexure is given by[6],

$$W_b = 2\pi M_p [2\pi R_1 + 2h(1-2m)] \quad \dots (15)$$

The energy dissipated in circumferential stretching for the portion of the fold inside the initial diameter of the frustum and circumferential compression of the portion of the fold outside for rotation upto an angle θ_1 of the lower limb and θ_2 of the upper limb of the fold, $W_{c\theta}$, the total energy dissipated in circumferential deformation can be calculated by,

$$W_{c\theta} = \int_0^{\theta_1} \left(\frac{dW_{c1}}{d\theta_1} \right) d\theta_1 + \int_0^{\theta_2} \left(\frac{dW_{c2}}{d\theta_2} \right) d\theta_2 \quad \dots (16)$$

(Lower Limb) (Upper Limb)

Considering an element of width dy_1 at a distance y_1 from point b in the inside portion of the lower limb and another element of width dy_2 at a distance y_2 from point b in the outside portion of the lower limb, the energy dissipated in circumferential deformation can be calculated as,

$$\frac{dW_{c1}}{d\theta_1} = \int_0^{mh_1} \left(\left| \frac{d\varepsilon_1}{d\theta_1} \right| \right) dA_1 + \int_0^{(1-m)h_1} \left(\left| \frac{d\varepsilon_2}{d\theta_1} \right| \right) dA_2 \dots (17)$$

where, dA_1 and dA_2 are the area of elemental rings, given by

$$dA_1 = 2\pi \{ R_b - y_1 \sin(\theta_1 + \alpha) \} dy_1 \text{ and}$$

$$dA_2 = 2\pi \{ R_b + y_2 \sin(\theta_1 + \alpha) \} dy_2 \quad \dots (18)$$

and ε_1 and ε_2 are the circumferential strains in the two elements, given by

$$\varepsilon_1 = \frac{2\pi [\{ R_b - y_1 \sin(\theta_1 + \alpha) \}] - (R_b - y_1 \sin\alpha)}{2\pi (R_b - y_1 \sin\alpha)} \dots (19)$$

$$\varepsilon_2 = \frac{2\pi [\{ R_b + y_2 \sin(\theta_1 + \alpha) \}] - (R_b + y_2 \sin\alpha)}{2\pi (R_b + y_2 \sin\alpha)} \dots (20)$$

Differentiating the above two equations, we get,

$$\frac{d\varepsilon_1}{d\theta_1} = -\frac{y_1 \cos(\theta_1 + \alpha)}{R_b - y_1 \sin\alpha} \quad \dots (21)$$

$$\frac{d\varepsilon_2}{d\theta_1} = \frac{y_2 \cos(\theta_1 + \alpha)}{R_b + y_2 \sin\alpha} \quad \dots (22)$$

Using equation (18), (21) and (22), equation (17) gives,

$$\frac{dW_{c1}}{d\theta_1} = 2\pi f_y t_0 \left(\frac{R_b}{\sin\alpha} \right)^2 \left[\left\{ -\ln(A^r B) + (1-m-rm)\frac{h_1 \sin\alpha}{R_b} \right\} \cos(\theta_1 + \alpha) + \left\{ \begin{array}{l} 2\ln(A^r B) + rA^2 + B^2 - (1+r) \\ -4(1-m-rm)\dfrac{h_1 \sin\alpha}{R_b} \end{array} \right\} \frac{\sin 2(\theta_1 + \alpha)}{4\sin\alpha} \right] \dots (23)$$

where, $A = 1 - \dfrac{mh_1 \sin\alpha}{R_b}$ and

$$B = 1 + \frac{(1-m)h_1 \sin\alpha}{R_b} \quad \dots (24)$$

Considering an element of width dy_3 at a distance y_3 from point b in the inside portion of the lower limb and another element of width dy_4 at a distance y_4 from point b in the outside portion of the lower limb, the energy dissipated in circumferential deformation can be calculated as,

$$\frac{dW_{c2}}{d\theta_2} = \int_0^{(1-m)h_2} \left(\left| \frac{d\varepsilon_3}{d\theta_2} \right| \right) dA_3 + \int_0^{mh_2} \left(\left| \frac{d\varepsilon_4}{d\theta_2} \right| \right) dA_4 \dots (25)$$

