Research Paper

On the Kinetic Energy Spectra of Turbulence in the Thermally Stratified Atmospheric Surface Layer

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Turbulence energy spectra in a thermally stratified turbulent shear flow are determined by solving the spectral equation for turbulent kinetic energy budget in a simplified framework. A brief account of the Monin-Obukhov similarity theory, as appropriate for the description of such a turbulent flow is given. Following the classical statistical approach to the turbulent flow, the equation for the turbulent kinetic energy budget in the physical space and its version in the wave number space are derived in a straightforward manner. The closure of the spectral equation is achieved through the modeling of the terms describing production of the turbulent kinetic energy from the mean velocity shear, energy transfer through hierarchy of the eddies and the contribution of the buoyancy forces towards turbulent kinetic energy. A complete solution for the three-dimensional energy spectrum and the two one-dimensional energy spectra corresponding to the turbulent velocity components in the horizontal and vertical directions are obtained. All these spectra are evaluated, with the data from well known simulated experiment on the stably stratified atmospheric boundary layer in a meteorological wind tunnel. The behaviour of these spectra over a wide range of wave numbers, which includes both inertial and dissipation ranges, is discussed.

Key Words: Monin-Obukhov similarity theory, thermally stratified flow, turbulent kinetic energy spectrum, heat flux or buoyancy spectrum, viscous energy dissipation

Introduction

The study of turbulence phenomena in the atmospheric surface layer is considered important as it involves transportation of momentum, heat flux, moisture and dispersion of pollutants etc. The turbulence processes are of high interest to meteorologists as they comprise, generally of wind shear and thermal stratification as a minimum complexity [1]. Vertical mixing of eddies occurs in the layer adjacent to earth's surface and it is termed as the atmospheric boundary layer. In case of horizontally homogeneous turbulent boundary layer over a flat earth's surface, vertical flux of horizontal momentum $-\rho\overline{u'w'}$ and vertical flux of heat $\rho C_p \overline{w'T'}$ are important, where u' and w' are the fluctuating components of velocity, respectively in the horizontal and vertical directions, T' is the temperature fluctuation, ρ and C_p, are respectively the density and specific heat capacity of air. The velocity and temperature spectra and the related cospectra were calculated with data from different sites, heights and thermal instabilities, and analysed by several authors [2,3,4] within the framework of Monin-Obukhov similarity theory [5]. Experiments [6] were carried out in the large meteorological wind tunnel of Colorado State University on a stably stratified turbulent boundary layer developing over a cold plate. Turbulent energy spectra were calculated from the data of these experiments and analysed [7] also within the framework of similarity theory [5].

In the present paper, we obtain the general solution of the spectral equation of the kinetic energy balance [4,7] in a stably stratified turbulent flow and evaluate the three-dimensional kinetic energy spectrum and as well as kinetic energy spectra corresponding to fluctuating velocity components in the horizontal and vertical directions using the values of different similarity parameters from the experimental data [8].

Surface-Layer Similarity Framework

We concentrate our analysis here to the case of a statistically stationary and horizontally homogeneous turbulent flow under stable stratification. The similarity theory proposed by Monin and Obukhov [5] is considered to be appropriate for the description of this type of turbulent flow. Their similarity theory is based on the assumption that the flow is plane homogenous and the turbulent fluxes of momentum and heat are constant (independent of height). Out of the four independent variable e.g., z the height measured from the ground, $\frac{\tau_0}{\rho}$ the surface drag, $\frac{H_0}{\rho C_p}$ the surface

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kinematic heat flux and $\frac{g}{T_o}$ the stability parameter, a unique combination $\frac{z}{L}$ (the stability ratio) is constructed [5], where T_o is the reference temperature and L the Monin-Obukhov length scale, given by

$$L = -\frac{u_*^3}{\kappa \frac{g}{T_a} \cdot \frac{H_0}{\rho C_p}} \; ; \quad \text{...(1)}$$

$$u_* = \left(\tau_0 / \rho\right)^{\frac{1}{2}} \; ; \; H_0 = -\rho C_p u_* \theta_*$$

The velocity scale u_*, temperature scale θ_* and length scale L are assumed to specify the dynamics of the flow field. κ is the well known Von Ka′rma′n constant, C_p is as usual the specific heat of air at constant pressure and T_a is the absolute average temperature of the layer under consideration.

