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Research Paper

On The Characterization of Nonoscillatory Motions in Magnetorotatory Triply Diffusive Convection

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Condition for characterizing nonoscillatory motions which may be neutral or unstable for triply diffusive convection with uniform vertical rotation and magnetic field is established for rigid surfaces (which may be insulating or perfectly conducting). It is mathematically established that 'the principle of the exchange of stabilities' in a magnetorotatory triply diffusive convection is valid in the regime $\frac{R_1\sigma}{2\tau_1^2\pi^4}+\frac{R_2\sigma}{2\tau_2^2\pi^4}+\frac{T_a}{\pi^4}+\frac{Q\sigma_1}{\pi^2}\leq 1$, where R_1 and R_2 are the Rayleigh numbers for the two concentration components, τ_1 and τ_2 are the Lewis numbers for the two concentration components respectively, T_a is the Taylor number, Q is the Chandrasekhar number, σ is the Prandtl number and σ_1 is the magnetic Prandtl number.

Key Words: **Triply Diffusive Convection; The Principle of the Exchange of Stabilities; Oscillatory Motion; Rotation; Magnetic Field**

Introduction

The term double diffusive convection (also known as thermohaline convection) refers to convection in a fluid where there are two diffusing components contributing to the density having different rates of diffusion. To determine the conditions under which these convective motions will occur, the linear stability of two superposed concentration components (or one of them may be temperature component) has been studied by Stern 1960, Veronis 1965, Nield 1967, Turner 1968 and Baines and Gill 1969 etc. For the broad view of the subject one may be referred to Schmitt 1994 and Brandt and Fernando 1996.

Only doubly diffusive convection has been considered by these researchers. However it has been recognized later (Griffiths 1979a; Turner 1985) that there are many fluid mechanical systems in which the density depends on three or more stratifying agencies having different diffusivities. A few examples of such systems are saline waters of geothermally heated lakes, magmas and their laboratory models, solidification of molten alloys, the Earth's core and sea water. Experimental and theoretical studies of the case in which the density depends on three stratifying agencies include the work of Griffiths 1979a, b; Pearlstein *et al*. 1989 and Lopez 1990; Terrones and Pearlstein 1989; Ryzhkov and Shevtsova 2007, 2009 and Rionero 2013a, b. It has been established theoretically and observed experimentally by these researchers that oscillatory or stationary convection is possible in multicomponent systems even when the overall density stratification is hydrostatically stable. These researchers also found some basic differences between doubly and triply diffusive convection. One

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of these was the prediction that if the gradients of two of the stratifying agencies are held fixed, then three critical values of the Rayleigh number of the third agency are sometimes required to specify the linear stability criteria (in double diffusive convection only one critical Rayleigh number is required). Another is that the onset of convection may occur via a quassiperiodic bifurcation from the motionless basic state. Prakash *et al*. 2013 derived the upper bounds for the complex growth rate of an arbitrary motion of growing amplitude in triply diffusive convection.

The establishment of the nonoccurrence of any slow oscillatory motions which may be neutral or unstable implies the validity of 'the principle of the exchange of stabilities' (PES). The validity of this principle in stability problems eliminates the unsteady terms from the linearized perturbation equations which results in notable mathematical simplicity since the transition from stability to instability occurs via a marginal state which is characterized by the vanishing of both real and imaginary parts of the complex time eigenvalue associated with the perturbation. Pellew and Southwell 1940 proved the validity of PES (i.e. occurrence of stationary convection) for the classical Rayleigh-Benard instability problem. However no such results existed for other more general hydrodynamic configurations. Banerjee *et al*. 1985 established such a criterion for magnetohydrodynamic Rayleigh-Benard convection problem which has further been extended by Gupta *et al*. 1986 for magnetorotatory thermohaline convection problem. The extension of Gupta *et al*.'s 1986 results to magnetorotatory triply diffusive convection in the domains of geophysics, astrophysics and terrestrial physics, wherein the liquid concerned has the property of electrical conduction and the magnetic field and rotation are prevalent is very much sought after in the present context. The present communication which provides a sufficient condition for the occurrence of stationary convection in magnetorotatory triply diffusive convection may be regarded as a first step in this scheme of extended investigations.

It is proved in the present article that for magnetorotatory triply diffusive convection, if $\frac{R_1\sigma}{2\tau_1^2\pi^4}+\frac{R_2\sigma}{2\tau_2^2\pi^4}+\frac{T_a}{\pi^4}+\frac{Q\sigma_1}{\pi^2}\le 1,$ then an arbitrary neutral or unstable mode of the system is definitely non oscillatory in character and in particular PES is valid where R_1 and R_2 are concentration Rayleigh numbers for the two concentration components, τ_1 and τ_2 are the Lewis numbers for the two concentrations, T_a is the Taylor number, Q is the Chandrasekhar number, σ is the Prandtl number and σ_1 is the magnetic Prandtl number. The configuration considered herein for Magnetorotatory triply diffusive convection is of the most general kind in terms of concentrations of the contributing salts. The results obtained are uniformly applicable to any two salts with different mass diffusivities.

It is further proved in the present article that the above result is uniformly valid for insulating or perfectly conducting rigid boundaries and the results of Pellew and Southwell 1940 for Rayleigh-Benard convection, Banerjee *et al*. 1985 for magnetohydrodynamic Rayleigh-Benard convection, Gupta *et al*. 1986 for magneto thermohaline convection and magnetorotatory thermohaline convection follow as a consequence.

