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Research Paper

Weak Convergence Theorem for Nonexpansive and Monotone, Lipschitz Continuous Mappings

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In this paper, we shall prove a weak convergence theorem for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping. With the help of a numerical example, we shall show the existence of a fixed point and find a solution of a variational inequality problem using C++. Further, we consider the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of zeros of a monotone, Lipschitz continuous mapping.

Key Words: Fixed Points; Monotone Mappings; Nonexpansive Mappings; Variational Inequalities

Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|\cdot\|$, respectively. Let C be a closed convex subset of H. The variational inequality problem is to find $u \in C$ such that $\langle Au, v-u \rangle \geq 0, \forall\, v \in C$.

The set of solutions of variational inequality problem VI(C, A) is denoted by Ω. The variational inequality problem has been extensively studied in literature, see, for example, Browder and Petryshyn (1967), Liu and Nashed (1998), Takahashi (2000) and references therein.

Definitions: Let A: $C \rightarrow H$ be a mapping of C into H.

1. A is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0, \forall\, u, v \in C$$

2. A is called α-inverse-strongly-monotone (Browder and Petryshyn, 1967; Liu and Nashed, 1998) if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \, \| Au - Av \|^2 \ \forall u, v \in C.$$

It is easy to see that an α-inverse strongly mapping A is monotone and Lipschitz continuous but converse is not true.

3. A mapping S : $C \rightarrow C$ is called nonexpansive (Takahashi, 2000; Takahashi and Tamura, 1998) if

$$\| Su - Sv \| \leq \| u - v \| \quad \forall u, v \in C.$$

We denote by $F(S)$ the set of fixed points of S.

4. A mapping $S : C \rightarrow C$ is called Lipschitz continuous if there exists a real number $L > 0$ such that,

$$\| Su - Sv \| \leq L \| u - v \| \quad \forall u, v \in C.$$

Takahashi and Toyoda (2003) introduced the following iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for an α-inverse-strongly-monotone mapping in a real Hilbert space.

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Theorem 1. Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \phi$. Let $\{x_n\}$ be a sequence generated by

$$x_0 = x \in C,$$

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n), \qquad (1.1)$$

for every $n = 0, 1, 2, \ldots\ldots$ where $\{\lambda_n\} \subset [c, d]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequence $\{x_n\}$ converges weakly to some point $z \in F(S) \cap VI(C, A)$, where

$$z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n .$$

In this paper, we shall prove the weak convergence of iterative scheme (1.1) for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone and Lipschitz continuous mapping in a real Hilbert space. Further, we consider the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of zeros of a monotone and Lipschitz continuous mapping.

Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|\cdot\|$. Let C be a closed convex subset of H. We shall denote $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. It is well known that for any $u \in H$, there exists a unique $y_0 \in C$ such that

$$\| u - y_0 \| - \inf\{\| u - y \| : y \in C\} = d(y, C) \quad (2.1)$$

We shall denote y_0 by $P_C u$, where P_C is called the metric projection of H onto C. The metric projection P_C of H onto C satisfies the following basic properties (for detail, see, e.g. the Book of F. Deutsch (2001)):

(*i*) $\| P_C x - P_C y \| \leq \| x - y \|$ for all $x, y \in H$,

(*ii*) $\langle x - y, P_C x - P_C y \rangle \geq \| P_C x - P_C y \|^2$, for every $x, y \in H$,

(iii) $\langle x - P_C x, y - P_C x \rangle \leq 0$, for all $x \in H, y \in C$

(iv) $\| x - y \|^2 \geq \| x - P_C x \|^2 + \| y - P_C x \|^2$, for all $x \in H$, $y \in C$.

Such properties of P_C will be crucial in the proof of our main result. Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see from the property (iv) that $u \in \Omega \Leftrightarrow u = P_C\ (u - \lambda A u)$, for all $\lambda > 0$. It is known that H satisfies the Opial condition (Opial 1967) that is, for any sequence $\{x_n\}$ with $x_n \to x$, the inequality,

$$\lim_{n\to\infty} \inf \| x_n - x \| < \lim_{n\to\infty} \inf \| x_n - y \|$$

holds for every $y \in H$ with $y \neq x$. We also know that, if $\{x_n\}$ is sequence of H with $x_n \to x$ and $\| x_n \| \to \| x \|$, then this implies that $x_n \to x$.