According to Monin-Obukhov similarity theory [5], all the mean flow and turbulent quantities, when non-dimensionalized by the appropriate combination of u_*, θ_* and L must be universal function of $\frac{z}{L}$. The dimensionless mean velocity gradient and the gradient of mean potential temperature are expressed as

$$\frac{\kappa z}{u_*}\left(\frac{\partial \overline{U}}{\partial z}\right) = \phi_m(\zeta) \quad \text{...(2)}$$

$$\frac{\kappa z}{\theta_*}\left(\frac{\partial \overline{\theta}}{\partial z}\right) = \phi_h(\zeta) \quad \text{...(3)}$$

where $\zeta = \frac{z}{L}$. $\phi_m(\zeta)$ and $\phi_h(\zeta)$ are the universal similarity functions which relate the constant fluxes $\tau = \tau_0 = \rho u_*^2$ and $H = H_0 = -\rho c_p u_* \theta_*$ to the mean gradient in the surface layer. It is to be noted that in the lowest layer close to the surface $(z << |L|)$, effects of wind shear usually dominate and effects of buoyancy become insignificant. Whereas, buoyancy effects may dominate over shear-generated turbulence for $(z >> |L|)$. Thus if the turbulent heat flux H be considered to decrease without limit and accordingly approaching the conditions of neutral stratification, L will increase in absolute value without bound, so that $\zeta = \frac{z}{L}$ tends to zero. In the limit, as $H \to 0$, we may recover the well known logarithmic equation from (2), as

$$\frac{d\overline{U}}{dz} = \frac{u_*}{\kappa z} \quad \text{...(4)}$$

with $\phi_m(0) = 1$. Equation (4) contains neither H nor $\frac{g}{T_0}$ and usual ‘Law of the wall’ can be derived from it.

Further, according to similarity theory [5], the Richardson number R_i, defined by

$$Ri = \frac{g}{\overline{\theta}} \frac{d\overline{\theta}/dz}{\left(d\overline{U}/dz\right)^2} \quad \text{...(5)}$$

is also universal function of $\frac{z}{L}$. The flux Richardson number is defined by $R_f = \alpha R_i$, where $\alpha = \frac{k_h}{k_m}$ is the ratio of the coefficients of turbulent mixing for heat and momentum.

It can be easily shown from the definition of ζ, L and the set of equations (2) and (3) that

$$R_i = \zeta \phi_h(\zeta) / \phi_m^2(\zeta) \quad \text{...(6)}$$

The forms for ϕ_m and ϕ_h have not been given in details in the similarity theory. Some possible forms for ϕ_m and ϕ_h, as emerged from practical measurements have been discussed in details in reference [9]. In the neutral limit i.e., for $\frac{z}{L} = 0$, ϕ_m must be constant (taken to be unity). ϕ_h is also constant and its value is expected to be determined from experiments. For $\frac{z}{L} < 0$, the stratification is unstable and for $\frac{z}{L} > 0$, stable stratification occurs. For practical purposes, Businger-Dyer flux-profile relationships [7,10] e.g. ,

$$\phi_m = 1 + \left(\frac{4.7z}{L}\right) \quad \text{for} \quad \frac{z}{L} > 0$$

$$\phi_m = 1 \quad \text{for} \quad \frac{z}{L} = 0 \quad \text{...(7)}$$

$$\phi_m = \left[1 - \left(\frac{15z}{L}\right)\right]^{-1/4} \quad \text{for} \quad \frac{z}{L} < 0$$

are found to be useful.

Similar forms have been estimated for the heat flux vs. the virtual potential temperature profile:

$$\phi_h = \frac{k_m}{k_h} + \frac{4.7z}{L} \quad \text{for} \quad \frac{z}{L} > 0$$

$$\phi_h = \frac{k_m}{k_h} \quad \text{for} \quad \frac{z}{L} = 0 \qquad \text{...(8)}$$

$$\phi_h = \frac{k_m}{k_h}\left[1 - \frac{9z}{L}\right]^{-1/4} \quad \text{for} \quad \frac{z}{L} < 0$$

The similarity form for the total rate of energy dissipation ε, is given by [4,5]

$$\varepsilon = \frac{u_*^3}{\kappa z}\phi_\varepsilon\left(\frac{z}{L}\right) \qquad \text{...(9)}$$

For stably stratified turbulent flow, concerned here, ϕ_ε has the form:

$$\phi_\varepsilon = \left(1 + 2.5\left|\frac{z}{L}\right|^{3/5}\right)^{3/2} \qquad \text{...(10)}$$

Spectral Equation for the Balance of Turbulent Kinetic Energy

Knowledge of the spectral behavior of the variables involved in the atmospheric turbulence processes is useful in many respects. Primarily information on the scales of motion that contribute to the production and dissipation of energy and the transport of various atmospheric properties, can be obtained from the spectra and co-spectra of turbulence. Also, decomposition of a series of measurements into frequency or wave number components can be effected to know how eddies of different time and space scales contribute to the overall turbulence state. Here, we shall embark upon the decomposition of the turbulent kinetic energy equation in a thermally stratified homogeneous, turbulent flow.