where, $dA_3 = 2\pi \{ R_d + y_3 \sin(\theta_2 - \alpha) \} dy_3$ and

$$dA_4 = 2\pi \{ R_d - y_4 \sin(\theta_2 - \alpha) \} dy_4 \quad \dots (26)$$

$$\varepsilon_3 = \frac{2\pi [\{ R_d + y_3 \sin(\theta_2 - \alpha) \}] - (R_d - y_3 \sin\alpha)}{2\pi (R_d - y_3 \sin\alpha)} \quad \dots (27)$$

$$\varepsilon_4 = \frac{2\pi [\{ R_d - y_4 \sin(\theta_2 - \alpha) \}] - (R_d + y_4 \sin\alpha)}{2\pi (R_d + y_4 \sin\alpha)} \quad \dots (28)$$

By differentiating equations (27) and (28) we get,

$$\frac{d\varepsilon_3}{d\theta_2} = \frac{y_3 \cos(\theta_2 - \alpha)}{R_d - y_3 \sin\alpha} \quad \dots (29)$$

$$\frac{d\varepsilon_4}{d\theta_2} = -\frac{y_4 \cos(\theta_2 - \alpha)}{R_d + y_4 \sin\alpha} \quad \text{... (30)}$$

Using equations (26), (29) and (30), equation (25) reduces to,

$$\frac{dW_{c2}}{d\theta_2} = -2\pi f_y t_0 \left(\frac{R_d}{\sin\alpha}\right)^2 \left[\left\{ \ln(C^r D) + (1-m-rm)\frac{h_2 \sin\alpha}{R_d} \right\} \cos(\theta_2 - \alpha) + \left\{ \begin{aligned} & 2\ln(C^r D) + rC^2 + D^2 - (1+r) \\ & + 4(1-m-rm)\frac{h_2 \sin\alpha}{R_d} \end{aligned} \right\} \frac{\sin 2(\theta_2 - \alpha)}{4 \sin\alpha} \right] \quad \text{... (31)}$$

where, $C = 1 - \frac{(1-m)h_2 \sin\alpha}{R_d}$ and

$$D = 1 + \frac{m h_2 \sin\alpha}{R_d} \quad \text{... (32)}$$

Using equations (23) and (31), equation (16) gives the energy absorbed in circumferential deformation during the rotation of lower limb upto θ_1 and upper limb upto θ_2 as,

$$W_{C\theta} = 2\pi f_y t_0 \left(\frac{R_b}{\sin\alpha}\right)^2 \left[\left\{ -\ln(A^r B) + (1-m-rm)\frac{h_1 \sin\alpha}{R_b} \right\} \{\sin(\theta_1 + \alpha) - \sin\alpha\} + \left\{ \begin{aligned} & 2\ln(A^r B) + rA^2 + B^2 - (1+r) \\ & - 4(1-m-rm)\frac{h_1 \sin\alpha}{R_b} \end{aligned} \right\} \left\{ \frac{\cos 2\alpha - \cos 2(\theta_1 + \alpha)}{8 \sin\alpha} \right\} \right]$$

$$- 2\pi f_y t_0 \left(\frac{R_d}{\sin\alpha}\right)^2 \left[\left\{ \ln(C^r D) + (1-m-rm)\frac{h_2 \sin\alpha}{R_d} \right\} \{\sin(\theta_2 - \alpha) - \sin\alpha\} + \left\{ \begin{aligned} & 2\ln(C^r D) + rC^2 + D^2 - (1+r) \\ & + 4(1-m-rm)\frac{h_2 \sin\alpha}{R_d} \end{aligned} \right\} \left\{ \frac{\cos 2\alpha - \cos 2(\theta_2 - \alpha)}{8 \sin\alpha} \right\} \right] \quad \text{... (33)}$$

