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Research Paper

Two Notes on Normal Hilbert Polynomial of Two Ideals

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Let I and J be m-primary ideals of an analytically unramified local ring (R, m). We prove that the difference of normal Hilbert function $\overline{H}(r,s)=\lambda(R/\overline{I^r J^s})$ and the corresponding normal Hilbert polynomial $\overline{P}(r,s)$ equals the Euler characteristic of the local cohomology modules of bigraded Rees algebra $\overline{\mathfrak{R}'}(I,J)$ for all integers r, s. We also prove that $\overline{I^{r+1}J^{s+1}}=a\overline{I^r J^{s+1}}+b\overline{I^{r+1}J^s}$ for all $r \geq r_0, s \geq s_0$ if $\overline{P}(r,s)=\overline{H}(r,s)$ for all $r \geq r_0, s \geq s_0$ in an analytically unramified Cohen-Macaulay local ring of dimension 2 with infinite residue field for a good joint reduction (a, b) of $\left\{\overline{I^r J^s}\right\}$.

Key Words : Normal Hilbert Polynomial of Two Ideals; Analytically Unramified Local Ring; Integral Closure of An Ideal; Good Joint Reductions; Local Cohomology of Rees Algebra

Introduction

Let R be a commutative ring and I be an ideal of R. An element $x \in R$ is called integral over I if x satisfies the equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

for some $a_i \in I^i$, $i = 1, 2, \ldots, n$. The set $\overline{I}$ of elements that are integral over I is an ideal, called the *integral closure* of I. If $I = \overline{I}$ then I is called *complete* or *integrally closed*. A Noetherian local ring (R, m) is said to be *analytically unramified* if its m-adic completion is reduced. For an m-primary ideal I, in an analytically unramified local ring R of dimension d, there exists a polynomial $\overline{P}_I(x) \in \mathbb{Q}[x]$ of degree d called as the *normal Hilbert polynomial* of I, such that

$$\lambda(R/\overline{I^n}) = \overline{P}_I(n) \text{ for } n >> 0,$$

where $\lambda(M)$ denotes the length of an R-module M, [Rees, 1961, Theorem 1.4] and [Rees, 1961, Theorem 1.1]. We write

$$\overline{P_I}(n+1) = \overline{e}_0(I)\binom{n+d}{d} - \overline{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{e}_d(I)$$

for some integers $\overline{e}_i(I)$ for $i = 0, \ldots, d$. The coefficient $\overline{e}_0(I) = e(I)$ is called the *multiplicity of I*. Let $\mathbb{N}$ denote the set of nonnegative integers. D. Rees studied the numerical function

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$\overline{H}_{I,J}:\mathbb{N}^2\rightarrow$ defined as $\overline{H}_{I,J}(r,s)=\lambda(R/\overline{I^rJ^s})$.

He proved [Rees, 1981] that there exists a polynomial $\overline{P}_{I,J}(x,y)\in\mathbb{Q}[x,y]$ of total degree d such that $\overline{P}_{I,J}(r,s)=\overline{H}_{I,J}(r,s)$ for r, $s >> 0$ in an analytically unramified local ring. We write

$$\overline{P}_{I,J}(x,y)=\sum_{i+j\leq d}(-1)^{d-(i+j)}\overline{e}_{(i,j)}(I,J)\binom{x+i-1}{i}\binom{y+j-1}{j}$$

for some integers $\overline{e}_{(i,j)}(I,J)$.

In Section 2, we prove the difference formula for the filtration $\{\overline{I^rJ^s}\}$ for all integers r, s. A. V. Jayanthan and J. Verma proved the formula for the filtration $\{I^rJ^s\}$ for all integers r, $s\geq 0$. See (Jayanthan and Verma, 2002). We prove the formula for all integers r, s. The proof also works for $\{I^rJ^s\}$.