Mathematical Formulation and Analysis

A viscous heat conducting Boussinesq fluid layer of infinite horizontal extension is statically confined between two horizontal boundaries $z = 0$ and $z = d$ which are respectively maintained at uniform temperatures T_0 and T_1 ($<T_0$) and uniform concentrations S_{10}, S_{20} and S_{11} ($<S_{10}$), S_{21} ($<S_{20}$) under the simultaneous presence of a uniform vertical rotation (with angular velocity $\vec{\Omega}$) and a uniform vertical magnetic field $\vec{H}$. (Fig. 1). It is assumed that the cross-diffusion effects of the stratifying agencies can be neglected.

The basic equations that govern the motion of magnetorotatory triply diffusive fluid layer under the

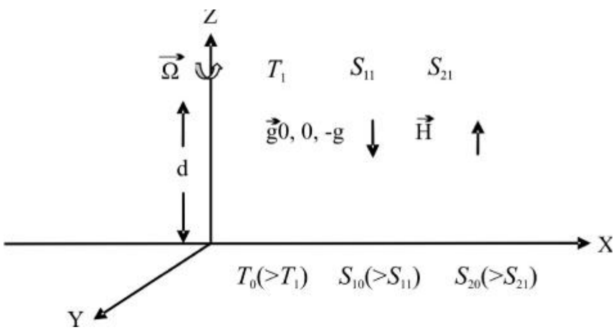

Fig. 1: Physical configuration

action of uniform vertical rotation and uniform vertical magnetic field are as follows:

Equation of continuity is

$$\frac{\partial u_j}{\partial x_j} = 0. \tag{1}$$

Equation of motion is

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - \frac{\mu_e}{4\pi\rho_0} H_j \frac{\partial H_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{p_i}{\rho_0} + \frac{|H|^2 \mu_e}{8\pi\rho_0} - \frac{1}{2}|\vec{\Omega}\times r|^2\right) + \left(1 + \frac{\delta\rho}{\rho_0} + \frac{\delta\rho'}{\rho_0} + \frac{\delta\rho''}{\rho_0}\right) X_i + 2\in_{ijk} u_j \Omega_k + \nu \nabla^2 u_i. \tag{2}$$

Equation of heat conduction is

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \kappa \nabla^2 T. \tag{3}$$

Equations of mass diffusion for two concentration components are respectively

$$\frac{\partial S_1}{\partial t} + u_j \frac{\partial S_1}{\partial x_j} = \kappa_1 \nabla^2 S_1. \tag{4}$$

$$\frac{\partial S_2}{\partial t} + u_j \frac{\partial S_2}{\partial x_j} = \kappa_2 \nabla^2 S_2. \tag{5}$$

Equation of magnetic induction is

$$\frac{\partial H_i}{\partial t} + u_j \frac{\partial H_i}{\partial x_j} = H_j \frac{\partial u_i}{\partial x_j} + \eta \nabla^2 H_i. \tag{6}$$

Equation of solenoidal character of the magnetic field is

$$\frac{\partial H_i}{\partial x_i} = 0. \tag{7}$$

Equation of State is

$$\rho = \rho_0[1 + \alpha(T_0 - T) - \alpha_1(S_{10} - S_1) - \alpha_2(S_{20} - S_2)], \tag{8}$$

where:

$$\delta\rho = -\rho_0 \alpha (T - T_0), \tag{9}$$

$$\delta\rho' = \rho_0 \alpha_1 (S_1 - S_{10}), \tag{10}$$

$$\delta\rho'' = \rho_0 \alpha_2 (S_2 - S_{20}), \tag{11}$$

In the above equations ρ is density, t is time, u_j are the components of velocity in the *x*, *y*, *z*-directions respectively, $X_i (i = 1, 2, 3)$ are the components of the external force, $\vec{g}(0,0,-g)$ is acceleration due to gravity, $x_j (j = 1, 2, 3)$ are the cartesian coordinates, $\frac{P_1}{\rho_0} + \frac{|H|^2 \mu_e}{8\pi\rho_0} - \frac{1}{2}|\vec{\Omega}\times r|^2$ is magnetorotatory hydrodynamic pressure, $\vec{H} = (H_1, H_2, H_3)$ = (0, 0, *H*) is the uniform vertical magnetic field, $\vec{\Omega}$ is angular velocity, $\vec{r}$ is position vector, $T, S_1, S_2, \kappa, \kappa_1, \kappa_2, \mu_e, \nu$ and η are respectively, the temperature, the concentration of first component, concentration of second component, thermal diffusivity, mass diffusivity of first component, mass diffusivity of second component, the magnetic permeability, the kinematic viscosity, and the resistivity, α, α_1 and α_2 are respectively the coefficients of volume expansion due to temperature and concentration variations for the two concentration components, ρ_0 is the value of density ρ at some properly chosen mean temperature T_0 and concentrations S_{10} and S_{20}.

Now the initial state solution on the basis of initial state $(u, v, w) \equiv (0, 0, 0)$, $P \equiv P(z)$, $T \equiv T(z)$, $S_1 \equiv S_1(z)$, $S_2 \equiv S_2(z)$, $(H_1, H_2, H_3) \equiv (0, 0, H)$, $\rho \equiv \rho(z)$ is given by

$$(u,v,w)=(0,0,0) S_1 = S_{10}-\beta_1 z,\ S_2 = S_{20}-\beta_2 z,\ T = T_0-\beta z,(H_1,H_2,H_3)=(0,0,H),\ \rho=\rho_0 [1+(\alpha\beta-\alpha_1\beta_1-\alpha_2\beta_2)z],\ P=\frac{P_1}{\rho_0}+\frac{|H|^2\mu_e}{8\pi\rho_0} -\frac{1}{2}|\vec{\Omega}\times r|^2 = P_0 - g\rho_0\left[z+(\alpha\beta-\alpha_1\beta_1-\alpha_2\beta_2)\frac{z^2}{2}\right], \tag{12}$$

where, H is constant, P_0 is the pressure at the lower boundary $z=0$, $\beta = \frac{T_0 - T_1}{d}$ is the maintained uniform adverse temperature gradient, $\beta_1 = \frac{S_{10}-S_{11}}{d}$ and $\beta_2 = \frac{S_{20}-S_{21}}{d}$ are the maintained uniform non-adverse concentration gradients.

