Indian Journal of History of Science (2024) 59:341–350
https://doi.org/10.1007/s43539-024-00140-5

ARTICLE

Symmetry in Bharata's *Yamaka*: A mathematical analysis

Amba Kulkarni[1] · **Amruta Barbadikar**[1] · **Bharati Bhat**[1]

Received: 7 November 2023 / Accepted: 1 October 2024 / Published online: 20 January 2025

Abstract
Indian literature exhibits symmetry in abundance. The *śabdālaṅkāra*s (figure of sound) such as *Yamaka* present examples of symmetry in literature. The first classification of *Yamaka* is given by Bharata in his treatise *Nāṭyaśāstra*. It is interesting to note that all the patterns of *Yamaka* mentioned by Bharata are not only simple but also exhibit symmetry under the operations of reflection and rotation. In this paper, we show the simplicity of the classification of *Yamaka* by Bharata by proving that the patterns corresponding to each type of *Yamaka* form a group isomorphic to either the Klein four-group $\boldsymbol{K_4}$ or the groups $\mathbb{Z}_2$ or $\mathbb{Z}_1$.

Keywords Symmetry · Symmetry group · *Yamaka alaṅkāra* · Klein 4 group

1 Introduction

The word symmetry (originated from the Greek word symmetria) refers to the arrangement in dimensions or the due proportion. It refers to certain compositions of structural objects that preserve the structure. It acts as a measure of the beauty of the form or structure. Therefore it has always been considered as an important factor in visual aesthetics (Molnar & Molnar, 1986). Symmetry is abundant in nature. Several studies discuss various kinds of symmetry in living and non-living objects such as plants, living organisms, the structure of chemical molecules and organic compounds, etc. (Lederman & Hill, 2005; Bunker & Jensen, 2004). The symmetry in arts, architecture, music, and literature are also studied in great detail (Pavlovic & Triajstic, 1986).

Indian civilisation is no exception. According to Sihna (1989), 'Indian ethos has shaped through the centuries in evolving the facets of symmetry. Various shapes of Vedic altars constructed using the *Śulbasūtr* as are all symmetric.' Some of the finest examples of symmetry are the fractal symmetry found in the temples and wells across India. Indian literature also exhibits symmetry in abundance. The *śabdālaṅkāra*s such as *Yamaka*, and various *citrabandha kāvya*s (poetries composed in the form of drawings) present examples of symmetry in literature.

Yamaka is used to enrich the beauty of the poetry. *Yamaka* is a figure of sound where a group of syllabic patterns is repeated typically with a different meaning at different locations in the same verse. The first classification of *Yamaka* is given by Bharata in his treatise *Nāṭyaśāstra*. Later Daṇḍi, Rudraṭa, and many other scholars gave various classifications covering different patterns of repetitions. There is an elegance and simplicity in Bharata's classification.[1] It is interesting to note that all the types of *Yamaka* mentioned by Bharata are not only simple but also exhibit symmetry under the operations of reflection and rotation. In this paper, we show the simplicity of the classification of *Yamaka* by Bharata by proving that each type forms a group isomorphic to either the Klein four-group K_4 or the groups $\mathbb{Z}_2$, or $\mathbb{Z}_1$.

2 *Yamaka*

The embellishment in the poetry makes it different from the language which is used for regular communication. It enhances the beauty of the poetry. The systematic study of this ornamentation is known as the theory of *alaṅkāra*. It is one of the important disciplines in the *Kāvyaśāstra* tradition. Scholars of the *Alaṅkāra* school regard the poetic

✉ Amba Kulkarni
ambakulkarni@uohyd.ac.in

Amruta Barbadikar
amruta.barbadikar@gmail.com

Bharati Bhat
bharatimundgodi@gmail.com

1 Department of Sanskrit Studies, University of Hyderabad, Hyderabad 500046, India

[1] Bharata does not insist on different meanings.

ornaments (*alaṅkāras*) as essential to poetry. Jayadeva, a twelfth-century poet ridicules those who do not consider ornamentation as an essential characteristic of poetry as admitting fire without heat.[2] Thus, *alaṅkāras* are deemed a mandatory element in poetry.

*Alaṅkāra*s are of two types viz. *śabdālaṅkāra* and *arthālaṅkāra*. The ornamentation of the sound such as wordplay, phonemic repetition, and special syntactic arrangements are called *śabdālaṅkāra*. The *arthālaṅkāra*s are the ones where the involvement of the semantic beauty is traced. It consists of delightful comparisons, various kinds of arguments or cause-effect relations, or even expressions with multiple meanings employed in the poetic style.

Yamaka is the first *śabdālaṅkāra* in the literary tradition which was first propounded by Bharata with its systematic definition and classification. According to Bharata, *Yamaka* is the repetition of sound.[3] It is essentially employed in the metrical verses. It can occur at different positions in a verse. Based on the position of repetition either within or across the quarter of a verse Bharata classified them in 10 classes, providing the definition and examples for each class.

2.1 Bharata's classification

1. ***Pādānta*** - It is a formation of *Yamaka* when the same sequence occurs at the end (*anta*) of each quarter, or a foot (*pāda*).[4] For example,[5]

   दिनक्षयात् संहृतरश्मि**मण्डलं**
   दिवीव लग्नं तपनीय**मण्डलम्** |
   विभाति ताम्रं दिवि सूर्य**मण्डलं**
   यथा तरुण्याः स्तनभार**मण्डलम्** ||
   (16.65, नाट्यशास्त्र) (Ghosh, 1951)

   *dinakṣayāt saṃhṛtaraśmi**maṇḍalaṃ***
   *divīva lagnaṃ tapanīya**maṇḍalam*** |
   *vibhāti tāmraṃ divi sūrya**maṇḍalaṃ***
   *yathā taruṇyāḥ stanabhāra**maṇḍalam*** ||
   (16.65, *Nāṭyaśāstra*)

2. ***Kāñcī*** - In *Kāñcī-yamaka*, the repetition of the sequence occurs at the end as well as at the beginning of the same foot. The pattern repeated at the beginning and at the end or across different *pāda*s need not be the same. See the following example.

