

On the Realizability of Relativistic Acoustic Geometry Under A Generalized Perturbation Scheme for Axisymmetric Matter Flow Onto Black Holes

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We propose a novel linear perturbation scheme to study the stability properties of the stationary transonic integral solutions for axisymmetric matter flow around astrophysical black holes for the Schwarzschild as well as for rotating Rindler spacetime. We discuss the emergence of the relativistic acoustic geometry as a consequence of such stability analysis. Our work thus makes a crucial connection between two apparently non-overlapping fields of research - the accretion astrophysics and the analogue gravity phenomena.

Key Words : Black Holes; Schwarzschild Metric; Accretion.

Introduction

The stationary transonic accretion solutions have long been found useful to probe the spectral signature of astrophysical black holes, see, e.g., (Frank *et al.*, 2002) for a detail review. It is, however, important to ensure the stability of such stationary solutions - at least within a reasonable astrophysical time scale - since transient phenomena are not quite uncommon during accretion processes. One can accomplish such task by perturbing the corresponding spacetime dependent fluid dynamic equations governing the accretion process and by studying whether such perturbation converges to ensure the stability of the transonic solutions of the time independent part of the aforementioned fluid dynamic equations. In the present work, we provide a linear perturbation scheme for low angular momentum irrotational inviscid accretion to ensure that the corresponding stationary integral transonic solutions are stable and to demonstrate that the relativistic acoustic geometry emerges from such perturbation analysis. Such accretion phenomena are observed in OB stellar

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winds accretion onto detached binary systems, semi-detached low-mass non-magnetic binaries and matter flow onto the supermassive black hole situated at the dynamical heart of our Galaxy. The linear stability analysis employed here helps us to determine whether the acoustic geometric structure (sonic or critical points etc.) present in almost any astrophysical accretion phenomenon, can in principle be observed. We also note that the stability of the accretion process or those stationary solutions may also be relevant in the context of the stability of the spacetime itself. Because if an accretion process ‘grows’ unboundedly with time, its backreaction may change the background spacetime structure.

As of now, the acoustic geometry for any classical fluid configuration has been obtained by perturbing the corresponding velocity potential, see, e.g., (Bercelo *et al.*, 2011) for further detail. We, for the first time in literature, obtained the acoustic geometry for general relativistic axisymmetric accretion flow by perturbing a more physically realizable, astrophysically relevant, and most importantly, observationally measurable entity - the mass accretion rate. Since the density and the velocity fields are directly interlinked through the mass accretion rate, perturbation of the accretion rate would provide a more physically realizable stability analysis scheme. Accretion rate is associated with matter flow onto astrophysical black holes and hence the aforementioned formalism shows that accreting black hole systems can be considered as an example of classical analogue system naturally found in the Universe and is unique in the sense that only in such systems both the gravitational as well as acoustic horizons exist. This is a very important finding as we believe since it shows how the actual gravitational field determines the salient features of the emergent gravity phenomena.

In subsequent sections, we shall demonstrate the application of our perturbation scheme first for the Schwarzschild spacetime and then for rotating Rindler spacetime, describing the background fluid flow.

Axisymmetric Accretion in the Schwarzschild Metric

We consider the Schwarzschild line element of the form $ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$, where $f \equiv f(r) = 1 - \frac{2}{r}$. For a dissipationless perfect fluid, the energy momentum tensor can be written as $T^{\mu\nu} = (\epsilon + p) v^\mu v^\nu + p g^{\mu\nu}$ where ϵ, p, ρ and v^μ are the mass energy density, pressure, the fluid density and the four velocity of the accretion flow, respectively. Throughout this work, we will use $G = M_{BH} = c = 1$, M_{BH} being the mass of the black hole, other symbols represents the known constants. Any representative length and velocity will be normalized by GM_{BH}/c^2 and c , respectively. A polytropic equation of state of the form $p = K\rho^\gamma$, γ being the ratio of the specific heats at constant pressure and at constant volumes, respectively, will be used to describe the accreting fluid. The specific enthalpy is defined as $h = \frac{\epsilon + p}{\rho}$.

The polytropic sound speed $c_s = \sqrt{\left(\frac{\partial p}{\partial \epsilon}\right)_s}$, s being the specific entropy, can thus be obtained as $c_s =$

$\sqrt{\frac{\gamma k(\gamma - 1)}{\gamma k + (\gamma - 1)(\rho)^{1-\gamma}}}$. The normalization condition $v^\mu v_\mu = -1$ gives $v^t = \frac{\sqrt{v^2 + f + (v^\phi)^2 r^2 f}}{f}$.