To study the stability of the system, we perturb all the variables in the form

$$(\bar{u},\bar{v},\bar{w})=(+u',0+v',0+w'),\ \bar{P}=P+\delta P',\ \bar{T}=T_0-\beta z+\theta',\ \bar{S}_1=S_{10}-\beta_1 z+\phi_1',\ \bar{S}_2=S_{20}-\beta_2 z+\phi_2',\ (\bar{H}_1,\bar{H}_2,\bar{H}_3)=(0+h_x', 0+h_y', H+h_z'),\ \bar{\rho}=\rho_0[1+\alpha(T_0-T-\theta') -\alpha_1(S_{10}-S_1-\phi_1')-\alpha_2(S_{20}-S_2-\phi_2')) \tag{13}$$

where, $u', v', w', \theta', \phi_1', \phi_2', \delta P', h_x', h_y'$ and h_z' are perturbations in variables $u, v, w, T, S_1, S_2, P, H_1, H_2, H_3$ respectively and are assumed to be small.

Substituting (13) into equations (1)-(7), we obtain the following linearized perturbation equations

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0, \tag{14}$$

$$\frac{\partial u'}{\partial t}-\frac{\mu_e H}{4\pi\rho_0}\frac{\partial h_x'}{\partial z}=-\frac{\partial(\delta P')}{\partial x}+2\Omega v'+\nu\nabla^2 u', \tag{15}$$

$$\frac{\partial v'}{\partial t}-\frac{\mu_e H}{4\pi\rho_0}\frac{\partial h_y'}{\partial z}=-\frac{\partial(\delta P')}{\partial y}-2\Omega u'+\nu\nabla^2 v', \tag{16}$$

$$\frac{\partial w'}{\partial t}-\frac{\mu_e H}{4\pi\rho_0}\frac{\partial h_z'}{\partial z}=-\frac{\partial(\delta P')}{\partial z} +g\alpha\theta'-g\alpha_1\phi_1'-g\alpha_2\phi_2'+\nu\nabla^2 w', \tag{17}$$

$$\frac{\partial \theta'}{\partial t}-\beta w'=\kappa_0\nabla^2\theta', \tag{18}$$

$$\frac{\partial \phi_1'}{\partial t}-\beta_1 w'=\kappa_{10}\nabla^2\phi_1', \tag{19}$$

$$\frac{\partial \phi_2'}{\partial t}-\beta_2 w'=\kappa_{20}\nabla^2\phi_2', \tag{20}$$

$$\frac{\partial h_x'}{\partial t}=H\frac{\partial u'}{\partial z}+\eta\nabla^2 h_x', \tag{21}$$

$$\frac{\partial h_y'}{\partial t}=H\frac{\partial v'}{\partial z}+\eta\nabla^2 h_y', \tag{22}$$

$$\frac{\partial h_z'}{\partial t}=H\frac{\partial w'}{\partial z}+\eta\nabla^2 h_z', \tag{23}$$

and

$$\frac{\partial h_x'}{\partial x}+\frac{\partial h_y'}{\partial y}+\frac{\partial h_z'}{\partial z}=0. \tag{24}$$

The normal mode expansion of the dependent variables $u', v', w', \theta', \phi_1', \phi_2', \delta P', h_x', h_y'$ and h_z' is assumed in the form

$$F'(x,y,z,t) = F''(z)\exp[i(k_x x + k_y y) + nt], \quad (25)$$

where k_x and k_y are the wave numbers along x and y directions respectively, and $k = \sqrt{k_x^2 + k_y^2}$ is the resultant wave number. For functions with this dependence on x, y and t, we have

$$\frac{\partial}{\partial t} = n, \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2 \text{ and } \nabla^2 = \frac{d^2}{dz^2} - k^2. \quad (26)$$

Equations (14)-(24), then becomes

$$ik_x u'' + ik_y v'' + \frac{dw''}{dz} = 0, \quad (27)$$

$$nu'' - \frac{\mu_e H}{4\pi\rho_0}\frac{dh_x''}{dz} = -ik_x(\delta P'') + 2\Omega v'' + \nu\left(\frac{d^2}{dz^2} - k^2\right)u'', \quad (28)$$

$$nv'' - \frac{\mu_e H}{4\pi\rho_0}\frac{dh_y''}{dz} = -ik_y(\delta P'') - 2\Omega u'' + \nu\left(\frac{d^2}{dz^2} - k^2\right)v'', \quad (29)$$

$$nw'' - \frac{\mu_e H}{4\pi\rho_0}\frac{dh_z''}{dz} = -\frac{d(\delta P'')}{dz} + g\alpha\theta'' - g\alpha_1\phi_1' - g\alpha_2\phi_2' + \nu\left(\frac{d^2}{dz^2} - k^2\right)w'', \quad (30)$$

$$n\theta'' - \beta w'' = \kappa_0\left(\frac{d^2}{dz^2} - k^2\right)\theta'', \quad (31)$$

$$n\phi_1'' - \beta_1 w'' = \kappa_{10}\left(\frac{d^2}{dz^2} - k^2\right)\phi_1'', \quad (32)$$

$$n\phi_2'' - \beta_2 w'' = \kappa_{20}\left(\frac{d^2}{dz^2} - k^2\right)\phi_2'', \quad (33)$$

$$nh_x'' = H\frac{du''}{dz} + \eta\left(\frac{d^2}{dz^2} - k^2\right)h_x'', \quad (34)$$