   <u>यामं यामं</u> **चन्द्रवतीनां द्रवतीनां**
   <u>व्यक्ताव्यक्ता</u> **सारजनीनां रजनीनाम्** |
   <u>फुल्ले फुल्ले</u> **सभ्भ्रमरे वाऽभ्रमरे वा**
   <u>रामा रामा</u> **विस्मयते च स्मयते च** ||
   (16.68, नाट्यशास्त्र)

   *<u>yāmaṃ yāmaṃ</u> can**dravatīnāṃ dravatīnāṃ***
   *<u>vyaktāvyaktā</u> sā**rajanīnāṃ rajanīnām*** |
   *<u>phulle phulle</u> sam**bhramare vābhramare vā***
   *<u>rāmā rāmā</u> vi**smayate ca smayate ca*** ||
   (16.68, *Nāṭyaśāstra*)

3. ***Samudgaka*** - In *Samudgaka yamaka*, the first half of a verse is repeated in the second half.

   For example,

   केतकीकुसुमपाण्डुरदन्तः
   शोभते प्रवरकाननहस्ती |
   <u>केतकीकुसुमपाण्डुरदन्तः</u>
   <u>शोभते प्रवरकाननहस्ती</u> ||
   (16.70, नाट्यशास्त्र)

   ketakīkusumapāṇḍuradantaḥ
   śobhate pravarakānanahastī |
   <u>ketakīkusumapāṇḍuradantaḥ</u>
   <u>śobhate pravarakānanahastī</u> ||
   (16.70, *Nāṭyaśāstra*)

4. ***Vikrānta*** - *Vikrānta* is the formation in which the alternate feet (that is, either the first and the third or the second and fourth) are the same. For example,

   स पूर्वं वारणो भूत्वा
   द्विशृङ्ग इव पर्वतः |
   अभवद्दन्तवैकल्या-
   द्विशृङ्ग इव पर्वतः ||
   (16.72, नाट्यशास्त्र)

   sa pūrvaṃ vāraṇo bhūtvā
   dviśruṅga iva parvataḥ |
   abhavaddantavaikalyā-
   dviśruṅga iva parvataḥ ||
   (16.72, *Nāṭyaśāstra*)

5. ***Cakravāla*** - The pattern at the end of the preceding foot is repeated at the beginning of the next foot. This kind of repetition is called as *Cakravāla*. For example,

   शैला यथा शत्रुभिराहता **हता**
   हताश्च भूयस्त्वनुपुङ्खपु**ङ्खगैः** |
   खगैश्च सर्वैर्युधि सञ्चिता**श्चिता-**
   श्चिताधिरूढा निहितास्तलैस्तलैः ||
   (16.75, नाट्यशास्त्र)

   *śailā yathā śatrubhirāhatā **hatā***
   hatā**śca bhūyastvanupuṅkhapuṅ**khagaiḥ |

---
[2] *aṅgīkaroti yaḥ kāvyaṃ śabdārthāvanalaṃkṛtī | asau na manyate kasmādanuṣṇamanalaṃ kṛtī* ||1.8, *Candrāloka* (Sharma, 1960).

[3] *śabdābhyāsastu yamakam* ||16.60, *Nāṭyaśāstra*.

[4] The description for this and the following types does not insist on the repetition in all four feet. But the examples provided have the repetition in all four feet.

[5] We have not provided the English translations of the verses, since the main focus here is the repetition of sound patterns.

***khagai**śca sarvairyudhi sañcitā**ścitā**-*
***ścitā**dhirūḍhā nihitāstalaistalaiḥ ||*
(16.75, *Nāṭyaśāstra*)

6. ***Sandaṣṭaka*** - In *Sandaṣṭaka*, the repetition of the sequence is at the beginning of the foot. For example,

पश्य पश्य रमणस्य मे गुणान्
येन येन वशगां करोति माम् |
येन येन हि समेति दर्शनं
तेन तेन वशगां करोति माम् ||
(16.77, नाट्यशास्त्र)

***paśya paśya** ramaṇasya me guṇān*
***yena yena** vaśagāṃ karoti mām |*
***yena yena** hi sameti darśanaṃ*
***tena tena** vaśagāṃ karoti mām ||*
(16.77, *Nāṭyaśāstra*)

7. ***Pādādi*** - The sequence is repeated once at the beginning of each foot in *Pādādi-yamaka*. For example,

विष्णुः सृजति भूतानि
विष्णुः संहरते प्रजाः |
विष्णुः प्रसूते त्रैलोक्यं
विष्णुर्लोकाधिदैवतम् ||
(16.79, नाट्यशास्त्र)

***viṣṇuḥ** sṛjati bhūtāni*
***viṣṇuḥ** saṃharate prajāḥ |*
***viṣṇuḥ** prasūte trailokyaṃ*
***viṣṇu**rlokādhidaivatam ||*
(16.79, *Nāṭyaśāstra*)

8. ***Pādāntāmreḍita*** - The repetition of the sequence is at the end of each foot in *Pādāntāmreḍita*. For example,