We begin with the equation for the second-order correlation between the fluctuating velocity components, pertaining to the space points, say $P(\vec{X})$ and $P'(\vec{X}')$ at the same instant of time in such a turbulent flow:

$$\frac{\partial}{\partial t}\overline{u_i u'_k} + \left(\overline{U'_j} - \overline{U}_j\right)\frac{\partial}{\partial r_j}\overline{u_i u'_k} + \overline{u_j u'_k}\frac{\partial \overline{U}_i}{\partial x_j} + \overline{u_i u'_j}\frac{\partial \overline{U}'_k}{\partial r_j} + \frac{\partial}{\partial r_j}\left(\overline{u_i u'_j u'_k} + \overline{u_i u_j u'_k}\right)$$
$$= \frac{1}{\rho_0}\left(\frac{\partial}{\partial r_k}\overline{p' u_i} - \frac{\partial}{\partial r_i}\overline{p u'_k}\right) + \frac{g}{T_0}\left(\overline{T u'_k}\delta_{i3} + \overline{T' u_i}\delta_{k3}\right) + 2\nu\frac{\partial^2}{\partial r_j \partial r_j}\overline{u_i u'_k} \qquad \text{...(11)}$$

In obtaining equation (11), it is assumed that the instantaneous turbulent quantities can be split up, following Reynolds' [11] concept, into mean and fluctuating parts e.g.,

$$\tilde{u}_i = \overline{U}_i + u_i \;;\quad \tilde{p} = \overline{p} + p \;;\quad \tilde{T} = \overline{T} + T$$

and due to homogeneity condition :

$$-\frac{\partial}{\partial x_j} = \frac{\partial}{\partial r_j} = \frac{\partial}{\partial x'_j} \;;\quad \nabla_X^2 = \nabla_{X'}^2 = \frac{\partial^2}{\partial r_j \partial r_j} \;;$$

$$\frac{\partial \overline{U}_j}{\partial x_j} = \frac{\partial u'_j}{\partial x_j} = 0$$

where $r_j = X'_j - X_j$.

Assuming the motion of the wind to be flat down x-axis and with vertical shear, we may write

$$\overline{U}_2 = \overline{U}_3 = 0 \;,\quad \overline{U}_1 = \overline{U}_1(z) \;,\quad \frac{\partial \overline{U}_1}{\partial x_3} = \frac{d\overline{U}}{dz} \;,$$

say ...(12)

Let us introduce now the following Fourier transform relation for the present problem:

$$\overline{u_i u'_k} = \int E_{ik}(\vec{k})e^{i(\vec{k}\cdot\vec{r})}d\vec{k} \;;\quad i^2 = -1 \qquad \text{...(13a)}$$

$$-\frac{\partial}{\partial r_j}\left(\overline{u_i u'_j u'_k} - \overline{u_i u_j u'_k}\right) = \int w_{ik}(\vec{k})e^{i(\vec{k}\cdot\vec{r})}d\vec{k} \qquad \text{...(13b)}$$

$$\frac{1}{\rho_0}\left(\frac{\partial}{\partial r_k}\overline{p' u_i} - \frac{\partial}{\partial r_i}\overline{p u'_k}\right) = \int \pi_{ik}^{p}(\vec{k})e^{i(\vec{k}\cdot\vec{r})}d\vec{k} \qquad \text{...(13c)}$$

$$\left(\overline{T u'_k}\delta_{i3} + \overline{T' u_i}\delta_{k3}\right) = \int \pi_{ik}^{T}(\vec{k})e^{i(\vec{k}\cdot\vec{r})}d\vec{k} \qquad \text{...(13d)}$$

Taking the relations (12) and (13a,b,c,d) into account in equation (11), we derive, after some calculations

$$\frac{\partial}{\partial t}E_{ik} + \left(\delta_{i1}E_{3k} + \delta_{k1}E_{i3} - k_1\frac{\partial}{\partial k_3}E_{ik}\right)\frac{d\overline{U}}{dz} - w_{ik} = \pi_{ik}^{p} + \frac{g}{T_0}\pi_{ik}^{T} - 2\nu k^2 E_{ik} \qquad \text{...(14)}$$

In equation (14), the terms E_{ik}, w_{ik}, π_{ik}^{p} and π_{ik}^{T} are functions of $\vec{k}$. Applying a contraction with respect to i and k, we obtain $\pi_{ii}^{P} = 0$ due to homogeneity of the turbulence and, accordingly the dynamic equation for E_{ii} as

$$\frac{\partial}{\partial t}E_{ii} + \left(2E_{13} - k_1\frac{\partial}{\partial k_3}E_{ii}\right)\frac{d\overline{U}}{dz} - w_{ii} = \frac{g}{T_0}\pi_{ii}^{T} - 2\nu k^2 E_{ii} \qquad \text{...(15)}$$