In Section 3, we prove another difference formula in terms of the homology of a modified Koszul complex of a joint reduction of two ideals. In [Marley, 1989], T. Marley studied the modified Koszul complex for $\mathbb{Z}$-filtration. If depth $G(F)_+=\oplus_{n\geq1}I_n/I_{n+1}\geq d-1$, Marley obtained a nice relationship between the postulation number of F and reduction number of F in a Cohen-Macaulay local ring of dimension d. We generalize this for the $\mathbb{Z}^2$-graded filtration $\{\overline{I^rJ^s}\}$ in 2-dimensional Cohen-Macaulay analytically unramified local ring. We prove that if $\overline{P}_{I,J}(r,s)=\overline{H}_{I,J}(r,s)$ for all $r\geq r_0, s\geq s_0$ then $\overline{I^{r+1}J^{s+1}}=a\overline{I^rJ^{s+1}}+b\overline{I^{r+1}J^s}$ for all $r\geq r_0, s\geq s_0$ and for any good joint reduction (a, b) of $\{\overline{I^rJ^s}\}$. See section 3.

The Difference Formula

The difference formula expresses the difference of Hilbert polynomial and Hilbert function in terms of Euler characteristic of the local cohomology of Rees algebra. A. V. Jayanthan and J. K. Verma (Jayanthan and Verma, 2002) proved the formula for the filtration $\{I^rJ^s\}$ for all integers $r, s\geq 0$. We prove the formula for all integers r, s for the filtration $\{\overline{I^rJ^s}\}$. The proof also works for $\{I^rJ^s\}$. Let us set up the notation for various Rees algebras:

The Rees ring of the filtration $\{I^rJ^s\}:=\mathfrak{R}(I,J)=\oplus_{r,s\geq0}I^rJ^st_1^rt_2^s$,

The Rees ring of the filtration $\{\overline{I^rJ^s}\}:=\overline{\mathfrak{R}}(I,J)=\oplus_{r,s\geq0}\overline{I^rJ^s}t_1^rt_2^s$,

The extended Rees ring of the filtration $\{I^rJ^s\}:=\mathfrak{R}'(I,J)=\oplus_{r,s\in Z}I^rJ^st_1^rt_2^s$,

The extended Rees ring of the filtration $\{\overline{I^rJ^s}\}:=\overline{\mathfrak{R}'}(I,J)=\oplus_{r,s\in Z}\overline{I^rJ^s}t_1^rt_2^s$,

where Z denotes the set of integers. Let the extended associated graded ring with respect to I of the filtration

$$\{I^rJ^s\} \text{ be } \mathfrak{R}'(I,J\,|\,I)=\oplus_{r,s\in Z}\frac{I^rJ^s}{I^{r+1}J^s} \text{ and}$$

$$\{\overline{I^rJ^s}\} \text{ be } \overline{\mathfrak{R}'}(I,J\,|\,I)=\oplus_{r,s\in Z}\frac{\overline{I^rJ^s}}{\overline{I^{r+1}J^s}}.$$

Similarly let the extended associated graded ring with respect to J of the filtration

$$\{I^rJ^s\} \text{ be } \mathfrak{R}'(I,J\,|\,J)=\oplus_{r,s\in Z}\frac{I^rJ^s}{I^rJ^{s+1}} \text{ and}$$

$\left\{\overline{I^r J^s}\right\}$ be $\overline{\mathfrak{R}'}(I, J \mid J) = \oplus_{r,s\in Z} \frac{\overline{I^r J^s}}{\overline{I^r J^{s+1}}}$.

For a bigraded $B_{(0,0)}$-algebra $B = \oplus B_{(r,s\in Z^2)}$, let B_{++} denote the ideal generated by $\oplus_{r,s\geq 1} B_{(r,s)}$.

Theorem 1: *[The Difference Formula] Let (R, m) be an analytically unramified local ring of dimension d and I, J be m-primary ideals in R. Put* $\mathfrak{R} = \mathfrak{R}(I, J)$, $\mathfrak{R}' = \mathfrak{R}'(I, J)$ *and* $\overline{\mathfrak{R}'} = \overline{\mathfrak{R}'}(I, J)$. *Then for all integers r, s,*

1. $\lambda_R([H^i{}_{\mathfrak{R}'_{++}}(\overline{\mathfrak{R}'})]_{(r,s)}) < \infty$ for all $i \geq 0$.