विजृम्भितं निःश्वसितं **मुहुर्मुहुः**
कथं विधेयं स्मरणं **पदे पदे** |
यथा च ते ध्यानमिदं **पुनः पुन-**
र्ध्रुवंगता ते रजनी **विना विना** ||
(16.81, नाट्यशास्त्र)

*vijṛmbhitaṃ niḥśvasitaṃ **muhurmuhuḥ***
*kathaṃ vidheyaṃ smaraṇaṃ **pade pade |***
*yathā ca te dhyānamidaṃ **punaḥ puna-***
*rdhruvaṃgatā te rajanī **vinā vinā ||***
(16.81, *Nāṭyaśāstra*)

9. ***Caturvyavasita*** - In *Caturvyavasita*, all four feet of a stanza are the same. For example,

वारणानामयमेव कालो
वारणानामयमेव कालः |
वारणानामयमेव कालो
वारणानामयमेव कालः ||
(16.83, नाट्यशास्त्र)

vāraṇānāmayameva kālo
vāraṇānāmayameva kālaḥ |
vāraṇānāmayameva kālo
***vāraṇānāmayameva kālaḥ** ||*
(16.83, *Nāṭyaśāstra*)

10. ***Mālā*** - In *Mālā-yamaka* instead of an ordered sequence, the repetition of syllables like a garland is considered.

    The repetition can occur at any position in the verse. For example,

असौ हि **रा**मा **र**ति**वि**ग्र**ह**प्रि**या**
रहः**प्र**गल्भा **र**मणं मनोगतम् |
रतेन **रा**त्रिं **र**मयेत् परेण वा न
चेदुदेष्यत्त**रु**णः प**रो रि**पुः ||
(16.86, नाट्यशास्त्र)

*asau hi **rā**mā **ra**tivig**ra**hap**ri**yā*
***ra**haḥ**pra**galbhā **ra**maṇaṃ manogatam |*
***ra**tena **rā**tri**ṃ** **ra**mayet pa**re**ṇa vā*
*na cedudeṣyatta**ru**ṇaḥ pa**ro** **ri**puḥ ||*
(16.86, *Nāṭyaśāstra*)

The kind of syllabic repetition considered in *Mālā-yamaka* developed into a completely new *alaṅkāra* named '*Anuprāsa*'. Out of these ten classes, the last one being more an *Anuprāsa*, we ignore it. Hence we study only the first nine.

We describe each of the nine types by providing the abstract pattern associated with it. Let 'a' be the common or the repeated sequence of syllables in the pattern and x,y,z,t be the remaining part of the sequence of syllables in the four lines in order. Then the *Pādādi* and *Pādānta Yamaka*s may be represented as,

$$\begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix} \text{ and } \begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix} \text{ respectively.}$$

In Table 1 we have expressed the pattern corresponding to each of the above types replacing the common pattern with the same letter, and the remaining by different letters. This representation abstracts the pattern of repetition, and thus it is easy to work with various symmetries on these patterns rather than the words. We have also deviated from the Bharata's order of the *alaṅkāras* in Table 1 by grouping similar patterns.

The questions we ask are: While there can be other patterns of repetition possible, why Bharata consider only these nine? What is special about them? Among these nine patterns, are some of them more symmetric compared to others?

To seek answers to these questions, we look at the symmetries each of these classes exhibits and the properties associated with such a set of symmetries. Further, to find

Table 1 Symmetry patterns in Bharata's classification

Name	Repetition in *pādas*	Pattern
Pādādi	Across	ax ay az at
Pādānta	Across	xa ya za ta
Sandaṣṭaka	Within	xx a yy b zz c tt d
Pādāntāmreḍita	Within	a xx b yy c zz d tt
Kāñcī	Within	xx a yy zz b tt pp c qq rr d ss
Vikrānta (i)	Across	x y z y
Vikrānta (ii)	Across	y x y z
Samudgaka	Across	x y x y
Cakravāla	Cyclic	a x x y y z z b
Caturvyavasita	Across	a a a a

out whether the symmetries involved in these nine different patterns are of the same order, or are some of them 'more' symmetric than the others, we use group theory which provides rigorous tools to describe symmetries of shapes. We also study other patterns with repetition of words that are not considered by Bharata to find out why Bharata did not find those patterns interesting.

To the best of our knowledge, we are not aware of any such work where the *Yamaka*s in Sanskrit are studied from the mathematical, especially the group theoretic, perspective. In the next section, for the sake of completeness, we define a group. In section 4 we look at the symmetries associated with various *Yamaka*s. In Sect. 5 we discuss the symmetries that are not considered by Bharata, and finally, we conclude that what Bharata has considered are the symmetries found in the literature, and they correspond to symmetry groups of type $\mathbb{Z}_1$ or $\mathbb{Z}_2$ or $\mathbb{Z}_2$ x $\mathbb{Z}_2$, specifically the Klein four-group K_4.

3 Group

A group $(G, *)$ is a set G, closed under a binary operation $*$, such that the following axioms are satisfied (Dummit & Foote, 2004).

- G is associative under the binary operation'$*$'.
  That is, for all $a, b, c \in G$, we have
  $(a * b) * c = a * (b * c)$.
- G has an identity element corresponding to '$*$'.
  There is an element I in G such that for all $x \in G$, $I * x = x * I = x$.
- G is closed under inverse with respect to '$*$'.
  That is, corresponding to each $a \in G$ there is an element a' in G such that $a * a' = a' * a = I$.

In addition to the above axioms, if G is commutative under $*$, that is, $a * b = b * a, \quad \forall\ a, b \in G$, then G is called an Abelian group.

