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ARTICLE

Revisiting the delian problem with dynamic geometry: Historical roots and modern innovations

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Abstract
The objective of this paper is to explore a geometric construction involving successive perpendiculars on the sides of a right-angled triangle to achieve a length of the cube root of 2, thereby solving the Delian problem in Greek mathematics and its connection*s to Indian mathematics.* Given a right-angled triangle ABC with unit length perpendicular *AB* and base angle *C,* perpendiculars are drawn alternately on the hypotenuse AC and base BC. The lengths of these perpendiculars follow a specific pattern: *BD/1 = DE/BD = EF/DE = FG/EF = GH/FG = HI/GH* maintaining a constant ratio equal to the cosine of the base angle *C*. Consequently, the lengths of successive perpendiculars (*BD, DE, EF,* etc.) correspond to cosine of angle *C*, square of cosine of angle *C*, cube of cosine of angle C... so on respectively. By adjusting angle C while keeping *AB* fixed at unity, to have results corresponding to the given condition of doubling a cube or a hypercube, as the case may be. This paper presents a dynamic geometric approach, distinct from traditional Euclidean methods, providing a novel perspective on these geometric constructions, particularly in relation to the impossibility of duplicating the cube using only a straightedge and compass.

Keywords Delian problem · Doubling the cube · Dynamic geometry · Greek mathematics · Indian mathematics · n-dimensional hypercube

MSC 2020 01A20 · 01A32 · 97G19

1 Introduction

Ancient Greek mathematics spans a period from the fifth century BCE to the sixth century CE. During this era, mathematicians were dispersed across the Mediterranean—from Anatolia to Italy and North Africa—yet shared a common Hellenic culture. Among the earliest of these thinkers was Thales of Miletus, who is credited with offering deductive proofs of five geometric propositions. Recognised as one of the Seven Sages of Greece, Thales is particularly known for what later came to be called Thales' Theorem: if three points lie on the circumference of a circle and the line joining two of them passes through the center, then that line subtends a right angle at the third point (Heath, 1956) (Fig. 1).

It is believed that Pythagoras may have been a student of Thales. While he might not have discovered the famous Pythagorean Theorem, he is thought to have provided its deductive proof. In addition to this, Pythagoras proposed several geometric axioms, including:

i. *The sum of the internal angles of a triangle is equal to two right angles.*
ii. *The sum of the external angles of a triangle is equal to four right angles.*
iii. *The sum of the internal angles of an n-sided polygon is equal to (2n—4) right angles.*
iv. *The sum of the external angles of any polygon equals four right angles.*
v. *Only three regular polygons—the triangle, square, and hexagon—can tessellate the plane around a point,*

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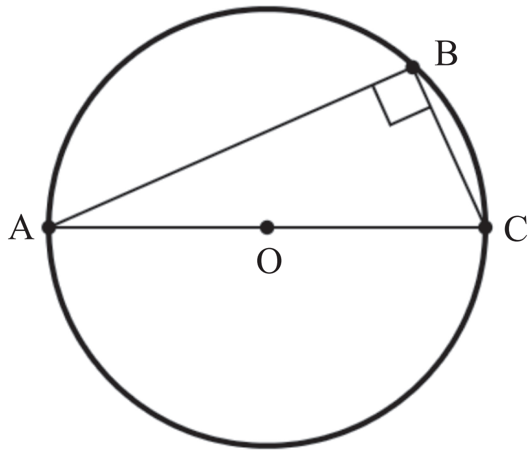

Fig. 1 Thales' Theorem of subtending a right angle by diameter at any point on a circle

fitting six triangles, four squares, or three hexagons perfectly around it.[1]

These foundational ideas were further shaped by Plato (427–347 BCE), a philosopher whose views deeply influenced Greek mathematics. Plato emphasised that geometry should be practiced using only a straightedge and compass, rejecting the use of mechanical aids such as graduated scales. He viewed such tools as belonging to artisans rather than mathematicians. This Platonic ideal gave rise to three famous geometric construction problems that adhered to strict classical constraints:

i. *Doubling the volume of a cube (the Delian problem)*(Yates, 1941)*,*
ii. *Squaring the circle,*
iii. *Trisecting an arbitrary angle* (see footnote 1).