A set valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T: H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \text{ x } H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $N_C v$ be the normal cone to C at $v \in C$, that is,

$$N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}$$

If we define,

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C \\ \phi, & \text{if } v \notin C \end{cases}$$

Then, T is maximal monotone (Rockafellar 1970) and $0 \in Tv$ if and only if $v \in VI(C, A)$.

Now we give below several lemmas and proposition which will be used in the proof of our main result in this paper.

Lemma 1. (given by Schu (1991)) Let H be a real Hilbert space, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \le \alpha_n \le b < 1$ for all $n = 0, 1, 2 \ldots$ and let $\{v_n\}$ and $\{w_n\}$ be sequences of H such that,

$$\lim_{n\to\infty} \sup \| v_n \| \le c, \quad \lim_{n\to\infty} \sup \| w_n \| \le c,$$

and

$$\lim_{n\to\infty} \| \alpha_n v_n + (1-\alpha_n) w_n \| = c, \text{ for some } c > 0.$$

Then, $\lim_{n\to\infty} \| v_n - w_n \| = 0.$

Lemma 2. (given by Takahashi and Toyoda (2003)) Let H be a real Hilbert space and let D be a non empty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,

$$\| x_{n+1} - u \| \le \| x_n - u \|, \text{ for every } n = 0, 1, 2 \ldots$$

Then, the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

Lemma 3. (given by Goebel and Kirk (1990)) Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $T : C \to E$ be a nonexpansive mapping. Then, the mapping $I - T$ is demiclosed on C, where I is the identity mapping: that is, $x_n \to x$ in E and $(I - T)\, x_n \to y$ imply that $x \in C$ and $(I - T)\, x = y$.

Browder F E (1965) proved the following proposition.

Proposition: Let C be a bounded closed convex subset of a real Hilbert space H and let A be an α-inverse-strongly-monotone mapping of C into H. Then, $VI\,(C, A)$ is nonempty.

Weak Convergence Theorem

In this section, we prove a weak convergence theorem for nonexpansive mapping and monotone mapping.

Theorem 2. Let C be a closed convex subset of a real Hilbert space H. Let A be a monotone k-Lipschitz continuous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \phi$. Let $\{x_n\}$ be sequence generated by

$$x_0 = x \in C,$$

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \quad (3.1.1)$$

for every $n = 0, 1, 2, \ldots$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequence $\{x_n\}$ converges weakly to some point $z \in F(S) \cap VI(C, A)$, where

$$z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n .$$

Proof: Put $y_n = P_C(x_n - \lambda_n A x_n)$, for every $n = 0, 1, 2, 3, \ldots$

Let $u \in F(S) VI(C, A)$.

Now,

$$\| y_n - u \|^2$$
$$\le \| x_n - \lambda_n A x_n - u \|^2 - \| x_n - \lambda_n A x_n - y_n \|^2$$
$$= \| x_n - u \|^2 + \| \lambda_n A x_n \|^2 - 2\lambda_n \langle x_n - u, A x_n \rangle$$
$$- \| x_n - y_n \|^2 - \| \lambda_n A x_n \|^2 + 2\lambda_n \langle x_n - y_n, A x_n \rangle$$
$$= \| x_n - u \|^2 - \| x_n - y_n \|^2 + 2\lambda_n$$
$$\langle x_n - y_n - x_n + u, A x_n \rangle$$
$$= \| x_n - u \|^2 - \| x_n - y_n \|^2 + 2\lambda_n \langle A x_n, u - y_n \rangle$$
$$= \| x_n - u \|^2 - \| x_n - y_n \|^2$$
$$+ 2\lambda_n (\langle A x_n - A u, u - x_n \rangle$$
$$+ \langle A u, u - x_n \rangle + \langle A x_n, x_n - y_n \rangle)$$
$$\le \| x_n - u \|^2 - \| x_n - y_n \|^2 + 2\langle \lambda_n A x_n, x_n - y_n \rangle$$
$$= \| x_n - u \|^2 - \| x_n - y_n \|^2$$
$$+ 2\langle x_n + \lambda_n A x_n - y_n, x_n - y_n \rangle + \langle y_n - x_n, x_n - y_n \rangle$$
$$\le \| x_n - u \|^2 - \| x_n - y_n \|^2 - \| x_n - y_n \|^2$$
$$= \| x_n - u \|^2 - 2\| x_n - y_n \|^2$$
$$\le \| x_n - u \|^2, \text{ for every } n = 0, 1, 2, \ldots \quad (1)$$