In order to express the spectrum functions of (15) in terms of the scalar wave number k, we average them over all directions of $\vec{k}$ in the wave number space i.e., we take mean values of the spectrum functions over spherical surface of radius k = constant. For example,

$$\tilde{E}_{ik}(k) = \frac{1}{4\pi k^2} \int_{k=|\vec{k}|} E_{ik}(\vec{k})\, d\sigma(k) \; ;$$

$$|\vec{k}| = k \qquad \text{...(16)}$$

where $d\sigma(k)$ is an element of the spherical surface of radius k. Thus, effecting spherical averaging, we obtain the transformed version of equation (15), as

$$\frac{\partial}{\partial t}\tilde{E}_{ii} + \left(2\tilde{E}_{13} - k_1 \frac{\partial}{\partial k_3}\tilde{E}_{ii}\right)\frac{d\overline{U}}{dz} - \tilde{W}_{ii} = \frac{g}{T_0}\tilde{\pi}_{ii}^{T} - 2\nu k^2 \tilde{E}_{ii} \qquad \text{...(17)}$$

We now introduce the new spectrum functions [12]:

$$E(k) = 4\pi k^2\left(\frac{1}{2}\tilde{E}_{ii}\right) \; ; \quad F(k) = 4\pi k^2\left(\frac{1}{2}\tilde{W}_{ii}\right)$$

$$\phi(k) = 4\pi k^2\left(\frac{k_1}{2}\frac{\partial}{\partial k_3}\tilde{E}_{ii}\right) \; ; \quad E_{wT}(k,t) = 4\pi k^2\left(\frac{1}{2}\tilde{\pi}_{ii}^{T}\right)$$

$$E_{uw}(k) = 4\pi k^2 \tilde{E}_{13} \qquad \text{...(18)}$$

In view of the relations (18), equation (17) reduces to

$$\frac{\partial}{\partial t}E + \left(E_{uw} - \phi\right)\frac{d\overline{U}}{dz} - F = \frac{g}{T_0}E_{wT} - 2\nu k^2 E \qquad \text{...(19)}$$

Since the transfer to energy due to deformation of the mean flow is rather smaller than the transfer of energy due to inertial forces (except, may be at the smallest wave number or at nonhomogeneity condition), the term involving ϕ can be neglected, and in the steady state equation (19) is simplified to :

$$E_{uw}\frac{d\overline{U}}{dz} - F = \frac{g}{T_0}E_{wT} - 2\nu k^2 E \qquad \text{...(20)}$$

Integrating equation (20) from k to ∞, we obtain

$$\frac{d\overline{U}}{dz}\int_k^\infty E_{uw}(k')\,dk' - \int_k^\infty F(k')\,dk' = \frac{g}{T_0}\int_k^\infty E_{wT}(k')\,dk' - 2\nu\int_k^\infty E(k')\,k'^2\,dk' \qquad \text{...(21)}$$

Now invoking the expression for the total rate of energy dissipation e.g.,

$$\varepsilon = 2\nu\int_0^\infty k'^2 E(k')\,dk' \qquad \text{...(22)}$$

in equation (21), we obtain, after some simple manipulation

$$\varepsilon = -\frac{d\overline{U}}{dz}\int_k^\infty E_{uw}(k')\,dk' + G(k) + \frac{g}{T_0}\int_k^\infty E_{wT}(k')\,dk' + 2\nu\int_0^k k'^2 E(k')\,dk' \qquad \text{...(23)}$$

where $G(k) = \int_k^\infty F(k')\,dk' \left(= -\int_0^k F(k')\,dk' \right)$,

Since, due to assumption of homogeneity of the turbulence $\int_0^\infty F(k')\,dk' = 0$

Equation (23) is the spectral version of the equation for the turbulent kinetic energy in a horizontally homogenous, stratified turbulent flow.

Two important points [13] are to be noted :

i) In case if the temperature in homogeneities are small in comparison with the mean temperature, we can write $T_0 = \overline{T}$

ii) Strictly speaking, in ordinary stratified flow $\frac{\partial\theta}{\partial z} = \frac{\partial T}{\partial z} - G_a$, where θ and T are the potential temperature and the ordinary temperature, and G_a is the adiabatic temperature gradient, given by $G_a = \frac{g}{C_p}$. In many situations G_a is negligible in comparison with $\frac{\partial T}{\partial z}$ e.g., in few tens of meters of the earth's atmosphere the typical values of the vertical temperature gradient are usually some hundreds of times greater than G_a. Thus, in many occasions it is possible to make no distinction between the ordinary temperature and the potential temperature [13].