$$nh_y'' = H\frac{dv''}{dz} + \eta\left(\frac{d^2}{dz^2} - k^2\right)h_y'', \quad (35)$$

$$nh_z'' = H\frac{dw''}{dz} + \eta\left(\frac{d^2}{dz^2} - k^2\right)h_z'', \quad (36)$$

and

$$ik_x h_x'' + ik_y h_y'' + \frac{dh_z''}{dz} = 0. \quad (37)$$

Now eliminating u'' and v'' from left hand side of equations (28) and (29) by multiplying equations (28) and (29) by k_x and k_y respectively, adding the resulting equations and using equations (27) and (37), and then eliminating $\delta P''$ between resulting equation and equation (30), we get

$$\left(\frac{d^2}{dz^2} - k^2\right)\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\nu}\right)w'' = \frac{g\alpha k^2\theta''}{\nu} - \frac{g\alpha_1 k^2\phi_1''}{\nu} - \frac{g\alpha_2 k^2\phi_2''}{\nu} - \frac{\mu_e H}{4\pi\rho_0\nu}\frac{d}{dz}\left(\frac{d^2}{dz^2} - k^2\right)h_z'' + \frac{2\Omega}{\nu}\frac{d\zeta''}{dz}, \quad (38)$$

where

$$\zeta'' = i(k_x v'' - k_y u''), \quad (39)$$

is the z–component of vorticity.

Further equations (31), (32), (33) and (36) can be written as:

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\kappa_0}\right)\theta'' = -\frac{\beta}{\kappa_0}w'', \quad (40)$$

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\kappa_{10}}\right)\phi_1'' = -\frac{\beta_1}{\kappa_{10}}w'', \quad (41)$$

$$\left(\frac{d^2}{dz^2}-k^2-\frac{n}{\kappa_{20}}\right)\phi_2'' = -\frac{\beta_2}{\kappa_{20}}w'', \tag{42}$$

and

$$\left(\frac{d^2}{dz^2}-k^2-\frac{n}{\eta}\right)h_z'' = -\frac{H}{\eta}\frac{dw''}{dz}. \tag{43}$$

In order to obtain an equation governing ζ'', multiplying equations (28) and (29) by k_y and k_x respectively, subtracting the former resulting equation from the latter resulting equation and then making use of equation (27) and (39), we obtain

$$\left(\frac{d^2}{dz^2}-k^2-\frac{n}{\nu}\right)\zeta'' = -\frac{\mu_e H}{4\pi\nu\rho_0}\frac{d\xi''}{dz}-\frac{2\Omega}{\nu}\frac{dw''}{dz}, \tag{44}$$

where

$$\xi'' = i(k_x h_y'' - k_y h_x''), \tag{45}$$

is the z-component of current density.

Similarly, we obtain an equation governing ζ'', by multiplying equations (34) and (35) by k_y and k_x respectively, subtracting the former resulting equation from the latter resulting equation and then making use of equation (39), in the form

$$\left(\frac{d^2}{dz^2}-k^2-\frac{n}{\eta}\right)\xi'' = -\frac{H}{\eta}\frac{d\zeta''}{dz}. \tag{46}$$

Now by introducing non-dimensional quantities defined by

$$a_* = kd, z_* = \frac{z}{d}, \tau_{1*} = \frac{\kappa_{10}}{\kappa_0}, \tau_{2*} = \frac{\kappa_{20}}{\kappa_0}, p_* = \frac{nd^2}{\kappa_0},$$

$$D_* = d\frac{z}{dz}, \sigma_* = \frac{\nu}{\kappa_0}, R_* = \frac{g\alpha\beta d^4}{\kappa_0\nu}, R_{1*} = \frac{g\alpha_1\beta_1 d^4}{\kappa_0\nu},$$

$$R_{2*} = \frac{g\alpha_2\beta_2 d^4}{\kappa_0\nu},$$

$$Q_* = \frac{\mu_e H^2 d^2}{4\pi\rho_0\nu\eta}, T_{a*} = \frac{4\Omega^2 d^4}{\nu^2}, w_* = \frac{\beta d^2}{\kappa_0}w'', \theta_* = \frac{\theta''}{\beta d},$$

$$\xi_* = \frac{\beta\nu\eta}{2\Omega\kappa_0 H}\xi'', h_{z*} = \frac{\eta\beta d}{H\kappa_0}h_z'', \sigma_{1*} = \frac{\nu}{\eta}. \tag{47}$$

we can reduce equations (38), (40)-(44) and (46) in the following non-dimensional forms (dropping the asterisks for simplicity)

$$(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right)w = Ra^2\theta$$
$$-R_1a^2\phi_1 - R_2a^2\phi_2 - QD(D^2-a^2)h_z + T_aD\zeta \tag{48}$$

$$(D^2-a^2-p)\theta = -w, \tag{49}$$

$$\left(D^2-a^2-\frac{p}{\tau_1}\right)\phi_1 = -\frac{w}{\tau_1}, \tag{50}$$

$$\left(D^2-a^2-\frac{p}{\tau_2}\right)\phi_2 = -\frac{w}{\tau_2}, \tag{51}$$

$$\left(D^2-a^2-\frac{p\sigma_1}{\sigma}\right)h_z = -Dw, \tag{52}$$

$$\left(D^2-a^2-\frac{p}{\sigma}\right)\zeta = -QD\xi - Dw, \tag{53}$$

and

$$\left(D^2-a^2-\frac{p\sigma_1}{\sigma}\right)\xi = -D\zeta. \tag{54}$$

The equations (48)-(54) are to be solved by using the following boundary conditions

$$w = \theta = \phi_1 = \phi_2 = Dw = h_z = \zeta = D\xi = 0$$
$$\text{at} \quad z=0 \quad \text{and} \quad z=1, \tag{55}$$