2. $\overline{P}_{I,J}(r,s) - \overline{H}_{I,J}(r,s) = \sum_{i=0}^{d} (-1)^i \lambda_R ([H^i{}_{\mathfrak{R}'_{++}}(\overline{\mathfrak{R}'})]_{(r,s)})$

Proof : (1) Since $\overline{\mathfrak{R}}(I, J)$ is a finitely generated $\mathfrak{R}(I, J)$-module, by Theorem [Jayanthan and Verma, 2002, Theorem 2.3], for all $i \geq 0$, $[H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}}(I, J))]_{(r,s)}$ is a finitely generated R-module for all integers r, s and vanishes for large r, s. By (Jaynathan and Verma, 2002, Proposition 4.5), for all $r, s \geq 0$ and $i \geq 0$, we have

$$[H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}'})]_{(r,s)} \cong [H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}}(I, J))]_{(r,s)}.$$

In order to prove that $\lambda_R([H^i{}_{\mathfrak{R}'_{++}}(\overline{\mathfrak{R}'})]_{(r,s)}) < \infty$ for all integers r, s we use decreasing induction on r, s. By change of ring principle, $H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}'}) \cong H^i{}_{\mathfrak{R}'_{++}}(\overline{\mathfrak{R}'})$. Thus it suffices to prove that $\lambda_R([H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}'})]_{(r,s)}) < \infty$. As $[H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}'})]_{(r,s)} = 0$ for large r, s, the result is true for large r, s. Let $u_i = t_i^{-1}$ for $i = 1, 2$. Let $M(p, q)$ denote the module M with $M(p, q)_{(m,n)} = M_{(m+p,\, n+q)}$. Consider the exact sequence of bigraded $\mathfrak{R}'$-modules

$$0 \to \overline{\mathfrak{R}'}(1,0) \xrightarrow{u_1} \overline{\mathfrak{R}'} \to \overline{\mathfrak{R}'}(I, J \mid I) \to 0 \text{ and}$$

$$0 \to \overline{\mathfrak{R}'}(0,1) \xrightarrow{u_2} \overline{\mathfrak{R}'} \to \overline{\mathfrak{R}'}(I, J \mid J) \to 0.$$

Let $\mathfrak{R}(I, J \mid I) = \oplus_{r,s\geq 0} \mathfrak{R}'(I, J \mid I)_{(r,s)}$ and $\overline{\mathfrak{R}}(I, J \mid I) = \oplus_{r,s\geq 0} \overline{\mathfrak{R}'}(I, J \mid I)_{(r,s)}$. By the change of ring princple,

$$H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}'}(I, J \mid I)) = H^i{}_{\mathfrak{R}(I,J|I)_{++}}(\overline{\mathfrak{R}'}(I, J \mid I))$$

for all $i \geq 0$.

From the above short exact sequence we obtain the long exact sequence of R-modules

$$\begin{aligned} &\to [H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}'})]_{(r+1,s)} \to [H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}'})]_{(r,s)} \\ &\to [H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}'}(I, J \mid I))]_{(r,s)} \to \cdots \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\to [H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}'})]_{(r,s+1)} \to [H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}'})]_{(r,s)} \\ &\to [H^i{}_{\mathfrak{R}_{++}}(\overline{\mathfrak{R}'}(I, J \mid J))]_{(r,s)} \to \cdots. \end{aligned} \tag{2.3}$$

Consider the exact sequence

$$0 \to \overline{\mathfrak{R}}(I, J \mid I) \to \overline{\mathfrak{R}'}(I, J \mid I) \to \frac{\overline{\mathfrak{R}'}(I, J \mid I)}{\overline{\mathfrak{R}}(I, J \mid I)} \to 0.$$