Substituting, $\theta_1 = \frac{\pi}{2} - \alpha$ and $\theta_2 = \frac{\pi}{2} + \alpha$ in the above equation, we get the total energy absorbed in circumferential deformation during complete crushing of the fold as,

$$W_C = 2\pi f_y t_0 \left(\frac{R_b}{\sin\alpha}\right)^2 \left[\left\{ -\ln(A^r B) + (1-m-rm)\frac{h_1 \sin\alpha}{R_b} \right\} (1 - \sin\alpha) + \left\{ \begin{aligned} & 2\ln(A^r B) + rA^2 + B^2 - (1+r) \\ & - 4(1-m-rm)\frac{h_1 \sin\alpha}{R_b} \end{aligned} \right\} \frac{\cos^2\alpha}{4 \sin\alpha} \right]$$

$$- 2\pi f_y t_0 \left(\frac{R_d}{\sin\alpha}\right)^2 \left[\left\{ \ln(C^r D) + (1-m-rm)\frac{h_2 \sin\alpha}{R_d} \right\} (1 - \sin\alpha) + \left\{ \begin{aligned} & 2\ln(C^r D) + rC^2 + D^2 - (1+r) \\ & + 4(1-m-rm)\frac{h_2 \sin\alpha}{R_d} \end{aligned} \right\} \frac{\cos^2\alpha}{4 \sin\alpha} \right] \quad \text{... (34)}$$

which for total outside fold model (i.e. m=0) reduces to,

$$W_c = 2\pi f_y t_0 \frac{1 - \sin\alpha}{\sin^2\alpha} \left[\left\{ R_b{}^2 A_1 - R_d{}^2 A_2 \right\} + \frac{1 + \sin\alpha}{2 \sin\alpha} \left\{ R_b{}^2 B_1 - R_d{}^2 B_2 \right\} \right] \quad \text{... (35)}$$

where,

$$A_1 = \frac{h_1 \sin\alpha}{R_b} - \ln\left(1 + \frac{h_1 \sin\alpha}{R_b}\right) \quad \text{... (36)}$$

$$A_2 = \frac{h_2 \sin\alpha}{R_d} + \ln\left(1 - \frac{h_2 \sin\alpha}{R_b}\right) \quad \text{... (37)}$$

$$B_1 = -\frac{h_1 \sin\alpha}{R_b} + \frac{1}{2}\left(\frac{h_1 \sin\alpha}{R_b}\right)^2 + \ln\left(1 + \frac{h_1 \sin\alpha}{R_b}\right) \quad \text{... (38)}$$

$$B_2 = \frac{h_2 \sin\alpha}{R_d} + \frac{1}{2}\left(\frac{h_2 \sin\alpha}{R_d}\right)^2 + \ln\left(1 - \frac{h_2 \sin\alpha}{R_d}\right) \quad \text{...(39)}$$

Equation (34) is not valid for $\alpha = 0$. This is the case of a cylinder, and to derive the expression for energy absorbed in circumferential deformation, we begin with equation (16) and after integration substitute $\theta_1 = \theta_2 = \pi/2$ and obtain,

$$W_c = 2\pi f_{yt} t_0 h^2 \left[rm^2 \left(1 - \frac{mh}{3R_1}\right) + (1-m)^2 \left\{ 1 + \frac{(1-m)h}{3R_1} \right\} \right] \quad \text{... (40)}$$

which is same as that given in Ref.[5] for cylinder of radius R_1 and size of fold, h.

Average crushing load

Assuming that the energy dissipation in the axi-symmetric axial crushing of frusta takes place in the form of flexural and circumferential deformations, therefore, the external work done can be equated to the energy absorbed in bending and circumferential stretching. The average crushing load, P_m, can, therefore, be calculated:

$$P_m = \frac{W_b + W_c}{(1+K)h_2 \cos\alpha} \qquad \text{... (41)}$$

where, W_b is given by equation (13) and W_c is given by equation (34) for frusta and equation (40) for tubes.

Size of Fold and folding parameter, m

Determination of the size of fold, h_1 and h_2, and the folding parameter, m, requires the minimization of external work done for crushing unit length of frusta during the fold formation or the minimization of average crushing load of the fold i.e.,

$$\frac{\partial P_m}{\partial h_2} = 0 \quad \text{and} \quad \frac{\partial P_m}{\partial m} = 0 \qquad \text{... (42)}$$

where, P_m is given by Eq. (41).