Modeling of Transfer, Production and Buoyancy terms

In thus section, our aim is to achieve closure of the system of equation (23) in terms of turbulent kinetic energy spectrum E(k). The first and the second term on the right hand side of equation describe, respectively the production of turbulent kinetic energy by the mean velocity shear and the transfer of turbulent kinetic energy through hierarchy of eddies. The third term describes the contribution of the buoyancy forces towards the turbulent kinetic energy. The last term describes the dissipation of turbulent kinetic energy under the influence of viscosity. The homogenous, stratified turbulence under consideration, may generally be pictured as follows: the energy produced by the mean velocity shear is fed into the energy containing range of the spectrum of turbulence and extracted by the buoyancy forces (under stable stratification). The net energy is then transferred through a cascade process to larger and larger wave numbers until ultimately the rate of viscous dissipation become significant.

For spectral closure, some relation among the nonlinear transfer spectrum $G(k)$, the energy spectrum $E(k)$ and the wave number k is necessary. Based on the intuitive physical picture of the mechanism of energy transfer across the wavenumbers and / or dimensional reasoning, a large number of proposals for G(k) have been made. We discuss here some interesting forms for $G(k)$:

***(i) Obukhov* [14]:**

$$G(k)=\alpha_O\left\{\int_0^k k'^2 E(k')dk'\right\}^{1/2}.\int_k^\infty E(k')dk'$$

***(ii) Heisemberg* [15]:**

$$G(k)=\alpha_H\left\{\int_0^k k'^2 E(k')dk'\right\}.\int_k^\infty \sqrt{\frac{E(k')}{k'^3}}dk'$$

***(iii) Kovasznay* [16]:**

$$G(k)=\alpha_K k^{5/2}[E(k)]^{3/2}$$

***(iv) Modified Obukhov* [17]:**

$$G(k)=\alpha_E\left\{\int_0^k k'^2 E(k')dk'\right\}^{1/2}.kE(k)$$

***(v) Pao* [18]:**

$$G(k)=\alpha_P\varepsilon^{1/3}k^{5/3}E(k)$$

where α_O, α_H, α_K, α_E and α_P are constants.

The physical idea behind Obukhov's from (i) for $G(k)$ is that the small eddies act on the large eddies like a Reynolds stress. Whereas, Heisenberg's from (ii) is based on the assumption that the small eddies act like an eddy-viscosity on the big eddies. Both these forms express $G(k)$ as product of two terms, one involving an integral over the wavenumber range 0 to k and the other involving an integral over the wavenumber range k to ∞. The second term represents the effects of the smaller eddies corresponding to wavenumbers $k'' > k$ on the eddy corresponding to wavenumber k. While the first term represents the transfer of energy from the bigger eddies corresponding to wavenumbers $k' < k$ to the eddy corresponding to wavenumber k. Thus, we may envisage an interaction between two eddies corresponding to $k'' > k$ and $k' < k$. However, this is not satisfactory from a theoretical point of view.

In Kovasznay' proposal (iii), $G(k)$ is assumed to depend on k and the local value of $E(k)$. It takes into account only the dimension of $G(k)$ and does not involve any physical consideration.

As some of the consequence of the Obukhov's original from (i) are physically unlikely, Ellison replaced the factor $\int_k^\infty E(k'')dk''$ by $kE(k)$ and suggested the modified obukhov from (iv) for $G(k)$. According to this form, the Reynolds stress interacting with the mean rate of shear or vorticity of the wavenumbers smaller than k is entirely restricted to k.

Pao's from (v) for $G(k)$ is based on a continuous cascading concept of turbulent kinetic energy in a decaying homogeneous turbulent flow. According to Pao [14] if $\xi(k)$ represents the rate of transfer of energy spectral element E(k) across k, then G(k) can be put as

G(k) = $\xi(k)$E(k), where $\xi(k)$ would not depend on E(k) but depend only on ε and k. From dimensional consideration

$$\xi(k)=P^{-1}\varepsilon^{1/3}k^{5/3}E(k)$$

and hence

$$G(k)=P^{-1}\varepsilon^{1/3}k^{5/3}E(k), \qquad ...(24)$$

where P is the Kolmogorov constant [19].

Although a question may be raised that why $\xi(k)$ should not depend on ν also, besides its dependence on ε and k but the proposal (v) leads to an expression for $E(k)$ which is in better agreement with the available experimental data than most of the others. We accept Pao's form for $G(k)$, for the present analysis.