Since $\frac{\overline{\mathfrak{R}'}(I, J \mid I)}{\overline{\mathfrak{R}}(I, J \mid I)}$ is $\mathfrak{R}(I, J \mid I)_{++}$-torsion,

$$H^0{}_{\mathfrak{R}(I,J|I)_{++}}(\overline{\mathfrak{R}'}(I, J \mid I) / \overline{\mathfrak{R}}(I, J \mid I)) = \frac{\overline{\mathfrak{R}'}(I, J \mid I)}{\overline{\mathfrak{R}}(I, J \mid I)}$$

and

$$H^i{}_{\mathfrak{R}(I,J|I)_{++}}(\overline{\mathfrak{R}'}(I, J \mid I) / \overline{\mathfrak{R}}(I, J \mid I)) = 0 \text{ for all } i > 0.$$

Therefore $H^i{}_{\mathfrak{R}(I,J|I)_{++}}(\overline{\mathfrak{R}}(I, J \mid I)) \cong H^i{}_{\mathfrak{R}(I,J|I)_{++}}(\overline{\mathfrak{R}'}(I, J \mid I))$ for all $i > 1$ and the sequence

$$0 \to H^0_{\Re(I,J|I)_{++}}(\overline{\Re}(I,J\mid I)) \to H^0_{\Re(I,J|I)_{++}}(\overline{\Re'}(I,J\mid I)) \to \frac{\overline{\Re'}(I,J\mid I)}{\overline{\Re}(I,J\mid I)}$$

$$\to H^1_{\Re(I,J|I)_{++}}(\overline{\Re}(I,J\mid I)) \to H^1_{\Re(I,J|I)_{++}}(\overline{\Re'}(I,J\mid I)) \to 0$$

is exact. Thus for all $r < 0$ and any integer s or for all $r, s \geq 0$

$$[H^i_{\Re(I,J|I)_{++}}(\overline{\Re}(I,J\mid I))]_{(r,s)} \cong [H^i_{\Re(I,J|I)_{++}}(\overline{\Re'}(I,J\mid I))]_{(r,s)} \text{ for all } i \geq 0.$$

By Theorem [Jayanthan and Verma, 2002, Theorem 2.3], $[H^i_{\Re(I,J|I)_{++}}(\overline{\Re}(I,J\mid I))]_{(r,s)}$ is a finite $\Re(I,J\mid I)_{(0,0)}$-module for all integers r, s and $i \geq 0$. Since $\Re(I,J\mid I)_{(0,0)}$ is artinian,

$\lambda_R[H^i_{\Re(I,J|I)_{++}}(\overline{\Re}(I,J\mid I))]_{(r,s)} < \infty$ for all integers r, s.

Therefore

$\lambda_R[H^i_{\Re_{++}}(\overline{\Re'}(I,J\mid I))]_{(r,s)} < \infty$ for $r < 0$ and any integer s or $r, s \geq 0$. (2.4)

Similar argument shows that

$\lambda_R[H^i_{\Re_{++}}(\overline{\Re'}(I,J\mid J))]_{(r,s)} < \infty$ for any integer r and $s < 0$ or $s \geq 0$. (2.5)

Using exact sequences 2.2 and 2.3 and induction hypothesis result follows for all integers r, s.

(2) For a bigraded module M over the bigraded ring $\Re$ let $H_M : \mathbb{Z}^2 \to$ be the numerical function defined by $H_M(r, s) := \lambda_R(M_{(r,s)})$ and $P_M(x, y) \in \mathbb{Q}[x, y]$ be the polynomial associated with H_M. We set

$$\chi_M(r,s) = \sum_{i \geq 0} (-1)^i \lambda_R([H^i_{\Re_{++}}(M)]_{(r,s)}) \text{ and}$$

$$g_M(r,s) = P_M(r,s) - H_M(r,s).$$

Let $\overline{g}(r,s) = \overline{P}_{I,J}(r,s) - \overline{H}_{I,J}(r,s)$. For $r < 0$ and any integer s or $r, s \geq 0$ we have