4 Symmetry analysis of various types of *Yamaka*

For each type of *Yamaka*, now we consider the transformations that result in a structure that is isomorphic to the original structure and explore whether these transformations form a group.

1. ***Pādādi* - *Pādānta***
   We consider both *Pādādi* and *Pādānta* together since they are the reflections of each other along the vertical axis. In these types of *Yamaka*s, the pattern repeats either at the beginning or at the end of each *pāda*. So, let 'a' be that common or the repeated sequence of syllables and x,y,z, and t be the remaining part of the sequence of syllables in the four lines in order. The *Pādādi* and *Pādānta Yamaka*s may be represented as,

   $$\begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix} \text{ and } \begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix} \text{ respectively.}$$

   (a) Reflection along the horizontal axis, H, divides the verse into two halves resulting in a pattern that is isomorphic to the original one retaining the class of the *Yamaka*. Thus we have,[6]

$$H : \begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix} \rightarrow \begin{pmatrix} at \\ az \\ ay \\ ax \end{pmatrix}, \quad H : \begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix} \rightarrow \begin{pmatrix} ta \\ za \\ ya \\ xa \end{pmatrix},$$

$$H : \begin{pmatrix} at \\ az \\ ay \\ ax \end{pmatrix} \rightarrow \begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix}, \quad H : \begin{pmatrix} ta \\ za \\ ya \\ xa \end{pmatrix} \rightarrow \begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix}.$$

(b) Vertical reflection V, maps a *Pādādi* into *Pādānta* and vice versa as shown below :

$$V : \begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix} \rightarrow \begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix}, \quad V : \begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix} \rightarrow \begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix},$$

$$V : \begin{pmatrix} at \\ az \\ ay \\ ax \end{pmatrix} \rightarrow \begin{pmatrix} ta \\ za \\ ya \\ xa \end{pmatrix}, \quad V : \begin{pmatrix} ta \\ za \\ ya \\ xa \end{pmatrix} \rightarrow \begin{pmatrix} at \\ az \\ ay \\ ax \end{pmatrix}.$$

(c) Let R represent rotation by an angle of $180°$. That is,

$$R : \begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix} \rightarrow \begin{pmatrix} ta \\ za \\ ya \\ xa \end{pmatrix}, \quad R : \begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix} \rightarrow \begin{pmatrix} at \\ az \\ ay \\ ax \end{pmatrix},$$

$$R : \begin{pmatrix} ta \\ za \\ ya \\ xa \end{pmatrix} \rightarrow \begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix}, \quad R : \begin{pmatrix} at \\ az \\ ay \\ ax \end{pmatrix} \rightarrow \begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix}.$$

Note that, R also maps a *pādādi* pattern to *pādānta* and vice versa.

(d) Finally, let I be an identity mapping that maps an element into itself. Thus,

$$I : \begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix} \rightarrow \begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix}, \quad I : \begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix} \rightarrow \begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix},$$

$$I : \begin{pmatrix} at \\ az \\ ay \\ ax \end{pmatrix} \rightarrow \begin{pmatrix} at \\ az \\ ay \\ ax \end{pmatrix}, \quad I : \begin{pmatrix} ta \\ za \\ ya \\ xa \end{pmatrix} \rightarrow \begin{pmatrix} ta \\ za \\ ya \\ xa \end{pmatrix}$$

Now, let $S = \{1, 2, 3, 4\}$ where

$$1 = \begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix}, 2 = \begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix}, 3 = \begin{pmatrix} ta \\ za \\ ya \\ xa \end{pmatrix}, and \quad 4 = \begin{pmatrix} at \\ az \\ ay \\ ax \end{pmatrix}.$$

Let S_4 denote a symmetric group with all permutations of S. We notice that S with the above operations, viz. horizontal reflection, vertical reflection, rotation through 180°, and the identity relation forms a subgroup of S_4, termed as permutation group represented compactly as $G = \{\mathrm{I}, (12)(34), (13)(24), (14)(23)\}$ where $(12)(34)$, $(13)(24)$, $(14)(23)$ are the cycle decompositions of S_4 representing vertical symmetry(σ_v), horizontal symmetry(σ_h) and rotational symmetry(σ_r) respectively, which are defined as below.

$$\sigma_v(1) = 2, \sigma_v(2) = 1, \sigma_v(3) = 4, \quad and \quad \sigma_v(4) = 3. \tag{1}$$

$$\sigma_h(1) = 3, \sigma_h(3) = 1, \sigma_h(2) = 4, \quad and \quad \sigma_h(4) = 2. \tag{2}$$

$$\sigma_r(1) = 4, \sigma_r(4) = 1, \sigma_r(2) = 3, \quad and \quad \sigma_r(3) = 2. \tag{3}$$

Let $*$ be the binary composition operation defined as $(a * b)(X) = a(b(X))$, where X is any verse of the type *Pādādi* or *Pādānta-yamaka* and $a, b \in G$.

The composition table for the set G is shown in Table 2. From this composition matrix, we can check that G satisfies all the axioms of the group given in Sect. 3. Further, we can also check that $G \simeq \mathbb{Z}_2$ x $\mathbb{Z}_2$ and specifically G is a Klein four-group K_4.