Further, the energy transfer spectrum G(k) can be expressed in the form :

$$G(k)=\nu_T(\text{Vorticity})^2$$

Since, the velocity is approximated [19] by $(\varepsilon k^2)^{1/3}$, the turbulent kinematic viscosity ν_T takes the form :

$$\nu_T = P^{-1}k^{1/3}\varepsilon^{-1/3}E(k)$$

In Boussinesq's sense, ν_T must be expressible as the product of a characteristic length l_k, and a characteristic velocity v_k we may write $\nu_T \sim l_k v_k$. Now as v_k can be determined [19] by approximating the amount of kinetic energy around a wave number k as $v_k = [kE(k)]^{1/2}$, l_k is estimated from (25) as $l_k \sim \left[k^{-1/3}\varepsilon^{-2/3}E(k)\right]^{1/2}$. The time scale t_k, as follows from l_k and v_k e.g., $t_k \sim \varepsilon^{-1/3}k^{-2/3}$ is the same as that conjectured by Lin[21] for the cascade of turbulent kinetic energy. Assuming the interaction between the mean and turbulent motions to be weak, the production and buoyancy terms i.e., the first and the third terms on the right hand side of (23) are modeled, as

$$\frac{d\overline{U}}{dz}\int_k^\infty E_{uw}(k')\,dk' = -\nu_T\left(\frac{d\overline{U}}{dz}\right)^2 \quad \text{...(26)}$$

$$\beta\int_k^\infty E_{wT}(k')\,dk' = -\nu_T^*\left(\beta\frac{d\overline{T}}{dz}\right) \quad \text{...(27)}$$

where $\beta = \frac{g}{T_0}$ and $\nu_T^* = \alpha\nu_T$.

It is also to be noted that $\beta\frac{d\overline{T}}{dz}$ has the dimension of square of the vorticity.

Substituting (24), (26) and (27) in equation (23), we obtain the desired form of spectral equation for the turbulent kinetic energy in a stratified, horizontally homogeneous turbulent flow, as

$$\varepsilon = 2\nu\int_0^k k'E(k')\,dk' + P^{-1}\varepsilon^{1/3}k^{5/3}E(k) + bP^{-1}\varepsilon^{-1/3}k^{1/3}E(k) \quad \text{...(28)}$$

where $b = \left(\frac{d\overline{U}}{dz}\right)^2 - \alpha\beta\frac{d\overline{T}}{dz}$

It is to be pointed out that 'b' can be written in the form :

$$b = N^2\left(\frac{1-\alpha R_i}{R_i}\right) \quad \text{...(29)}$$

where $N = \left(\beta\frac{d\overline{T}}{dz}\right)^{1/2}$ is called the Brunt-Väisälä frequency.

Solution for the Turbulent Energy Spectra

To find energy spectrum E(k), we differentiate equation (28) with respect to k and obtain

$$\frac{d\left[k^{5/3}E(k)\right]}{k^{5/3}E(k)} = \frac{\frac{4}{3}b\varepsilon^{-1/3}k^{-7/3} - 2\nu Pk^{1/3}}{\varepsilon^{1/3} + b\varepsilon^{-1/3}k^{-4/3}}dk \quad \text{...(30)}$$

Solution of (30), is given by

$$\log\left[k^{5/3}E(k)\right] = \begin{bmatrix} -\frac{3\nu P}{2\varepsilon^{1/3}}k^{4/3} + \frac{4}{3}\log k \\ -\left\{\frac{2b\varepsilon^{1/3} + 3\nu Pb^2\varepsilon^{-2/3}}{2\varepsilon^{\frac{1}{3}}b}\right\} \\ \log\left\{b\varepsilon^{-1/3} + \varepsilon^{1/3}k^{4/3}\right\} \end{bmatrix} + C \quad \text{...(31)}$$

To determine the constant of integration C, we put E(k)=E(k_d) at the Kolmogovov wave number $k = k_d$, where

$$k_d = \left(\varepsilon/\nu^3\right)^{\frac{1}{4}} \text{ and } E(k_d) = \left(\varepsilon\nu^5\right)^{\frac{1}{4}}. \quad \text{...(32)}$$

Determining C, we obtain after some algebraic calculation the following equation in non-dimensional form :

$$\log\left[\hat{k}^{5/3}\hat{E}(\hat{k})\right] = -\frac{3}{2}P\left[\hat{k}^{4/3} - 1\right] + \frac{4}{3}\log\hat{k} - \left\{\frac{2\varepsilon^{1/3} + 3\nu Pb\varepsilon^{-2/3}}{2\varepsilon^{1/3}}\right\}\log\left[\frac{b\varepsilon^{-2/3}\nu + \varepsilon^{1/3}\hat{k}^{4/3}}{b\varepsilon^{-2/3}\nu + \varepsilon^{1/3}}\right] \quad \text{...(33)}$$