(when both the boundaries are rigid and perfectly conducting) or

$$w = \theta = \phi_1 = \phi_2 = Dw = Dh_z \mp ah_z = \zeta = \xi = 0$$
$$\text{at} \quad z=0 \quad \text{and} \quad z=1, \tag{56}$$

(when both the boundaries are rigid and insulating) where z is the real independent variable such that $0 \le z \le 1$. $D = \frac{d}{dz}$ is the differentiation along the vertical coordinate, $a^2 > 0$ is a constant, $\sigma > 0$ is a constant, $\sigma_1 > 0$ is a constant, $\tau_1 > 0$ is a constant, $\tau_2 > 0$ is a constant, $R > 0$, $R_1 > 0$, $R_2 > 0$ are constants, $Q > 0$ is a constant, $T_a > 0$ is a constant, $p = p_r + ip_i$ is a complex constant such that p_r are p_i real constants and as a consequence the dependent variables $w(z) = w_r(z) + iw_i(z)$, $\theta(z) = \theta_r(z) + i\theta_i(z)$, $\phi_1(z) = \phi_{1r}(z) + i\phi_{1i}(z)$, $\phi_2(z) = \phi_{2r}(z) + i\phi_{2i}(z)$, $\zeta(z) = \zeta_r(z) + i\zeta_i(z)$, $h_z(z) = h_{zr}(z) + ih_{zi}(z) = \xi(z) = \xi_r(z) + i\xi_i(z)$ are complex valued functions of the real variable z such that $w_r(z)$, $w_i(z)$, $\theta_r(z)$, $\theta_i(z)$, $\phi_{1r}(z)$, $\phi_{1i}(z)$, $\phi_{2r}(z)$, $\phi_{2i}(z)$, $h_{zr}(z), h_{zi}(z)$, $\xi_r(z), \xi_i(z)$, $\zeta_r(z)$, and $\zeta_i(z)$ are real valued functions of the real variable z. The meaning of the symbols from the physical point of view are as follows: z is the vertical coordinate, $\sigma = \frac{\nu}{\kappa}$ is the Prandtl number, $\sigma_1 = \frac{\nu}{\eta}$ is the magnetic Prandtl number $\tau_1 = \frac{\kappa_1}{\kappa}$ and $\tau_2 = \frac{\kappa_2}{\kappa}$ are the Lewis numbers for the two concentration components with mass diffusivities κ_1 and κ_2 respectively and κ is thermal diffusivity, Q is the Chandrasekhar number, T_a is the taylor number, R is the Rayleigh number, R_1 and R_2 are concentration Rayleigh numbers for the two concentration components, p is the complex growth rate, w is the vertical velocity, θ is the temperature, ϕ_1, ϕ_2 are the two concentrations, ξ is the vertical vorticity, ζ is the z-component of current density and $h_z(z)$ is the vertical component of the perturbation in the initially external imposed magnetic field. It may be further noted that equations (48)-(56) describes an eigenvalue problem for p and govern magnetorotatory triply diffusive convection for rigid surfaces (which may be insulating or perfectly conducting).

We now provide a proof of the following theorem:

Theorem: If $(w, \theta, \phi, \phi_1, \phi_2, \zeta, \xi, h_z, p)$, $p_r \ge 0$ with $R > 0$, $R_1 > 0$, $R_2 > 0$, $T_a > 0$, $Q > 0$, is a solution of equations (48)-(54) together with either of the boundary conditions (55) or (56) and $\frac{R_1\sigma}{2\tau_1^2\pi^4} + \frac{R_2\sigma}{2\tau_2^2\pi^4} + \frac{T_a}{\pi^4} + \frac{Q\sigma_1}{\pi^2} \le 1$, then $p_i = 0$. In particular,

$p_r = 0 \Rightarrow p_i = 0$ if

$$\frac{R_1\sigma}{2\tau_1^2\pi^4} + \frac{R_2\sigma}{2\tau_2^2\pi^4} + \frac{T_a}{\pi^4} + \frac{Q\sigma_1}{\pi^2} \le 1.$$

Proof: Multiplying equation (48) by w^* (the superscript * here denotes the complex conjugation) throughout and integrating the resulting equation over vertical range of z, we obtain

$$\int_0^1 w^*(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) w\,dz = Ra^2\int_0^1 w^*\theta\,dz - R_1a^2\int_0^1 w^*\phi_1 dz - R_2a^2\int_0^1 w^* \phi_2 d_2 - Q\int_0^1 w^* D(D^2 - a^2)\,h_z dz + T_a\int_0^1 w^* D\zeta\,dz. \tag{57}$$

Making use of equations (49) - (54) and the fact that $w(0) = 0 = w(1)$ and $\zeta(0) = 0 = \zeta(1)$ we can write

$$Ra^2\int_0^1 w^*\theta\,dz = -Ra^2\int_0^1 \theta(D^2 - a^2 - p^*)\theta^*dz, \tag{58}$$

$$-R_1a^2\int_0^1 w^*\phi_1 dz = R_1a^2\tau_1\int_0^1 \phi_1\left(D^2 - a^2 - \frac{p^*}{\tau_1}\right)\phi_1^* dz, \tag{59}$$

$$-R_2a^2\int_0^1 w^*\phi_2 dz = R_2a^2\tau_2\int_0^1\phi_2\left(D^2-a^2-\frac{p^*}{\tau_2}\right)\phi_2^* dz, \tag{60}$$

$$T_a\int_0^1 w^* D\zeta\, dz = -T_a\int_0^1\zeta\, Dw^* dz = T_a\int_0^1\zeta$$

$$\left(D^2-a^2-\frac{p^*}{\sigma}\right)\zeta^* dz + QT_a\int_0^1\zeta\, D\xi^* dz$$

$$= T_a\int_0^1\zeta\left(D^2-a^2-\frac{p^*}{\sigma}\right)\zeta^* dz - QT_a\int_0^1\xi^* D\xi\, dz$$

$$= T_a\int_0^1\zeta\left(D^2-a^2-\frac{p^*}{\sigma}\right)\zeta^* dz + QT_a\int_0^1\xi^*$$

$$\left(D^2-a^2-\frac{p\sigma_1}{\sigma}\right)\xi\, dz, \tag{61}$$

$$-Q\int_0^1 w^* D(D^2-a^2)h_z dz = Q\int_0^1 Dw^*(D^2-a^2)h_z dz$$
$$= -Q\int_0^1\left(D^2-a^2-\frac{p^*\sigma_1}{\sigma}\right)h_z^*\cdot(D^2-a^2)h_z dz. \tag{62}$$