Table 2 Composition matrix : *Pādādi - Pādānta Yamaka*

$*$	I	σ_v	σ_h	σ_r
I	I	σ_v	σ_h	σ_r
σ_v	σ_v	I	σ_r	σ_h
σ_h	σ_h	σ_r	I	σ_v
σ_r	σ_r	σ_h	σ_v	I

[6] One may wonder about the necessity of the latter two mappings out of the four for H, since patternwise the first two are exactly the same as the latter two. However as an element of a set, the two elements

$\begin{pmatrix} ax \\ ay \\ az \\ at \end{pmatrix}$ and $\begin{pmatrix} xa \\ ya \\ za \\ ta \end{pmatrix}$ being different from $\begin{pmatrix} at \\ az \\ ay \\ ax \end{pmatrix}$ and $\begin{pmatrix} ta \\ za \\ ya \\ xa \end{pmatrix}$,

the mappings of each of these are specified separately.

2. ***Sandaṣṭaka* and *Pādāntāmreḍita***

Now we consider the two types viz. *Sandaṣṭaka* and *Pādāntāmreḍita*, where the same sequence of letters repeats twice at the beginning or the end of each line of a verse. As in the above case, these patterns are vertical reflections of each other, we consider them together. So let *xx*, *yy*, *zz*, and *tt* be that common or the repeated sequence of syllables in each line in order. Let *a*,*b*,*c*,and *d* be the remaining part of the sequence of syllables in the four lines in order. Then the patterns for *Sandaṣṭaka* and *Pādāntāmreḍita-Yamaka* would correspond to

$$\begin{pmatrix} xxa \\ yyb \\ zzc \\ ttd \end{pmatrix} \text{ and } \begin{pmatrix} axx \\ byy \\ czz \\ dtt \end{pmatrix} \text{ respectively.}$$

As in the previous case, here also we consider four transformations viz. Vertical reflection, Horizontal reflection, rotation through 180°, and identity mapping defined as below.

(1) Let *H* represent the horizontal reflection defined as follows:

$$H : \begin{pmatrix} xxa \\ yyb \\ zzc \\ ttd \end{pmatrix} \to \begin{pmatrix} ttd \\ zzc \\ yyb \\ xxa \end{pmatrix}, H : \begin{pmatrix} ttd \\ zzc \\ yyb \\ xxa \end{pmatrix} \to \begin{pmatrix} xxa \\ yyb \\ zzc \\ ttd \end{pmatrix},$$
$$H : \begin{pmatrix} axx \\ byy \\ czz \\ dtt \end{pmatrix} \to \begin{pmatrix} dtt \\ czz \\ byy \\ axx \end{pmatrix}, H : \begin{pmatrix} dtt \\ czz \\ byy \\ axx \end{pmatrix} \to \begin{pmatrix} axx \\ byy \\ czz \\ dtt \end{pmatrix}.$$

This transformation retains the type of *Yamaka*.

(2) The vertical reflection (*V*) is defined as

$$V : \begin{pmatrix} xxa \\ yyb \\ zzc \\ ttd \end{pmatrix} \to \begin{pmatrix} axx \\ byy \\ czz \\ dtt \end{pmatrix}, V : \begin{pmatrix} axx \\ byy \\ czz \\ dtt \end{pmatrix} \to \begin{pmatrix} xxa \\ yyb \\ zzc \\ ttd \end{pmatrix},$$
$$V : \begin{pmatrix} ttd \\ zzc \\ yyb \\ xxa \end{pmatrix} \to \begin{pmatrix} dtt \\ czz \\ byy \\ axx \end{pmatrix}, V : \begin{pmatrix} dtt \\ czz \\ byy \\ axx \end{pmatrix} \to \begin{pmatrix} ttd \\ zzc \\ yyb \\ xxa \end{pmatrix}.$$

Here the *Sandaṣṭaka Yamaka* gets transformed to *Pādāntāmreḍita* and vice versa.

(3) Let *R* represent the rotation by an angle of 180°.

$$R : \begin{pmatrix} xxa \\ yyb \\ zzc \\ ttd \end{pmatrix} \to \begin{pmatrix} dtt \\ czz \\ byy \\ axx \end{pmatrix}, R : \begin{pmatrix} axx \\ byy \\ czz \\ dtt \end{pmatrix} \to \begin{pmatrix} ttd \\ zzc \\ yyb \\ xxa \end{pmatrix},$$
$$R : \begin{pmatrix} dtt \\ czz \\ byy \\ axx \end{pmatrix} \to \begin{pmatrix} xxa \\ yyb \\ zzc \\ ttd \end{pmatrix}, R : \begin{pmatrix} ttd \\ zzc \\ yyb \\ xxa \end{pmatrix} \to \begin{pmatrix} axx \\ byy \\ czz \\ dtt \end{pmatrix}.$$

Here also we see that *R* transforms a *Sandaṣṭaka* to *Pādāntāmreḍita* and vice versa.

(4) Let *I* be an identity element that maps an element to itself.

Now, let $S = \{1, 2, 3, 4\}$ where

$$1 = \begin{pmatrix} xxa \\ yyb \\ zzc \\ ttd \end{pmatrix}, 2 = \begin{pmatrix} axx \\ byy \\ czz \\ dtt \end{pmatrix}, 3 = \begin{pmatrix} ttd \\ zzc \\ yyb \\ xxa \end{pmatrix}, and\ 4 = \begin{pmatrix} dtt \\ czz \\ byy \\ axx \end{pmatrix}.$$

Then, as in the previous case, here also the permutation group *G* of S_4 represented compactly as $G = \{\mathrm{I}, (12)(34), (13)(24), (14)(23)\}$, where (12)(34), (13)(24), and (14)(23) are the cycle decompositions of S_4 representing vertical symmetry(σ_v), horizontal symmetry(σ_h) and rotational symmetry(σ_r) respectively, which are defined as in Eqs. (1)–(3) with the same composition matrix defined in Table 2.