where $\hat{k} = \frac{k}{k_d}$ and $\hat{E}(\hat{k}) = \frac{E(k)}{E(k_d)}$

The three-dimensional energy spectrum function $\hat{E}(\hat{k})$ can be computed from (33), once the values of the parameters ν, ε, b, N and the constant P and α are determined. One-dimensional spectra which are important for practical applications can be obtained from (33). As observed turbulence spectra appear to satisfy the assumptions about isotropy at the very high frequency of the inertial and dissipation sub ranges [22], we accept the following relations between one-dimensional energy spectra and three-dimensional energy spectrum:

$$\hat{E}_{11}\left(=\frac{E_{11}}{E(k_d)}\right) = \int_{\hat{k}}^\infty\left(1-\frac{\hat{k}^2}{\hat{\eta}^2}\right)\frac{\hat{E}(\hat{\eta})}{\hat{\eta}}d\hat{\eta} \quad \text{...(34)}$$

and

$$\hat{E}_{33}\left(=\frac{E_{33}}{E(k_d)}\right)=\hat{E}_{22}\left(=\frac{E_{22}}{E(k_d)}\right)=\frac{1}{2}\int_{\hat{k}}^{\infty}\left(1+\frac{\hat{k}^2}{\hat{\eta}^2}\right)\frac{\hat{E}(\hat{\eta})}{\hat{\eta}}d\hat{\eta} \quad ...(35)$$

where $\hat{E}_{11}, \hat{E}_{22}$ *and* $\hat{E}_{33}$ are, respectively the non-dimensional u-component, v-component and w-component energy spectra.

Putting $\sigma=\frac{\hat{k}}{\hat{\eta}}$ and noting the conditions $\sigma\to 1$ as $\hat{\eta}\to\hat{k}$ and $\sigma\to 0$ as $\hat{\eta}\to\infty$, the relations (34) and (35) are transformed to:

$$\hat{E}_{11}=\int_0^1\frac{(1-\sigma^2)}{\sigma}\hat{E}\left(\frac{\hat{k}}{\sigma}\right)d\sigma \quad ...(36)$$

and

$$\hat{E}_{33}=\frac{1}{2}\int_0^1\frac{(1+\sigma^2)}{\sigma}\hat{E}\left(\frac{\hat{k}}{\sigma}\right)d\sigma \quad ...(37)$$

where $\hat{E}\left(\frac{\hat{k}}{\sigma}\right)$ is expressed as

$$\hat{E}\left(\frac{\hat{k}}{\sigma}\right)=Exp\left\{\begin{array}{l}-\frac{3}{2}P\sigma^{\frac{-4}{3}}\left(\hat{k}^{\frac{4}{3}}-\sigma^{\frac{4}{3}}\right)\\ -\frac{1}{3}\left(\log\hat{k}-\log\sigma\right)-\left[1+\frac{3}{2}P\frac{b\nu}{\varepsilon}\right]\\ \log\left[\frac{\sigma^{\frac{-4}{3}}\left(\hat{k}^{\frac{4}{3}}-\frac{b\nu}{\varepsilon}\sigma^{\frac{4}{3}}\right)}{1+(b\nu/\varepsilon)}\right]\end{array}\right\} \quad ...(38)$$

Computation of the Turbulence Energy Spectra

In order to compute the three-dimensional energy spectrum $\hat{E}$ and the two one-dimensional energy spectra $\hat{E}_{11}$ and $\hat{E}_{33}$, we consider the values of $\frac{z}{L}, \varepsilon, R_f(=\alpha R_i)$ and N as presented by plate and Arya[8] for three different sets of values of u_∞, u_* and L. For the sake of completeness, we include relevant part of their table, comprising data of different parameters, used in the present calculations.

The values of the constant P are chosen [18] as 1.70. The value of α is taken as 1.35 (Kansas data). For the Von Ka′rma′n constant, we take its standard value, as $\kappa = 0.42$.

Now, the values of 'b' can be estimated for different cases from its formula (29) e.g., $b=N^2\left(\frac{1-\alpha R_i}{R_i}\right)$. Three-dimensional energy spectrum $\hat{E}(\hat{k})$ is calculated from equation (33) for different cases with $\frac{z}{L}$, as given below (table I). The plots of $\log\hat{E}$ vs. $\log\hat{k}$ are shown in figures 1-3, for the entire range of wavenumbers $\hat{k}<\frac{k_b}{k_d}$, $\hat{k}\geq\frac{k_b}{k_d}$.