$$\chi_{\overline{\Re'}}(r+1,s) - \chi_{\overline{\Re'}}(r,s) = -\chi_{\overline{\Re'}(I,J|I)}(r,s) \text{ by (2.2)}$$

$$= -\chi_{\overline{\Re}(I,J|I)}(r,s)$$

$= g_{\overline{\Re}(I,J|I)}(r,s)$ by [Jayanthan and Verma, 2002, Theorem 2.4]

$$= \overline{g}(r+1,s) - \overline{g}(r,s).$$

Thus for $r < 0$ and any integer s or $r, s \geq 0$

$$\chi_{\overline{\Re'}}(r,s) - \overline{g}(r,s) = \chi_{\overline{\Re'}}(r+1,s) - \overline{g}(r+1,s) \quad (2.6)$$

Similarly for any integer r and $s < 0$ or $r, s \geq 0$

$$\chi_{\overline{\Re'}}(r,s) - \overline{g}(r,s) = \chi_{\overline{\Re'}}(r,s+1) - \overline{g}(r,s+1)$$

But $\chi_{\overline{\Re'}}(r,s) = \overline{g}(r,s) = 0$ for large r, s. Therefore

$\chi_{\overline{\Re'}}(r,s) = \overline{g}(r,s) = \overline{P}_{I,J}(r,s) - \overline{H}_{I,J}(r,s)$ for all integers r, s.

The modified Koszul Complex

In this section we study a modified Koszul complex. Before defining the complex we prove properties of numerical functions.

Given a numerical function $f : \mathbb{Z}^2 \to$ we define $\Delta_{e_i}(f) : \mathbb{Z}^2 \to$ as

$$\Delta_{e_i}(f)(\mathbf{n}) = f(\mathbf{n} + e_i) - f(\mathbf{n})$$

where $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We say f is of *polynomial type* of degree d if there exists a polynomial $P(x, y) \in \mathbb{Q}[x, y]$ of total degree d such that $P(r, s) = f(r, s)$ for, $r, s >> 0$. We write

$$P(x,y)=\sum_{i+j\le d}(-1)^{d-(i+j)}a_{(i,j)}\binom{x+i-1}{i}\binom{y+j-1}{j}$$

Proposition 2: *Let* $f:\mathbb{Z}^2\to$ *be a numerical function of polynomial type of degree 2. Let P(x, y) be a polynomial associated with f of degree 2. Then for all* $r,s\in\mathbb{Z}$,

$$\Delta_{e_2}\Delta_{e_1}(P(r,s)-f(r,s))=a_{(1,1)}$$
$$-[f(r+1,s+1)-f(r+1,s)-f(r,s+1)+f(r,s)].$$

Proof : We have

$$\Delta_{e_1}P(r,s)=a_{(2,0)}(r+1)+a_{(1,1)}s-a_{(1,0)}$$

Therefore

$$\Delta_{e_2}\Delta_{e_1}(P(r,s))=a_{(1,1)}$$

Also,

$$\Delta_{e_2}(\Delta_{e_1})f(r,s)=\Delta_{e_2}(f(r+1,s)-f(r,s))$$

$$=f(r+1,s+1)-f(r+1,s)-f(r,s+1)+f(r,s)$$

Therefore for all $r,s\in\mathbb{Z}$,

$$\Delta_{e_2}\Delta_{e_1}(P(r,s)-f(r,s))=a_{(1,1)}$$
$$-[f(r+1,s+1)-f(r+1,s)-f(r,s+1)+f(r,s)].$$

Let (R, m) be a Noetherian local ring of dimension 2 and I and J be m-primary ideals in R. Rees introduced joint reductions in [Rees, 1984] for the filtration $\{I^rJ^s\}$. A sequence (a, b) is a *joint reduction* of the filtration $\{\overline{I^rJ^s}\}$ if $a\in I$ and $b\in J$ and for r, s >> 0

$$\overline{I^rJ^s}=a\overline{I^{r-1}J^s}+b\overline{I^rJ^{s-1}}.$$

Thus $IJ\subseteq\sqrt{(a,b)}$. Rees [Rees, 1984] proved the existence of joint reductions for the filtration $\{I^rJ^s\}$ if residue field is infinite. The same proof shows the existence of joint reductions for the filtration $\{\overline{I^rJ^s}\}$ if R is an analytically unramified local ring with infinite residue field.