Variation of Crushing Load

The variation of crushing load can now be found from the following relation

$$P_\theta = \frac{d(W_{b\theta} + W_{c\theta})}{dz} = \frac{\frac{d}{d\theta}(W_{b\theta} + W_{c\theta})}{\frac{dz}{d\theta}} \qquad \text{... (43)}$$

where, $W_{b\theta}$ and $W_{c\theta}$ are the work done in bending and circumferential stretching in rotation of lower limb of fold upto θ_1 and upper limb upto θ_2 given by equations (12) and (33); and z is the crushing distance in the direction of the load which is given by,

$$z = h_2 \{(1+K)\cos\alpha - K\cos(\theta_1 + \alpha) - \cos(\theta_2 - \alpha)\} \qquad \text{... (44)}$$

therefore,

$$\frac{dz}{d\theta_1} = Kh_2 \sin(\theta_1 + \alpha) + h_2 \sin(\theta_2 - \alpha)\frac{d\theta_2}{d\theta_1} \qquad \text{... (45)}$$

where, $$\frac{d\theta_2}{d\theta_1} = \frac{K\cos(\theta_1 + \alpha)}{\cos(\theta_2 - \alpha)} \qquad \text{... (46)}$$

Comparison with Experimental Observations

An Aluminium frustum, 1.85 *mm* thick, 130.3 *mm* long, and with end diameters of 43.9 and 57.5 *mm*, tested in axial compression[7] has been taken for experimental validation. The value of yield strength of the material of the frustum was found to be 92.0 *MPa*. The crushing load variation obtained experimentally for first fold has been plotted in Fig. 2. The variation of non-dimensional crushing load $\boldsymbol{P_\theta}/P_0$ for first fold (i.e. a = 0) and for folds other than first fold (i.e. a = 1) has also been plotted in this figure, where, $P_0 = \pi D f_{yt} t$. The analytical load-deformation curves do not start from zero load level due to the neglect of the elastic deformation in the beginning. It is observed that the consideration of first fold brings the calculated curve close to experimental curve.

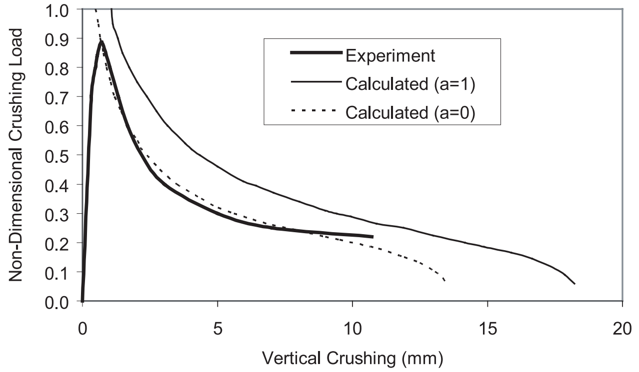

Figure 2: *Load deformation curves for frustum*

One steel cylindrical tube of 43.0 mm diameter and 1.8 mm thick tested in axial compression[6] has been used for the validation of the analysis presented in earlier sections. A parametric study has also been carried out for studying the influence of the difference in the compressive and tensile strength of both the materials by taking the parameter r as 1.0, 1.5 and 2.0. The values of size of fold and folding parameter were first determined numerically and these values were used for finding out the variation of crushing load. The variation of non-dimensional crushing load $\boldsymbol{P_\theta}/P_0$ with r = 1.0 along with the experimental curve has been plotted in figure 3.

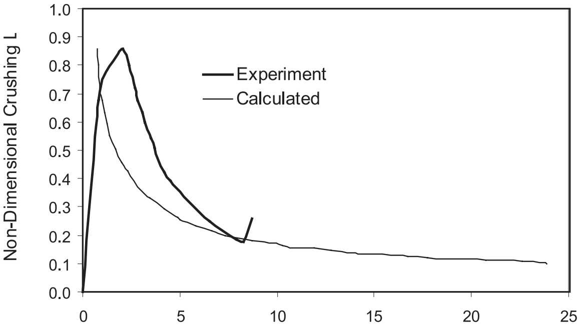

Figure 3: *Load deformation curve of a steel tube of D = 43.0 mm, t = 1.8 mm*

Conclusions

A mathematical model for the axi-symmetric crushing of frusta, normally observed in frusta of low semi-apical angles, with partly inside and partly outside folding has been presented. The existing total outside fold model of frusta and partly inside and partly outside fold model of tube can be derived from this model. The variation in crushing load and mean collapse load have been computed.

The neglect of lower hinge in the first fold of frusta, which is based on the experiments wherein it has been observed that the first limb of the first fold remains straight, brings the calculated crushing load curve close to the experimental curve.

The results have been compared with experiments and good agreement has been observed. The results are of help in understanding the phenomenon of actual fold formation.

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