We consider low angular momentum inviscid accretion in our work. We show that the effective potential for fluid accretion in the Schwarzschild metric is $V_{eff} = \sqrt{\frac{r^2(r-2)}{\lambda^2(r-2) + r^3}}$, λ being the specific angular momentum of the flow. If $\lambda < 3.674$ matter plunges through the event horizon even if one neglects the viscous transport of the angular momentum (Chakrabarti, 1996). Such a choice of sub Keplerian angular momentum distribution prefers that the geometric configuration of matter is of conical type, see, e.g., (Tarafdar and Das, 2013) and references therein for details about various geometrical flow configurations accreting matter can have), and any flow variable of the fluid F , is assumed to be a slowly varying function of the local flow thickness. The vertical averaging of F or any other accretion variable can be performed by integrating that variable over θ and ϕ as $\int d\theta d\phi \sqrt{-g} F = 4\pi F H_\theta$, where H_θ is the characteristic angular scale of the flow about the equatorial plane and is assumed to be the same for all averaged flow variables. The continuity equation $(\rho v^\mu)_{;\mu} = 0$ can thus be expressed as

$$\partial_t (\rho v^t \sqrt{-g} H_\theta) + \partial_r (\rho v \sqrt{-g} H_\theta) = 0. \quad (1)$$

Finding the off-equatorial accretion solution is beyond the scope of this work and we concentrate on the accretion flow on the equatorial plane only. Accordingly, the continuity equation finally takes the form

$$\partial_t (\rho v^t r^2) + \partial_r (\rho v r^2) = 0. \quad (2)$$

With the help of the continuity equation, the linear energy momentum conservation equation on the equatorial plane can be obtained as

$$v^\nu \partial_\nu v^\mu + v^\nu v^\lambda \Gamma_{\nu\lambda}^\mu + \frac{c_s^2}{\rho} (v^\mu v^\nu + g^{\mu\nu}) \partial_\nu \rho = 0, \quad (3)$$

where $\Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\eta} (\partial_\nu g_{\eta\lambda} + \partial_\lambda g_{\eta\nu} - \partial_\eta g_{\nu\lambda})$.

For $\mu = t$, using the vertically averaged accretion variables one can have

$$v^t \partial_t v^t + v \partial_r v^t + \frac{v^t v}{f} \partial_r f + \frac{c_s^2}{\rho} \left[\left((v^t)^2 - \frac{1}{f} \right) \partial_t \rho + v v^t \partial_r \rho \right] = 0. \quad (4)$$

For $\mu = r$, the relativistic Euler equation is found to be

$$v^t \partial_t v + v \partial_r v + \frac{1}{2} \partial_r f (1 + (v^\phi)^2 r^2) - (v^\phi)^2 f r + (v^2 + f) \frac{c_s^2}{\rho} \partial_r \rho + \frac{v v^t c_s^2}{\rho} \partial_t \rho = 0. \quad (5)$$

Similarly, $\mu = \phi$ provides

$$v^t \partial_t v^\phi + v \partial_r v^\phi + \frac{2v v^\phi}{r} + \frac{v^\phi c_s^2}{\rho} [v^t \partial_t \rho + v \partial_r \rho] = 0. \quad (6)$$

Irrotational condition can now be introduced by making vorticity operator $w_{\mu\nu} = P_\mu^\eta P_\nu^\eta \nabla_\nu v_\eta = 0$, where $P_\nu^\mu = \delta_\nu^\mu - v^\mu v_\nu$ is the projection operator which projects an arbitrary vector in space-time into its component in the subspace orthogonal to v^μ . Hence for irrotational flow, the Euler equation provides $\partial_\mu (h v_\nu) - \partial_\nu (h v_\mu) = 0$ and for $[(\mu = r, \nu = t), (\mu = r, \nu = \phi), (\mu = t, \nu = \phi)]$ provides

$$\begin{aligned} \frac{1}{f} \partial_t v + \partial_r (f v t) + \frac{c_s^2}{\rho} \left[\frac{v}{f} \partial_t \rho + v^t f \partial_r \rho \right] &= 0, \\ f \left[\partial_r (v^\phi r^2) + v^\phi r^2 \frac{c_s^2}{\rho} \partial_r \rho \right] &= 0, \\ \partial_t \left[v^\phi g_{\phi\phi} \frac{\epsilon + p}{\rho} \right] &= 0. \end{aligned} \quad (7)$$

The last part of the above equation implies the conservation of the angular momentum since $v_\phi \left(\frac{\epsilon + p}{\rho} \right) = L$. The second part of the above equation helps to write the Euler equation in the following form

$$v^t \partial_t v + \partial_r \left(\frac{v^2 + f + (v^\phi)^2 r^2 f}{2} \right) + \frac{c_s^2}{\rho} \left[v v^t \partial_t \rho + (v^2 + f + v^{\phi^2} r^2 f) \partial_r \rho \right] = 0. \quad (8)$$

The stationary solution can thus be obtained by integrating the time independent part of the Euler and the continuity equations. We find that the total specific energy $\mathcal{E} = h v_t$ and the angular momentum $L = h v_\phi$ remain constant along the fluid world line. For steady state, the mass accretion rate $\dot{M} = 2\pi \rho v r^2$ is also found to be a first integral of motion.