Let (a, b) be a joint reduction of the filtration $\{\overline{I^rJ^s}\}$. Consider the Koszul complex $K.(a, b; R)$:

$$0\longrightarrow R\xrightarrow{\psi_1}R^{\binom{2}{1}}\xrightarrow{\psi_0}R\longrightarrow 0.$$

This induces the complex :

$$C.((a,b),r,s):0\longrightarrow\frac{R}{\overline{I^rJ^s}}\xrightarrow{\phi_1}\frac{R}{\overline{I^{r+1}J^s}}\oplus\frac{R}{\overline{I^rJ^{s+1}}}\xrightarrow{\phi_0}\frac{R}{\overline{I^{r+1}J^{s+1}}}\longrightarrow 0$$

where ϕ_0 and ϕ_1 are defined as

$$\phi_1(\bar{x})=(\overline{ax},\overline{bx}),\phi_0(\bar{u},\bar{v})=\overline{bu-av}.$$

Let $H_i((a, b), r, s)$ denote the ith homology of the complex $C.((a, b), r, s)$. In next Proposition we derive a relation between homology of this complex and the difference $\overline{P}_{I,J}(r,s)-\overline{H}_{I,J}(r,s)$.

Proposition 3: *Let (R, m) be an analytically unramified local ring of dimension 2 with infinite field and I, J be m-primary ideals. Let (a, b) be a joint reduction of* $\{\overline{I^rJ^s}\}$. *Then for all integers r, s*

$$\Delta_{e_2}\Delta_{e_1}(\overline{P}_{I,J}(r,s)-\overline{H}_{I,J}(r,s))=\bar{e}_{(1,1)}(I,J)$$
$$-\sum_{i=0}^{2}(-1)^i\lambda(H_i((a,b),r,s)).$$

Proof : By Propositon 2,

$$\Delta_{e_2}\Delta_{e_1}(\overline{P}_{I,J}(r,s)-\overline{H}_{I,J}(r,s))=\bar{e}_{(1,1)}(I,J)$$
$$-[\overline{H}_{I,J}(r+1,s+1)-\overline{H}_{I,J}(r+1,s)$$
$$-\overline{H}_{I,J}(r,s+1)+\overline{H}_{I,J}(r,s)].$$

But

$$\overline{H}_{I,J}(r+1,s+1)-\overline{H}_{I,J}(r+1,s)-\overline{H}_{I,J}(r,s+1)$$
$$+\overline{H}_{I,J}(r,s)=\sum_{i=0}^{2}(-1)^i\lambda(C_i((a,b),r,s)$$
$$=\sum_{i=0}^{2}(-1)^i\lambda(H_i((a,b),r,s))$$

Hence the result follows.

We say (a, b) is a *good joint reduction* of $\{\overline{I^rJ^s}\}$ if (a, b) is a joint reduction of $\{\overline{I^rJ^s}\}$ and

$$(a)\cap\overline{I^rJ^s}=a\overline{I^{r-1}J^s} \text{ for all } s\in\mathbb{Z} \text{ and } r>0$$

and

$$(b)\cap\overline{I^rJ^s}=b\overline{I^rJ^{s-1}} \text{ for all } r\in\mathbb{Z} \text{ and } s>0.$$

Rees (Rees, 1981) proved that good joint reductions exist in an analytically unramified Cohen-Macaulay local ring of dimension 2 with infinite residue field.