Now, by (1), we have

$$\| x_{n+1} - u \|^2 = \| \alpha_n (x_n - u) + (1 - \alpha_n)(S y_n - u \|^2$$

$\leq \alpha_n \| x_n - u\|^2 + (1 - \alpha_n)\| y_n - u\|^2$

$\leq \alpha_n \| x_n - u\|^2 + (1 - \alpha_n)(\| x_n - u\|^2 - 2\| x_n - y_n\|^2)$

$\leq \| x_n - u\|^2 - 2(1 - \alpha_n)\| x_n - y_n\|^2 \leq \| x_n - u\|^2$ (2)

So, $\| x_{n+1} - u\| = \| x_n - u\|$

Therefore, there exists $c = \lim_{n\to\infty} \| x_n - u\|$ and the sequences $\{x_n\}$, $\{y_n\}$ are bounded. From equation (2),

$2(1 - \alpha_n) \| x_n - y_n\|^2 \leq \| x_n - u\|^2 - \| x_{n+1} - u\|^2$

Since, $\lim_{n\to\infty} \| x_n - u\|^2 = \lim_{n\to\infty} \| x_{n+1} - u\|^2$ so above equation implies that $x_n - y_n \to 0$.

Since, A is Lipschitz continuous, so $Ax_n - Ay_n \to 0$. As $\{x_n\}$ is bounded, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to z. Then, we shall obtain that

$z \in F(S) \cap VI(C, A)$.

Firstly, we shall show that $z \in VI(C, A)$. Since $x_n - y_n \to 0$, we have, $y_{n_i} \to z$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C \\ \phi, & \text{if } v \notin C \end{cases}$$

Then T is maximal monotone. Let $(v, w) \in G(T)$. Since $w - Av \in N_C v$ and $y_n \in C$, so we get $\langle v - y_n, w - Av\rangle \geq 0$.

On the other hand, from $y_n = P_C(x_n - \lambda_n Ax_n)$, we have, $\langle x_n - \lambda_n Ax_n - y_n, y_n - v\rangle \geq 0$ and hence, $\langle v - y_n, (y_n - x_n)/\lambda_n + Ax_n\rangle \geq 0$.

Therefore, we have

$\langle v - y_{n_i}, w\rangle \geq \langle v - y_{n_i}, Av\rangle \geq \langle v - y_{n_i}, Av\rangle$

$-\langle v - y_{n_i}, (y_{n_i} - x_{n_i})/\lambda_{n_i} + Ax_{n_i}\rangle$

$= \langle v - y_{n_i}, Av - Ax_{n_i} - (y_{n_i} - x_{n_i})/\lambda_{n_i}\rangle$

$= \langle v - y_{n_i}, Av - Ay_{n_i}\rangle + \langle v - y_{n_i}, A - Ay_{n_i} - Ax_{n_i}\rangle$

$-\langle v - y_{n_i}, (y_{n_i} - x_{n_i} / \lambda_{n_i}\rangle \geq \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i}\rangle$

$-\langle v - y_{n_i}, (y_{n_i} - x_{n_i})/\lambda_{n_i}\rangle$.

Hence, we get $\langle v - z, w\rangle \geq = 0$, as $i \to \infty$.

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$.

Next we shall show that $z \in F(S)$. Let

$u \in F(S) \cap VI(C, A)$.

Since, $\|Sy_n - u\| \leq \| y_n - u\| \leq \|x_n - u\|$,

so, we have, $\lim_{n\to\infty} \sup \| Sy_n - u\| \leq c$,

where, $C = \lim_{n\to\infty} \| x_n - u\|$,

Further, we have

$\lim_{n\to\infty} \| \alpha_n (x_n - u) + (1 - \alpha_n)(Sy_n - u)\|$

$= \lim_{n\to\infty} \| x_{n+1} - u\| = c$.

By lemma 2.1, we have, $\lim_{n\to\infty} \| Sy_n - x_n\| = 0$.

We also have,

$\|Sx_n - x_n\| \leq \|Sx_n - Sy_n\| + \|Sy_n - x_n\|$

$\leq \|x_n - y_n\| + \|Sy_n - x_n\|$

Hence, we have $\lim_{n\to\infty} \| Sx_n - x_n\| = 0$.