Linear Perturbation Analysis in the Schwarzschild Metric

The dynamical velocity, flow density and the mass accretion rates are perturbed about their background steady state value. $\Psi_0 = \rho_0 v_0 r^2$ is defined as the background steady state value of the variable Ψ , which actually is the accretion rate (apart from the geometric constants). We write

$$v = v_0 + v', \quad v^\phi = v_0^\phi + v^{\phi'}, \quad \rho = \rho_0 + \rho', \quad \Psi' = (\rho' v_0 + \rho_0 v') r^2, \quad (9)$$

where the subscript '0' stands for the stationary values. The derivatives of ρ' and v' can be expressed in terms of derivatives of Ψ' as

$$\frac{\partial_t v'}{v_0} = \frac{v_0^t f^2}{\left[f + (v_0^\phi)^2 r^2 f (1 - c_{s_0}^2) \right]} \left[\left(v_0^t - \frac{(v_0^\phi)^2 r^2 c_{s_0}^2 f}{v_0^t f^2} \right) \frac{\partial_t \Psi'}{\Psi_0} + v_0 \frac{\partial_r \Psi'}{\Psi_0} \right], \quad (10)$$

$$\frac{\partial_t \rho'}{\rho_0} = \frac{-v_0}{\left[f + (v_0^\phi)^2 r^2 f (1 - c_{s_0}^2) \right]} \left[v_0 \frac{\partial_t \Psi'}{\Psi_0} + v_0^t f^2 \frac{\partial_r \Psi'}{\Psi_0} \right]. \quad (11)$$

The differential equation corresponding to the first order linearly perturbed Ψ thus comes out to be

$$\partial_t \left(h^{tt} \partial_t \Psi' \right) + \partial_t \left(h^{tr} \partial_r \Psi' \right) + \partial_r \left(h^{rt} \partial_t \Psi' \right) + \partial_r \left(h^{rr} \partial_r \Psi' \right) = 0, \quad (12)$$

where $h^{tt} = \frac{v_0}{v_0^t \Lambda f} \left[\frac{(v_0^t)^2 f^2 - c_{s_0}^2 (v_0^2 + (v_0^\phi)^2 r^2 f)}{f^2} \right]$, $h^{tr} = h^{rt} = \frac{v_0}{v_0^t \Lambda f} [v_0 v_0^t (1 - c_{s_0}^2)]$, and $h^{rr} = \frac{v_0}{v_0^t \Lambda f} [v_0^2 - c_{s_0}^2 (v_0^2 + f)]$.

We now construct

$$f^{\mu\nu} \equiv -\frac{v_0 c_{s_0}^2}{v_0^t \Lambda f} \begin{bmatrix} -\frac{1}{f} + (v_0^t)^2 \left(1 - \frac{1}{c_{s_0}^2}\right) & v_0 v_0^t \left(1 - \frac{1}{c_{s_0}^2}\right) \\ v_0 v_0^t \left(1 - \frac{1}{c_{s_0}^2}\right) & f + v_0^2 \left(1 - \frac{1}{c_{s_0}^2}\right) \end{bmatrix},$$

where $\Lambda = (1 + (v_0^\phi)^2 r^2 (1 - c_{s_0}^2))$.

The wave equation in Ψ' can now be written in a more compact invariant way as

$$\partial_\mu \left(f^{\mu\nu} \partial_\nu \Psi' \right) = 0. \quad (13)$$

One may try to identify this with the Klein-Gordon equation so as to define the acoustic metric, but there is an ambiguity associated with divergent conformal factor for defining acoustic metric in (1+1) dimension (Barcelo *et al.*, 2004, 2005). So we define an effective metric, $f_{\mu\nu}$, which still gives the desired causal structure.

$$f_{\mu\nu} \equiv -\frac{f v_0^t}{v_0} \begin{bmatrix} f + v_0^2 \left(1 - \frac{1}{c_{s_0}^2}\right) & -v_0 v_0^t \left(1 - \frac{1}{c_{s_0}^2}\right) \\ -v_0 v_0^t \left(1 - \frac{1}{c_{s_0}^2}\right) & -\frac{1}{f} + (v_0^t)^2 \left(1 - \frac{1}{c_{s_0}^2}\right) \end{bmatrix},$$

Setting $f_{tt} = 0$, gives the sonic point condition $u_0^2 = c_{s_0}^2$, where u is the radial velocity in local rest frame. We thus demonstrate that for the background flow of relativistic fluid in strong gravity, the analogue spacetime forms an intrinsic manifold configuration of which remains invariant for various perturbation schemes.