From now onwards we assume that R is an analytically unramified Cohen-Macaulay local ring of dimension 2 with infinite residue field unless otherwise stated. Let (a, b) be a good joint reduction of $\{\overline{I^rJ^s}\}$.

Proposition 4: *Let (R, m) be an analytically unramified Cohen-Macaulay local ring of dimension 2 with infinite residue field and I, J be m-primary ideals. Let (a, b) be a good joint reduction of $\{\overline{I^rJ^s}\}$.*

Then for all integers r, s

1. $H_0((a,b),r,s)\cong\dfrac{R}{((a,b),\overline{I^{r+1}J^{s+1}})}$

2. $H_1((a,b),r,s)\cong\dfrac{\overline{I^{r+1}J^{s+1}}\cap(a,b)}{a\overline{I^rJ^{s+1}}+b\overline{I^{r+1}J^s}}$

3. $H_2((a,b),r,s)=0.$

Proof : (1) Note that $H_0((a,b),r,s)$ is the homology of

$$\frac{R}{\overline{I^{r+1}J^s}}\oplus\frac{R}{\overline{I^rJ^{s+1}}}\xrightarrow{\phi_0}\frac{R}{\overline{I^{r+1}J^{s+1}}}\longrightarrow 0$$

where $\phi_0(\bar{u},\bar{v})=\overline{bu-av}$. Therefore

$$\text{im}(\phi_0)=\frac{(a,b)+\overline{I^{r+1}J^{s+1}}}{\overline{I^{r+1}J^{s+1}}}$$

and hence

$$H_0((a,b),r,s)\cong\frac{R}{((a,b),\overline{I^{r+1}J^{s+1}})}$$

(2) Consider the commutative diagram :

$$\begin{array}{ccccccc}
0\to \text{im}\,\phi_1 \to & \dfrac{R}{\overline{I^{r+1}J^s}}\oplus\dfrac{R}{\overline{I^rJ^{s+1}}} & \xrightarrow{\delta} & \dfrac{(a,b)}{a\overline{I^rJ^{s+1}}+b\overline{I^{r+1}J^s}} & \to 0\\
\downarrow & \psi\downarrow & & \gamma\downarrow & \\
0\to \ker\phi_0 \longrightarrow & \dfrac{R}{\overline{I^{r+1}J^s}}\oplus\dfrac{R}{\overline{I^rJ^{s+1}}} & \xrightarrow{\phi_0} & \dfrac{R}{\overline{I^{r+1}J^{s+1}}} &
\end{array}$$

where $\psi = id$, $\delta(\bar{u},\bar{v})=\overline{bu-av}$ and γ is the natural map. To prove exactness of top row it is enough to prove exactness at $\dfrac{R}{\overline{I^{r+1}J^s}}\oplus\dfrac{R}{\overline{I^rJ^{s+1}}}$. Suppose $\delta(\bar{u},\bar{v})=0.$ Then $bu-av\in a\overline{I^rJ^{s+1}}+b\overline{I^{r+1}J^s}$. Let $bu - av = ap + bq$ for some $p\in\overline{I^rJ^{s+1}}, q\in\overline{I^{r+1}J^s}$. Therefore $b(u - q) = a(p + v)$. Since $IJ\subseteq\sqrt{(a,b)}$, (a, b) is m-primary. Hence a, b is a regular sequence in R. Thus $u - q = at$ for some $t\in R$. Hence $p + v = bt$. Then $\phi_1(\bar{t})=(\overline{at},\overline{bt})=(\bar{u},\bar{v})$. This proves exactness of top row. The bottom row is clearly exact.