Thus these two types of *Yamaka*s also confirm that $G \simeq \mathbb{Z}_2 \text{ x } \mathbb{Z}_2$ and specifically *G* is a Klein four-group K_4.

3. ***Kāñcī***

In the *Kāñcī* type of *Yamaka*, the same sequence of letters repeats twice at the beginning as well as at the end of each line of a verse. So let *xx*, *zz*, *pp*, and *rr* be that common or the repeated sequence of syllables at the beginning of each line in order. Let *yy*, *tt*, *qq*, and *ss* be the common or the repeated sequence of syllables at the end of each line in order. Let *a*,*b*,*c*, and *d* be the remaining part of the sequence of syllables in the middle of four lines in order. So the verse can be represented as

$$\begin{pmatrix} xxayy \\ zzbtt \\ ppcqq \\ rrdss \end{pmatrix}.$$

This being a combination of both *Sandaṣṭaka* as well as *Pādāntāmreḍita*, the same transformations *I*, *H*, *V* and *R* as described previously produce a K_4. For the sake of completeness, we describe below the transformations.

(a) Let H be the horizontal reflection which is defined as

$$H:\begin{pmatrix} xxayy \\ zzbtt \\ ppcqq \\ rrdss \end{pmatrix} \rightarrow \begin{pmatrix} rrdss \\ ppcqq \\ zzbtt \\ xxayy \end{pmatrix}, H:\begin{pmatrix} rrdss \\ ppcqq \\ zzbtt \\ xxayy \end{pmatrix} \rightarrow \begin{pmatrix} xxayy \\ zzbtt \\ ppcqq \\ rrdss \end{pmatrix},$$
$$H:\begin{pmatrix} yyaxx \\ ttbzz \\ qqcpp \\ ssdrr \end{pmatrix} \rightarrow \begin{pmatrix} ssdrr \\ qqcpp \\ ttbzz \\ yyaxx \end{pmatrix}, H:\begin{pmatrix} ssdrr \\ qqcpp \\ ttbzz \\ yyaxx \end{pmatrix} \rightarrow \begin{pmatrix} yyaxx \\ ttbzz \\ qqcpp \\ ssdrr \end{pmatrix}.$$

(b) The vertical reflection V is defined as

$$V:\begin{pmatrix} xxayy \\ zzbtt \\ ppcqq \\ rrdss \end{pmatrix} \rightarrow \begin{pmatrix} yyaxx \\ ttbzz \\ qqcpp \\ ssdrr \end{pmatrix}, V:\begin{pmatrix} yyaxx \\ ttbzz \\ qqcpp \\ ssdrr \end{pmatrix} \rightarrow \begin{pmatrix} xxayy \\ zzbtt \\ ppcqq \\ rrdss \end{pmatrix},$$
$$V:\begin{pmatrix} rrdss \\ ppcqq \\ zzbtt \\ xxayy \end{pmatrix} \rightarrow \begin{pmatrix} ssdrr \\ qqcpp \\ ttbzz \\ yyaxx \end{pmatrix}, V:\begin{pmatrix} ssdrr \\ qqcpp \\ ttbzz \\ yyaxx \end{pmatrix} \rightarrow \begin{pmatrix} rrdss \\ ppcqq \\ zzbtt \\ xxayy \end{pmatrix}.$$

(c) Let R represent the rotation of by an angle of $180°$ defined as

$$R:\begin{pmatrix} xxayy \\ zzbtt \\ ppcqq \\ rrdss \end{pmatrix} \rightarrow \begin{pmatrix} ssdrr \\ qqcpp \\ ttbzz \\ yyaxx \end{pmatrix}, R:\begin{pmatrix} ssdrr \\ qqcpp \\ ttbzz \\ yyaxx \end{pmatrix} \rightarrow \begin{pmatrix} xxayy \\ zzbtt \\ ppcqq \\ rrdss \end{pmatrix},$$
$$R:\begin{pmatrix} yyaxx \\ ttbzz \\ qqcpp \\ ssdrr \end{pmatrix} \rightarrow \begin{pmatrix} rrdss \\ ppcqq \\ zzbtt \\ xxayy \end{pmatrix}, R:\begin{pmatrix} rrdss \\ ppcqq \\ zzbtt \\ xxayy \end{pmatrix} \rightarrow \begin{pmatrix} yyaxx \\ ttbzz \\ qqcpp \\ ssdrr \end{pmatrix}.$$

(d) Let I be the identity mapping.

Now, let $S = \{1, 2, 3, 4\}$ where

$$1 = \begin{pmatrix} xxayy \\ zzbtt \\ ppcqq \\ rrdss \end{pmatrix}, 2 = \begin{pmatrix} yyaxx \\ ttbzz \\ qqcpp \\ ssdrr \end{pmatrix}, 3 = \begin{pmatrix} rrdss \\ ppcqq \\ zzbtt \\ xxayy \end{pmatrix},$$
$$and \ \ 4 = \begin{pmatrix} ssdrr \\ qqcpp \\ ttbzz \\ yyaxx \end{pmatrix}.$$

Then, as in the previous case, here also $G \simeq \mathbb{Z}_2$ x $\mathbb{Z}_2$ or Klein four-group K_4.

4. ***Vikrānta***

In *Vikrānta*, any one pair of alternative lines of a verse is the same. That is either the first and third lines are identical (type 1) or the second and fourth lines are identical (type 2). We represent the type 1 and type 2 of *Vikrānta* respectively as

$$\begin{pmatrix} x \\ y \\ z \\ y \end{pmatrix}, \text{ and } \begin{pmatrix} y \\ z \\ y \\ x \end{pmatrix}.$$

Now, let us consider the following transformations.