Combining equations (57) - (62), we get

$$\int_0^1 w^*(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right)w\, dz$$

$$= -Ra^2\int_0^1\theta(D^2-a^2-p^*)\theta^* dz + R_1a^2\tau_1$$

$$\int_0^1\phi_1\left(D^2-a^2-\frac{p^*}{\tau_1}\right)\phi_1^* dz + R_2a^2\tau_2\int_0^1\phi_2$$

$$\left(D^2-a^2-\frac{p^*}{\tau_2}\right)\phi_2^* dz - Q\int_0^1\left(D^2-a^2\frac{p^*\sigma_1}{\sigma}\right)$$

$$h_z^*(D^2-a^2)h_z dz + T_a\int_0^1\zeta\left(D^2-a^2-\frac{p^*}{\sigma}\right)$$

$$\zeta^* dz + QT_a\int_0^1\xi^*\left(D^2-a^2-\frac{p\sigma_1}{\sigma}\right)\xi\, dz. \tag{63}$$

Integrating the various terms of equation (63), by parts, for an appropriate number of times and making use of either of the boundary conditions (55) or (56), we get

$$\int_0^1(|D^2w|^2+2a^2|Dw|^2+a^4|w|^2)dz$$

$$+\frac{p}{\sigma}\int_0^1(|Dw|^2+a^2|w|^2)\,dz = Ra^2\int_0^1$$

$$(|D\theta|^2+a^2|\theta|^2+p^*|\theta|^2)\,dz - R_1a^2\tau_1\int_0^1$$

$$\left(|D\phi_1|^2+a^2|\phi_1|^2+\frac{p^*}{\tau_1}|\phi_1|^2\right)dz - R_2a^2\tau_2$$

$$\int_0^1\left(|D\phi_2|^2+a^2|\phi_2|^2+\frac{p^*}{\tau_2}|\phi_2|^2\right)dz - Q$$

$$\int_0^1|(D^2-a^2)h_z|^2 dz - \frac{Qp^*\sigma_1}{\sigma}$$

$$\left[a\{(|h_z|^2)_0+(|h_z|^2)_1\}+\int_0^1(|Dh_z|^2+a^2|h_z|^2)dz\right]$$

$$-T_a\int_0^1\left(|D\zeta|^2+a^2|\zeta|^2+\frac{p^*}{\sigma}|\zeta|^2\right)dz$$

$$-QT_a\int_0^1\left(|D\xi|^2+a^2|\xi|^2+\frac{p\sigma_1}{\sigma}|\xi|^2\right)dz. \tag{64}$$

Equating the imaginary parts of both sides of equation (64) and cancelling $p_i(\neq 0)$ throughout from the imaginary part, we get

$$\frac{1}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)\,dz = -Ra^2\int_0^1 |\theta|^2\,dz$$

$$R_1 a^2\int_0^1 |\phi_1|^2\,dz + R_2 a^2\int_0^1 |\phi_2|^2\,dz + \frac{Q\sigma_1}{\sigma}$$

$$\left[a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz\right]$$

$$+\frac{T_a}{\sigma}\int_0^1 |\zeta|^2\,dz - \frac{QT_a\sigma_1}{\sigma}\int_0^1 |\xi|^2\,dz. \qquad (65)$$

Now multiplying (50) and (51) by their respective complex conjugate and integrating over the vertical range of z, and making use of either of the boundary conditions (55) or (56), we get,

$$\int_0^1 (|D^2\phi_1|^2 + 2a^2|D\phi_1|^2 + a^4|\phi_1|^2\,dz + \frac{2p_r}{\tau_1}$$

$$\int_0^1 (|D\phi_1|^2 + a^2|\phi_1|^2)\,dz + \frac{|p|^2}{\tau_1^2}$$

$$\int_0^1 |\phi_2|^2\,dz = \frac{1}{\tau_1^2}\int_0^1 |w|^2\,dz. \qquad (66)$$

$$\int_0^1 (|D^2\phi_2|^2 + 2a^2|D\phi_2|^2 + a^4|\phi_2|^2)dz$$

$$+\frac{2p_r}{\tau_2}\int_0^1 (|D\phi_2|^2 + a^2|\phi_2|^2)dz$$

$$+\frac{|p|^2}{\tau_2^2}\int_0^1 |\phi_2|^2\,dz = \frac{1}{\tau_2^2}\int_0^1 |w|^2\,dz. \qquad (67)$$

Since $p_r \geq 0$ it follows from equations (66) and (67) that

$$2a^2\int_0^1 |D\phi_1|^2\,dz \leq \frac{1}{\tau_1^2}\int_0^1 |w|^2\,dz, \qquad (68)$$

$$2a^2\int_0^1 |D\phi_2|^2\,dz \leq \frac{1}{\tau_2^2}\int_0^1 |w|^2\,dz. \qquad (69)$$