Inspection of figures 1-3 indicates that at low wavenumbers the effects of buoyancy on $\hat{E}$ are significant in each of the cases studied. Such effects on $\hat{E}$ are found to increase with the increase of $\frac{z}{L}$ in each of the cases at lower wavenumbers. However buoyancy effects decrease with the increase of wavenumber. It may be observed further that there is tendency of the curves to collapse at higher wavenumbers. Exact collapsing of the curves for $\hat{E}$, corresponding to different $\frac{z}{L}$ is found to occur at higher wavenumbers for the first group of data (Fig.1). The energy spectrum $\hat{E}$ exhibits $\hat{k}^{-5/3}$ dependence in each of the cases studied here, but over very short inertial sub ranges only. The deviation from the $\hat{k}^{-5/3}$ law can be seen at slightly larger scales (Figures 1-3), wherein the energy spectrum $\hat{E}$ is found to obey $\hat{k}^{-1}$ power law [6,19]

We compute one-dimensional energy spectra $\hat{E}_{11}$ and $\hat{E}_{33}$, respectively from the equations (36) and (37), selecting the values of one set of the parameters $\frac{z}{L}, \varepsilon, R_f$ and N^2 from each of the three groups of data (Table I). The plotting of $log\hat{E}_{11}$ *vs.* $log\hat{k}$ and $log\hat{E}_{33}$ *vs.* $log\hat{k}$ for the entire range of wavenumbers $\hat{k}<\frac{k_b}{k_d}$, $\hat{k}\geq\frac{k_b}{k_d}$ are shown, respectively in figures 4-6. The energy spectra $\hat{E}_{11}$ and $\hat{E}_{33}$ exhibit the expected behaviour, in respect of the corresponding curves of the three-dimensional energy spectrum $\hat{E}$. As of practical utility, one-dimensional spectra are ought to receive much more attention and, accordingly would be the subject of separate study.

All the energy spectra studied correspond to small but positive values of $\frac{z}{L}$, we are concerned in the present analysis with the cases of stable stratification only.

We may remark that though the present approach of analyze homogeneous, stratified turbulent shear

Table I: Plate and Arya[8].

Cases	z/L	$\varepsilon.10^{-3}$ (cm^2/sec^3)	R_f	N_2 (rad/sec^2)
$u_\infty = 9.1 m/\sec$	0.0012	102	0.0012	10.8
$u_* = 30 cm/\sec$	0.0024	52	0.0022	6.7
	0.0095	10.8	0.0080	2.04
L=5.3 meters	0.0238	5.5	0.0125	1.04
$u_\infty = 6.1 m/\sec$	0.0013	32.5	0.0013	34.8
$u_* = 18 cm/\sec$	0.0052	10.0	0.0045	7.9
	0.0104	5.8	0.0085	4.2
L=2.44 meters	0.0521	0.88	0.028	1.1
$u_\infty = 3.0 m/\sec$	0.0061	2.23	0.0052	25.6
$u_* = 7.6 cm/\sec$	0.025	1.03	0.0155	6.8
	0.049	0.64	0.026	4.8
L=0.52 meters	0.246	0.1 1	0.054	1.2

$=2.44 mete$

Fig. 4: *One dimensional energy spectrum:*
$Log\ \hat{E}_{11}\ vs.\ Log\ \hat{K}\ ;\ Log\ \hat{E}_{33}\ vs.\ Log\ \hat{K}$
$\left(\mu_\infty = 9.1 m/\sec,\ \mu_* = 30 m/\sec,\ L = 5.3\ meters,\ z/L = 0.0238\right)$

Fig. 2: *Three dimensional energy spectrum:*
$Log \hat{E}\ vs.\ Log \hat{K}\ \left(\mu_\infty = 6.1 m/\sec, \mu_* = 18 cm/\sec, L = 2.44\ meters\right)$

Fig. 1: *Three dimensional energy spectrum:*
$Log\ \hat{E}\ vs.\ Log\ \hat{K}\ \left(\mu_\infty = 9.1 m/\sec,\ \mu_* = 30 m/\sec,\ L = 5.3\ meters\right)$

flow is within the framework of a statistical theory of turbulence but it is in full conformity with the Monin-Obukhov similarity method.

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Fig. 3: *Three dimensional energy spectrum:*
$Log\ \hat{E}\ vs.\ Log\ \hat{K}\ (\mu_\infty = 3.0m/\sec,\ \mu_* = 7.6cm/\sec,\ L = 0.52\ meters)$

Fig. 5: *One dimensional energy spectrum:*
$Log\ \hat{E}_{11}\ vs.\ Log\ \hat{K}\ ;\ Log\ \hat{E}_{33}\ vs.\ Log\ \hat{K}$
$(\mu_\infty = 3m/\sec,\ \mu_* = 7.6cm/\sec,\ L = 0.52\ meters,\ z/L = 0.246)$

Fig. 6: *One dimensional energy spectrum:*
$Log\ \hat{E}_{11}\ vs.\ Log\ \hat{K}\ ;\ Log\ \hat{E}_{33}\ vs.\ Log\ \hat{K}$
$(\mu_\infty = 6.1m/\sec,\ \mu_* = 18cm/\sec,\ L = 2.44\ meters,\ z/L = 0.0521)$

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