(a) Let H represent the horizontal reflection defined as follows.

$$H:\begin{pmatrix} x \\ y \\ z \\ y \end{pmatrix} \rightarrow \begin{pmatrix} y \\ z \\ y \\ x \end{pmatrix}, H:\begin{pmatrix} y \\ z \\ y \\ x \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \\ y \end{pmatrix}.$$

(b) Let I be the identity operation, that maps an element into itself. That is

$$I:\begin{pmatrix} x \\ y \\ z \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \\ y \end{pmatrix}, \text{ and } I:\begin{pmatrix} y \\ z \\ y \\ x \end{pmatrix} \rightarrow \begin{pmatrix} y \\ z \\ y \\ x \end{pmatrix}.$$

Thus, we notice that the horizontal reflection changes the type of the *yamaka* from type 1 to type 2 and type 2 to type 1 while retaining the main class - *Vikrānta*.

Now, let $S = \{1, 2\}$ where

$$1 = \begin{pmatrix} x \\ y \\ z \\ y \end{pmatrix}, \text{ and } 2 = \begin{pmatrix} y \\ z \\ y \\ x \end{pmatrix}.$$

Let S_2 denote a symmetric group with all permutations of S. We notice that S with the above operations, viz. horizontal reflection and identity forms a subgroup of S_2, termed as permutation group represented compactly as $G = \{\mathrm{I}, (12)\}$ where (12) is a cycle decomposition of S_2 representing horizontal symmetry(σ_h) defined as below.

$$\sigma_h(1) = 2, \sigma_h(2) = 1. \tag{4}$$

It is trivial to check that the group axioms are satisfied. This confirms that $G \simeq \mathbb{Z}_2$.

5. ***Samudgaka***

Here the alternative lines of a verse are the same. That is first and third lines are the same and the second and fourth lines are the same. Thus this is a combination of *Vikrānta* type 1 and type 2. So the verse may be represented as,

$$\begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix}.$$

Now, let us consider the following operations on this.

(a) Let H represent the horizontal reflection which is defined as follows :

$$H : \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ x \\ y \\ x \end{pmatrix}, \text{ and } H : \begin{pmatrix} y \\ x \\ y \\ x \end{pmatrix} \to \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix}.$$

(b) Let I be the identity mapping.

Now, let $S = \{1, 2\}$ where

$$1 = \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix}, \text{ and } 2 = \begin{pmatrix} y \\ x \\ y \\ x \end{pmatrix}.$$

It is trivial to check, as in the previous case, that $G \simeq \mathbb{Z}_2$.

6. ***Cakravāla***
In the *Cakravāla* type of *Yamaka*, the end sequence of letters of the first line is the beginning sequence of the second line, the end sequence of letters of the second line is the beginning sequence of the third line and the end sequence of letters of the third line is the beginning sequence of the fourth line. So the verse may be represented as

$$\begin{pmatrix} ax \\ xy \\ yz \\ zb \end{pmatrix}.$$

Now, let us consider the following transformations.

(a) Let R represent the rotational reflection, which rotates $\begin{pmatrix} ax \\ xy \\ yz \\ zb \end{pmatrix}$ by an angle of $180°$. That is,

$$R : \begin{pmatrix} ax \\ xy \\ yz \\ zb \end{pmatrix} \to \begin{pmatrix} bz \\ zy \\ yx \\ xa \end{pmatrix}, \text{ and } R : \begin{pmatrix} bz \\ zy \\ yx \\ xa \end{pmatrix} \to \begin{pmatrix} ax \\ xy \\ yz \\ zb \end{pmatrix}.$$

(b) Let I be the identity mapping.

Table 3 Repetition in the same foot

aax	axa	xaa	aa
bby	byb	ybb	bb
ccz	czc	zcc	cc
ddt	dtd	tdd	dd
(1)	(2)	(3)	(4)

Let S_2 denote a symmetric group with all permutations of S. We notice that S with the above operations, viz. rotational reflection and identity forms a subgroup of S_2, termed as permutation group represented compactly as $G = \{\mathrm{I}, (12)\}$ where (12) is a cycle decomposition of S_2 representing rotational symmetry(σ_r) defined as below.

$$\sigma_r(1) = 2, \sigma_r(2) = 1. \tag{5}$$

Thus we notice that *Cakravāla* type of *Yamaka* also is a symmetry group isomorphic to $\mathbb{Z}_2$.

7. ***Caturvyavasita***
In the *Caturvyavasita* type of *Yamaka*, all the lines are identical. This is not a very interesting pattern. The only distinct operation is the identity operation I, as any other operation yields the same element, all the lines being the same. So, in this type $G = \{I\}$ and hence, $G \simeq \mathbb{Z}_1$.

We observe that all 9 types of *Yamaka* with one or more symmetry operations like reflection about the vertical axis, reflection in the horizontal axis, and rotation through 180° form abelian groups that are either isomorphic to the Klein four-group K_4 or $\mathbb{Z}_2$ or $\mathbb{Z}_1$. To be specific, the *Pādādi-yamaka*, *Pādānta-yamaka*, *Sandaṣṭaka-yamaka*, *Pādāntāmreḍita-yamaka* and *Kāñcī-yamaka* are all isomorphic to the Klein four group K_4, *Vikrānta-yamaka*, *Samudgaka-yamaka* and *Cakravāla-yamaka* are all isomorphic to $\mathbb{Z}_2$, and the last one viz. *Caturvyavastita-yamaka* is isomorphic to $\mathbb{Z}_1$.