ϕ_1, ϕ_2 and w satisfy $\phi_1(0) = 0 = \phi_1(1)$, $\phi_2(0) = 0 = \phi_2(1)$ and $w(0) = 0 = w(1)$, respectively, then by Rayleigh-Ritz inequality (Schultz 1973) we get

$$\int_0^1 |D\phi_1|^2\,dz \geq \pi^2\int_0^1 |\phi_1|^2 dz, \qquad (70)$$

$$\int_0^1 |D\phi_2|^2\,dz \geq \pi^2\int_0^1 |\phi_2|^2 dz, \qquad (71)$$

$$\int_0^1 |Dw|^2\,dz \geq \pi^2\int_0^1 |w|^2 dz. \qquad (72)$$

Now, using inequalities (70) and (72) in equality (68) and inequalities (71) and (72) in equality (69), we have

$$a^2\int_0^1 |\phi_1|^2\,dz \leq \frac{1}{2\tau_1^2\pi^4}\int_0^1 |Dw|^2 dz, \qquad (73)$$

$$a^2\int_0^1 |\phi_2|^2\,dz \leq \frac{1}{2\tau_2^2\pi^4}\int_0^1 |Dw|^2 dz. \qquad (74)$$

Now for the case of rigid boundaries, $\zeta(0) = 0 = \zeta(1)$, again by using Rayleigh-Ritz inequality (Schultz 1973), we obtain

$$\int_0^1 |D\zeta|^2\,dz \geq \pi^2\int_0^1 |\zeta|^2\,dz. \qquad (75)$$

Multiplying equation (53) by ζ^* and integrating over the vertical range of z, by parts for a suitable number of times and making use of either of the boundary conditions (55) or (56), we have from the real parts of final equation

$$\int_0^1 \left(|D\zeta|^2 + a^2|\zeta|^2 + \frac{p_r}{\sigma}|\zeta|^2\right) = \text{Real part of}$$

$$\left(\int_0^1 \zeta^* Dw dz + Q\int_0^1 \zeta^* D\xi\, dz\right),$$

$$= \text{Real part of } \left(-\int_0^1 wD\zeta^* dz - Q\int_0^1 \xi D\zeta^* dz\right),$$

= Real part of

$$\left(-\int_0^1 wD\zeta^* dz + Q\int_0^1 \xi\left(D^2 - a^2 - \frac{p^*\sigma_1}{\sigma}\right)\xi^* dz\right),$$

(using equation 54)

= Real part of

$$\left(-\int_0^1 wD\zeta^* dz - Q\int_0^1\left(|D\xi|^2 + a^2|\xi|^2 + \frac{p^*\sigma_1}{\sigma}\right)|\xi|^2 dz\right),$$

= Real part of

$$\left(-\int_0^1 wD\zeta^* dz - Q\int_0^1\left(|D\xi|^2 + a^2|\xi|^2 + \frac{p_r\sigma_1}{\sigma}\quad|\xi|^2 dz\right)\right),$$

$$\leq \text{Re}\left(-\int_0^1 wD\zeta^* dz\right),$$

$$\leq \left|-\int_0^1 wD\zeta^* dz\right|,$$

$$\leq \left|\int_0^1 wD\zeta^* dz\right|,$$

$$\leq \left(\int_0^1 |w|^2\, dz\right)^{1/2}, \left(\int_0^1 |D\zeta|^2\, dz\right)^{1/2},$$ (using Schwartz inequality)

which implies that

$$\leq \left(\int_0^1 |D\zeta|^2\, dz\right)^{1/2} \leq \left(\int_0^1 |w|^2\, dz\right)^{1/2}, \qquad (76)$$

and thus using inequalities (72) and (75), we obtain

$$\int_0^1 |\zeta|^2\, dz \leq \frac{1}{\pi^4}\int_0^1 |Dw|^2\, dz. \qquad (77)$$

Now multiplying equation (52) by h_z^* and integrating the resulting equation by parts for a suitable number of times and making use of either of the boundary conditions (55) or (56), we have

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1\left(|Dh_z|^2 + a^2|h_z|^2 + \frac{p\sigma_1}{\sigma}|h_z|^2\right)$$

$$dz = \int_0^1 h_z^* Dw dz = -\int_0^1 (Dh_z^*) w dz.$$

Equating real parts on both sides, we obtain

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1 + \int_0^1\left(|Dh_z|^2 + a^2|h_z|^2\ \frac{p_r\sigma_1}{\sigma}|h_z|^2\right)dz$$

$$= -\text{Real part of } \int_0^1 (Dh_z^*) w dz,$$

$$\leq \left|\int_0^1 (Dh_z^*) w dz\right|,$$

$$\leq \int_0^1 |Dh_z||w|\, dz,$$

$$\leq \left(\int_0^1 |Dh_z|^2\, dz\right)^{1/2}, \left(\int_0^1 |w|^2\, dz\right)^{1/2}. \qquad (78)$$

(using Schwartz inequality)

Since $p_r \geq 0$, we have from inequality (78)

$$\int_0^1 |Dh_z|^2\, dz \leq \left(\int_0^1 |Dh_z|^2\, dz\right)^{1/2}\left(\int_0^1 |w|^2\, dz\right)^{1/2},$$

which gives

$$\left(\int_0^1 |Dh_z|^2\,dz\right)^{1/2} \le \left(\int_0^1 |w|^2\,dz\right)^{1/2}.$$

Using this inequality in (78), we obtain

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2\right)dz$$

$$\le \int_0^1 |w|^2\,dz \le \frac{1}{\pi^2}\int_0^1 |Dw|^2\,dz. \tag{79}$$

(using inequality 72)