Stability Analysis in the Schwarzschild Metric

We now study the stability of the stationary solutions using the trial acoustic wave solution $\Psi' = p(r) \exp(-i\omega t)$. Substituting this in the wave equation results in

$$\omega^2 p(r) f^{tt} + i\omega [\partial_r (p(r) f^{rt}) + f^{tr} \partial_r p(r)] - [\partial_r (f^{rr} \partial_r p(r))] = 0. \quad (14)$$

We use the trial power solution

$$p(r) = \exp \left[\sum_{k=1}^n \frac{k_n(r)}{\omega^n} \right]. \quad (15)$$

Substitute this in Eqn.(14) result in a series in ω and setting the coefficients of individual powers of ω to zero gives the leading coefficients. Equating the coefficient of ω^2 term to zero gives

$$k_{-1} = i \int \frac{v_0 v_0^t (1 - c_{s_0}^2) \pm c_{s_0} \sqrt{\Lambda}}{v_0^2 - c_{s_0}^2 (v_0^2 + f)} dr. \quad (16)$$

Similarly equating the coefficient of ω term to zero gives

$$k_0 = \frac{-1}{2} \ln \left(\frac{c_{s_0} v_0}{v_0^t f \sqrt{\Lambda}} \right). \quad (17)$$

In the asymptotic limit, we get $k_{-1} \sim r$, $k_0 \sim \ln r$ and $k_1 \sim r^{-1}$. Since $\omega \gg 1$ we can see that $\omega r \gg \ln r \gg \frac{\omega}{r}$, the power series converges. Hence the stationary solutions obtained are self-consistent.

Acoustic Geometry for the Rindler Spacetime

A uniformly accelerating observer in the Minkowski space is described by the Rindler co ordinates (Wald, 1984). We now introduce uniform rotation $\phi \rightarrow \phi - \Omega t$ and the line element comes out to be $ds^2 = -(a^2 x^2 - \Omega^2 \rho^2) dt^2 - 2\Omega \rho^2 dt d\phi + \rho^2 d\phi^2 + dx^2 + d\rho^2$, where a , and Ω are constants. We diagonalise the Rindler metric to find the acoustic geometry through the procedure what has been followed for the Schwarzschild spacetime. We define a Killing vector χ_a , which is a linear combination of both $(\partial_t)_a$ and $(\partial_\phi)_a$ and is orthogonal to both ∂_t and ∂_ϕ . Hence $\chi_a = (\partial_t)_a + \alpha (\partial_\phi)_a$. The orthogonality property provides $\alpha = \Omega$ resulting $\chi_a = (\partial_t)_a + \Omega (\partial_\phi)_a$. Hence in the new diagonalized basis $[\chi_a, (\partial_x)_a, (\partial_\phi)_a, (\partial_\rho)_a]$, one obtains $g_{\tau\tau} = -a^2 x^2$, $g_{xx} = 1$, $g_{\rho\rho} = 1$, $g_{\phi\phi} = \rho^2$. The velocity component along the χ , v^τ can be obtained from the normalization condition as $v^\tau = \frac{\sqrt{1 + v^2 + (\rho v^\phi)^2}}{ax}$. The metric in the transformed basis is a static one and hence following the linear perturbation analysis as has been done for the Schwarzschild spacetime, one obtains the elements of the effective metric for the Rindler spacetime as

$$f_{\mu\nu} \equiv -\frac{axv_0^\tau}{v_0} \begin{bmatrix} 1 + v_0^2 \left(1 - \frac{1}{c_{s_0}^2}\right) & -v_0 v_0^\tau \left(1 - \frac{1}{c_{s_0}^2}\right) \\ -v_0 v_0^\tau \left(1 - \frac{1}{c_{s_0}^2}\right) & -\frac{1}{ax} + (v_0^\tau)^2 \left(1 - \frac{1}{c_{s_0}^2}\right) \end{bmatrix}.$$

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References

1. Barcelo C, Liberati S and Visser M (2005) Analogue Gravity *Living Rev Relativity* **8** 12
2. Barcelo C, Liberati S and Visser M (2011) Analogue Gravity *Living Rev Relativity* **14** (2011) 3
3. Barcelo C, Liberati S, Sonogo S and Visser M (2004) Causal structure of acoustic spacetimes *New J Phys* **6** 186
4. Chakrabarti SK (1996) Accretion Processes On Black Holes *Phys Rept* **266** 229
5. Frank J, King A and Raine D (2002) *Accretion Power in Astrophysics*, Cambridge University Press, Cambridge
6. Tarafdar P and Das TK (2013) Dependence of acoustic surface gravity on geometric configuration of matter for axisymmetric background flow in Schwarzschild metric, arXiv:1305.7134 (2013)
7. Wald R (1984) *General Relativity*, University Of Chicago Press.