Using the snake lemma and the fact that ψ is an isomorphism we obtain,

$$\ker\gamma\cong\frac{\ker\phi_0}{\text{im}\,\phi_1}=H_1((a,b,r,s))$$

Therefore,

$$H_1((a,b),r,s) \cong \frac{\overline{I^{r+1}J^{s+1}} \cap (a,b)}{a\overline{I^r J^{s+1}} + b\overline{I^{r+1}J^s}}.$$

(3) If r, $s \leq 0$, then result is trivially true as $\frac{R}{\overline{I^r J^s}} = 0$. Suppose $r > 0$ and $s \in \mathbb{Z}$. Let $\bar{x} \in \ker\phi_1$. Then $ax \in \overline{I^{r+1}J^s} \cap (a) = a\overline{I^r J^s}$. Since a is a nonzerodivisor, $x \in \overline{I^r J^s}$. Thus $\bar{x} = 0$ and hence $H_2((a,b),r,s) = 0$. Similar argument implies that $H_2((a,b),r,s) = 0$ if $r \in \mathbb{Z}$ and $s > 0$.

Theorem 5: *Let (R, m) be an analytically unramified Cohen-Macaulay local ring of dimension 2 with infinite residue field. Let I, J be m-primary ideals. Let (a, b) be a good joint reduction of* $\left\{\overline{I^r J^s}\right\}$. *Then for all* $r,s \in \mathbb{Z}$,

$$\Delta_{e_2}\Delta_{e_1}(\overline{P}_{I,J}(r,s) - \overline{H}_{I,J}(r,s)) = \lambda\left(\frac{\overline{I^{r+1}J^{s+1}}}{a\overline{I^r J^{s+1}} + b\overline{I^{r+1}J^s}}\right)$$

Proof : By Proposition 3,

$$\Delta_{e_2}\Delta_{e_1}(\overline{P}_{I,J}(r,s) - \overline{H}_{I,J}(r,s)) = \bar{e}_{(1,1)}(I,J) - \sum_{i=0}^{2}(-1)^i \lambda(H_i((a,b),r,s))$$

By Rees' mixed multiplicity theorem [Rees, 1984, Theorem 2.4], $\bar{e}_{(1,1)}(I,J) = \lambda\left(\frac{R}{(a,b)}\right)$.

Therefore, by Proposition 3 and 4, for all $r,s \in \mathbb{Z}$ we obtain that,

$$\Delta_{e_2}\Delta_{e_1}(\overline{P}_{I,J}(r,s) - \overline{H}_{I,J}(r,s)) = \lambda\left(\frac{R}{(a,b)}\right) - \lambda\left(\frac{R}{(a,b),\overline{I^{r+1}J^{s+1}}}\right) + \lambda\left(\frac{\overline{I^{r+1}J^{s+1}} \cap (a,b)}{a\overline{I^r J^{s+1}} + b\overline{I^{r+1}J^s}}\right)$$

$$= \lambda\left(\frac{(a,b),\overline{I^{r+1}J^{s+1}}}{(a,b)}\right) + \lambda\left(\frac{\overline{I^{r+1}J^{s+1}} \cap (a,b)}{a\overline{I^r J^{s+1}} + b\overline{I^{r+1}J^s}}\right)$$

$$= \lambda\left(\frac{\overline{I^{r+1}J^{s+1}}}{(a,b) \cap \overline{I^{r+1}J^{s+1}}}\right) + \lambda\left(\frac{\overline{I^{r+1}J^{s+1}} \cap (a,b)}{a\overline{I^r J^{s+1}} + b\overline{I^{r+1}J^s}}\right)$$

$$= \lambda\left(\frac{\overline{I^{r+1}J^{s+1}}}{a\overline{I^r J^{s+1}} + b\overline{I^{r+1}J^s}}\right).$$

Corollary 6: *Let (R, m) be an analytically unramified Cohen-Macaulay local ring of dimension 2 with infinite residue field. Let I, J be m-primary ideals. Let (a, b) be a good joint reduction of* $\left\{\overline{I^r J^s}\right\}$. *Suppose* $\overline{P}_{I,J}(r,s) = \overline{H}_{I,J}(r,s)$ *for all* $r \geq r_0, s \geq s_0$ *Then for all* $r \geq r_0, s \geq s_0$,

$$\overline{I^{r+1}J^{s+1}} = a\overline{I^r J^{s+1}} + b\overline{I^{r+1}J^s}.$$

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