Now utilizing inequalities (73), (74), (77) and (79) in equation (65), we obtain

$$\left(\frac{1}{\sigma} - \frac{R_1}{2\tau_1^2\pi^4} - \frac{R_2}{2\tau_2^2\pi^4} - \frac{T_a}{\sigma\pi^4} - \frac{Q\sigma_1}{\pi^2\sigma}\right)$$

$$\int_0^1 |Dw|^2\,dz + \frac{a^2}{\sigma}\int_0^1 |w|^2\,dz < -Ra^2\int_0^1 |\theta|^2\,dz$$

$$-\frac{T_aQ\sigma_1}{\sigma}\int_0^1 |\xi|^2\,dz, \tag{80}$$

which clearly implies (for $p_i \neq 0$) that

$$\frac{R_1\sigma}{2\tau_1^2\pi^4} + \frac{R_2\sigma}{2\tau_2^2\pi^4} + \frac{T_a}{\pi^4} + \frac{Q\sigma_1}{\pi^2} > 1,$$

hence $\frac{R_1\sigma}{2\tau_1^2\pi^4} + \frac{R_2\sigma}{2\tau_2^2\pi^4} + \frac{T_a}{\pi^4} + \frac{Q\sigma_1}{\pi^2} > 1,$ then we must have $p_i = 0$, which proves the theorem.

The essential content of the theorem from physical point of view is that for the problem of magnetorotatory triply diffusive convection for rigid boundaries (perfectly conducting or insulating) an arbitrary neutral or unstable mode of the system is definitely nonoscillatory in character and in particular 'the principle of the exchange of stabilities' is valid

if $\frac{R_1\sigma}{2\tau_1^2\pi^4} + \frac{R_2\sigma}{2\tau_2^2\pi^4} + \frac{T_a}{\pi^4} + \frac{Q\sigma_1}{\pi^2} \le 1,$

Special Cases: It follows from theorem1 that an arbitrary neutral or unstable mode is non oscillatory in character and in particular PES is valid for:

1. Rayleigh-Benard convection ($R_1 = R_2 = T_a =$ Q = 0)
   (Pellew and Southwell 1940)

2. Magnetohydrodynamic Rayleigh-Benard convection ($R_1 = R_2 = 0 = T_a$) if $\frac{Q\sigma_1}{\pi^2} \le 1.$
   (Banerjee *et al*. 1985)

3. Rotatory Rayleigh-Benard convection ($R_1 = R_2$ = 0 = Q) if $\frac{T_a}{\pi^4} \le 1.$
   (Gupta *et al*. 1984)

4. Magnetorotatory Rayleigh-Benard convection ($R_1 = R_2 = 0,\ Q > 0,\ T_a > 0$) if $\frac{T_a}{\pi^4} + \frac{Q\sigma_1}{\pi^2} \le 1.$
   (Gupta *et al*. 1984)

5. Thermohaline convection ($R_2 = T_a =$ Q = 0) if $\frac{R_1\sigma}{2\tau_1^2\pi^4} \le 1.$
   (Gupta *et al*. 1986)

6. Magnetohydrodynamic thermohaline convection ($R_2 = 0 = T_a$) if $\frac{R_1\sigma}{2\tau_1^2\pi^4} + \frac{Q\sigma_1}{\pi^2} \le 1.$
   (Gupta *et al*. 1986)

7. Rotatory thermohaline convection ($R_2 = 0 = Q$) if $\frac{R_1\sigma}{2\tau_1^2\pi^4} + \frac{T_a}{\pi^4} \le 1.$
   (Gupta *et al*. 1986)

8. Magnetorotatory thermohaline convection (R_2 = 0) if $\frac{R_1\sigma}{2\tau_1^2\pi^4} + \frac{T_a}{\pi^4} + \frac{Q\sigma_a}{\pi^2} \le 1.$
   (Gupta *et al*. 1986)

9. Thermohaline convection of Stern (1960) type ($R < 0$, $R_1 < 0$, $R_2 = T_a = 0 = Q$) if $\frac{|R|\sigma}{2\pi^4} \le 1$.
(Gupta *et al*. 1986)

10. Magnetohydrodynamic thermohaline convection of Stern (1960) type ($R < 0$, $R_1 < 0$, $R_2 = 0 = T_a$, $Q > 0$) if $\frac{|R|\sigma}{2\pi^4} + \frac{Q\sigma_1}{\pi^2} \le 1$.
(Gupta *et al*. 1986)

11. Rotatory thermohaline convection of Stern (1960) type $R = |R|$, $R_1 = |R_1|$, $R_2 = -|R_2|$ if $\frac{|R|\sigma}{2\pi^4} + \frac{T_a}{\pi^2} \le 1$.
(Gupta *et al*. (1986))

12. Magnetorotatory thermohaline convection of Stern (1960) type ($R < 0$, $R_1 < 0$, $R_2 = 0$ $Q > 0$, $T_a, > 0$) if $\frac{|R|\sigma}{2\pi^4} + \frac{T_a}{\pi^4} + \frac{Q\sigma_1}{\pi^2} \le 1$.
(Gupta *et al*. 1986)

13. Magnetorotatory triply diffusive convection analogous to Stern (1960) type ($R < 0$, $R_1 < 0$, $R_2 < 0, Q > 0, T_a, > 0$) if $\frac{|R|\sigma}{2\pi^4} + \frac{T_a}{\pi^4} + \frac{Q\sigma_1}{\pi^2} \le 1$.

Proof: Putting, $R = |R|$, $R_1 = |R_1|$, $R_2 = -|R_2|$ in equation (48) and adopting the procedure exactly similar to the one used in proving the general theorem we obtain the desired result.

Conclusion

Linear stability theory has been used to derive a sufficient condition for the validity of 'the principle of the exchange of stabilities' in magnetorotatory triply diffusive convection. It is further proved that the result is uniformly valid for rigid boundaries either perfectly conducting or insulating.

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