Since $x_{n_i} \to z$ and $\lim_{n\to\infty} \| Sx_n - x_n\| = 0$, so by demiclosedness of $I - S$, we have $z \in F(S)$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$, such that $\{x_{n_j}\} \to z'$. Then, $z' \in F(S) \cap VI(C, A)$. Let us show that $z = z'$. Assume that $z \neq z'$. From the Opial condition, we have

$\lim_{n\to\infty} \| x_n - z \| \lim_{n\to\infty} \inf \| x_{n_i} - z \|$

$< \lim_{n\to\infty} \inf \| x_{n_i} - z' \|$

$= \lim_{n\to\infty} \| x_n - z' \| = \lim_{j\to\infty} \inf \| x_{n_j} - z' \|$

$< \lim_{j\to\infty} \inf \| x_{n_j} - z \| = \lim_{n\to\infty} \| x_n - z \|$.

Since, this is a contradiction, therefore,we have $z = z'$. This implies that

$x_n \to z \in F(S) \cap VI(C, A)$

Now, take $u_n = P_{F(S) \cap VI(C,A)} x_n$.

We show that $z = \lim_{n\to\infty} u_n$.

From, $u_n = P_{F(S) \cap VI(C,A)} x_n$ and $z \in F(S) \cap VI(C, A)$, we have, $\langle z - u_n, u_n - x_n \rangle \geq 0$.

By Lemma 2.2, $\{u_n\}$ converges strongly to some $z_0 \in F(S) \cap VI(C, A)$.

Then, we have, $\langle z - z_0, z_0 - z \rangle \geq = 0$ and hence $z = z_0$.

Remark: Since we know that every inverse-strongly-monotone mapping is monotone and Lipschitz continuous, but converse is not true. So our result is the generalization of Theorem 1.1 given by Takahashi and Toyoda (2003).

Numerical Example

Now we illustrate our main result with the help of a numerical example and a program in C++ .

Example 1. Let $H = R$ be a real Hilbert space with usual inner product defined on R. Let $C = [0, 1]$ be a closed convex subset of H. Let $Ax = \dfrac{x^2}{1+x}$ be a monotone, Lipschitz-continuous mapping of C into H and let $Sx = \dfrac{x}{2}$ be a nonexpansive mapping of C into itself. Let $\{\alpha_n\} = \{\lambda_n\} = \dfrac{1}{n+1}$ be two sequences satisfying the conditions of theorem 3.1.

Now using the iterative scheme

$x_0 = x \in C,$

$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n)$, (3.1.1),

we obtain the following data given in table 4.1 when initial approximation is taken as $x_0 = 0.6$.

Thus we see that the sequence $\{x_n\}$ converges to $z = 0$, which is a fixed point of S as well as a solution of variational inequality problem.

Application

Theorem: Let H be a real Hilbert space. Let A be a monotone k-Lipschitz continuous mapping of H into itself and let S be a nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \phi$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in H$ and let

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S(x_n - \lambda_n A x_n),$$

for every $n = 0, 1, 2, \ldots$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequence $\{x_n\}$ converges weakly to some point $z \in F(S) n$.

$A^{-1}0$, where

$$z = \lim_{n\to\infty} P_{F(S) \cap A_0^{-1} x_n}.$$

Proof: We have $A^{-1}0 = VI(H, A)$ and $P_H = I$. By Theorem 3.1, we obtain the desired result.

Remark: Notice that $F(S) \cap A^{-1}0 \subset VI(F(S), A)$. See Yamada (2001) for the case when A is strongly monotone and Lipschitz continuous mapping of H into itself.

Table 4.1:

n	x_{n+1}	Sx_n	$P_C(x_n - \lambda_n A x_n)$
0	0.6	0.1875	0.375
1	0.3	0.3	0.6
2	0.15	0.15	0.3
3	0.075	0.075	0.15
498	7.33185e-151	7.33185e-151	1.46637e-150
499	3.66592e-151	3.66592e-151	7.33185e-151
797	7.19854e-241	7.19854e-241	1.43971e-240
798	3.59927e-241	3.59927e-241	7.19854e-241
1072	9.88131e-324	9.88131e-324	2.47033e-323
1073	4.94066e-324	4.94066e-324	9.88131e-324
1074	0	0	4.94066e-324
1075	0	0	0
1076	0	0	0

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