5 Discussion on missing symmetries

Are these the only symmetries with 4 quarters(feet) of a verse? To answer this question, let us look at the possible repetitions in 4 feet. The repetitions can be

Table 4 One foot repeated

a	a	a	X	X	X
a	X	X	a	a	Y
X	a	Y	a	Y	a
Y	Y	a	Y	a	a
(1)	(2)	(3)	(4)	(5)	(6)

Table 5 Two feet repeated

a	a	a
a	b	b
b	a	b
b	b	a
(1)	(2)	(3)

(a) within the same foot resulting in a symmetry due to reflection across the vertical axis, or

(b) across the feet resulting in a symmetry due to reflection across the horizontal axis.

5.1 Repetition within the same foot

There are 4 possible patterns of repetition within the same foot as shown in Table 3.

Among these, the first and the third correspond to the *Sandaṣṭaka* and *Pādāntāmreḍita* of Bharata. The fourth is a special case since it results from the previous three patterns when 'x' is empty. The second one forms an abelian group $\mathbb{Z}_2$ with G defined as $G = \{I, (12)\}$ where

$$1 = \begin{pmatrix} axa \\ byb \\ czc \\ dtd \end{pmatrix}, 2 = \begin{pmatrix} dtd \\ czc \\ byb \\ axa \end{pmatrix}$$

.

The horizontal symmetry σ_h is defined as $\sigma_h(1) = 2, \sigma_h(2) = 1$ and I is the identity mapping. As discussed under *Vikrānta*, the set G, thus, is a group isomorphic to $\mathbb{Z}_2$.

5.2 Repetition across the feet

Now, let us consider the repetition across the feet. This repetition may involve 2 or 3 or all the 4 feet. Further, a complete foot may be repeated as another foot, or only a part of a foot is repeated as some other foot in a verse.

1. Repetition of complete foot:

   (a) With a complete foot repeated twice, there are C_2^4 (= 6) possible combinations as shown in Table 4. Out of these, Bharata has considered (2) and (5) which correspond to *Vikrānta*. Patterns (1) and (6) are horizontal reflections of each other, and they together result in a symmetry group isomorphic to $\mathbb{Z}_2$. Similarly, the patterns (3) and (4) where one repeated pattern is in the beginning of one hemistitch and the other one at the end of the other are horizontal reflections of each other. These two together also result in a symmetry group isomorphic to $\mathbb{Z}_2$. The patterns (1), (3), (4) and (6) do not seem to be appealing cases of symmetry from a literature point of view.

   There are 4 ways in which a pattern can repeat in 3 feet. All of them are isomorphic to $\mathbb{Z}_2$. Finally, there is only one pattern in which the pattern gets repeated in all four feet which is classified by Bharata under *Caturvyayasita*.

   (b) If there are two different repetitions we can have 3 patterns as shown in Table 5. Of these, the second one corresponds to *Samudgaka*. Patterns 1 and 3 would result in a symmetry group isomorphic to $\mathbb{Z}_2$ and $\mathbb{Z}_1$ respectively. But they are not of much interest from a literature point of view.

2. Repetition of partial foot

   The texts on *Alaṅkāraśāstra* provide examples of partial repetition across the feet where the repetition is found in all the feet. Hence we discussed only them. However, the definition of the *Yamaka* does not imply that the repetition should be across all the feet. Thus one may have partial repetition in 2 or 3 feet as well. However since examples of such patterns are not found, we do not study them here.

6 Conclusion

We observe that Bharata has considered only those patterns that are symmetric under the horizontal and vertical reflection and rotation through 180°. These symmetry operations of reflection and rotation form either a Klein four-group K_4 or $\mathbb{Z}_2$ group. *Caturvyavasita* is an exception, where the only possible operation is that of identity. A few patterns that are isomorphic to $\mathbb{Z}_2$ are not considered by Bharata, and these might have been ignored by him maybe because of the absence of any evidence of such patterns in earlier literature. We also notice that from the literature point of view, such patterns are not very interesting.

Acknowledgements The authors thank Dr. Archana Morey for the fruitful discussions on the symmetries that provided us a new insights to study the symmetries in *Yamaka*.

Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article. Financial interest The authors have no relevant financial or non-financial interests to disclose. All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript. The authors have no financial or proprietary interests in any material discussed in this article.

References

Bunker, P., & Jensen, P. (2004). *Fundamentals of Molecular Symmetry*. CRC Press.

Dummit, D. S., & Foote, R. M. (2004). *Abstract Algebra* (3rd ed.). Wiley.

Ghosh, M. (1951). *The Nāṭyaśāstra* (Vol. 1). Asiatic Society of Bengal.

Lederman, L., & Hill, C. T. (2005). *Symmetry and the Beautiful Universe*. Prometheus Books.

McManus, I. C. (2005). Symmetry and aymmetry in aesthetics and the arts. *European Review, 13*(2), 157–180.

Molnar, V., & Molnar, F. (1986). Symmetry making and breaking in visual art. *Computers and Mathematics with Applications, 12B*(1/2), 291–301.

Pavlovic, B., & Triajstic, N. (1986). On symmetry and asymmetry in literature. *Computers and Mathematics with Applications, 12B*(1/2), 197–227.

Sanchez-Behar, A. (2014). Symmetry in the music of John Adams. *Tempo, 68*(268), 46–60.

Sihna, D.K. (1989). Symmetry in the Indian context: A critique.

Sharma, N. (1960). *Candrālokaḥ*. Chowhamba Sanskrit Series.