Among the later mathematicians, Euclid of Alexandria (circa 325–265 BCE) compiled the *Elements*, a collection of 13 books that laid the foundations of what is now known as Euclidean geometry. His axiomatic system introduced principles such as two points determine a straight line, a finite line can be extended indefinitely, a circle can be drawn with any center and radius, and all right angles are equal. Euclid's work was so comprehensive that even aspects of algebra were expressed geometrically in his texts, eclipsing many previous contributions (see footnote 1). Archimedes of Syracuse (287–212 BCE), another monumental figure, further advanced Greek geometry and physics. Considered the founder of hydrostatics, his mathematical contributions included Measurement of a Circle and treatises on conoids and spheroids. His geometric insight was so profound that it later inspired Renaissance thinkers like René Descartes and Pierre de Fermat (see footnote 1).

With this understanding of the key Greek contributors, I return to the focal point of this paper: the problem of *doubling the cube.* According to an ancient Greek legend, the island of Delos, sacred as the birthplace of Apollo and Artemis, was devastated by a plague. Seeking guidance, the Delians consulted the Oracle at Delphi, who proclaimed that the gods would be appeased only if the altar of Apollo—originally in the shape of a cube—were doubled in volume. What began as a religious edict transformed into a profound geometrical challenge.

This mythological problem evolved into one of the central pursuits of Greek geometry. The task was to construct, using only a straightedge and compass, a new cube with exactly twice the volume of a given cube. Mathematically, this means finding a side x such that,

$$x^3 = 2a^3,$$

where, a is the original side length.

This leads to $x = a2^{1/3}$ (Heath, 1931). While the problem is easy to state, constructing such a value under the strict classical tools proved elusive. Early Greek mathematicians such as Hippocrates of Chios approached the problem by attempting to find two mean proportionals between a and 2a, that is:

$$a : x = x : y = y : 2a$$

Although this was theoretically valid, such a construction could not be achieved using only a straightedge and compass.

Later, Menaechmus introduced the use of conic sections, showing that the intersection of a parabola and a hyperbola could yield the necessary cube root. This marked a significant step forward in Greek mathematical thought. Still more advanced was Archytas of Tarentum, who tackled the problem using three-dimensional constructions involving a cylinder, cone, and torus—an early venture into solid geometry.

These efforts, despite their sophistication, lay beyond the constraints of Euclidean tools and thus could not offer a classical solution. Nevertheless, the problem of doubling the cube remains a landmark of mathematical curiosity and a testament to the rich intellectual legacy of ancient Greece.

1.1 Indian mathematics

In contrast, Indian geometry has a much earlier origin, dating back to the *Indus Valley Civilisation* (3300 BCE – 1300 BCE), with its mature phase between 2600 and 1900 BCE. Excavations at *Harappa*, *Mohenjo-daro*, and other sites

[1] History of Geometry, Wikipedia at https://en.wikipedia.org/wiki/History_of_geometry

reveal that the civilisation produced bricks in the ratio of 4:2:1 and manufactured coins in geometrical shapes such as hexahedra, barrels, cones, and cylinders—clear evidence of practical geometric knowledge (Sergent, 1997, p. 113).

Following the decline of the Indus Valley Civilisation, the Vedic period (circa 1500 BCE – fifth century BCE) emerged. Mathematical knowledge from this era is preserved in the *Śulbasūtras*, which form appendices to the *Śrautasūtras,* themselves part of the broader Vedic literature. These texts prescribe the construction of *Vedis*—sacred fire altars—with unique shapes associated with specific divine blessings. For instance, a falcon-shaped altar promised ascension to heaven, a tortoise-shaped one symbolised conquest, and a rhombus-shaped altar aimed to destroy enemies (Fig. 2).

The four major *Śulbasūtras* are attributed to *Baudhayana*, *Manava*, *Apastamba*, and *Katyayana*. A falcon-shaped altar was uncovered during excavations at Kosambi in 1957–59. Please refer to Fig. 2. Scholars like David Pingree and George Gheverghese Joseph have debated the connections between the *Śulbasūtras* and Greek geometry. Historian Staal even proposed a common ritual origin, citing similar interests in geometric transformations such as doubling the cube (Plofker, 2007). The *Baudhayana Śulbasūtra* provides a specific statement equivalent to the Pythagorean Theorem:

The rope which is stretched across the diagonal of a square produces an area double the size of the original square," mathematically interpreted as $l^2 = a^2 + a^2$, where l is the diagonal and a the side of the square. The *Katyayana Śulbasūtras* offers a generalised version: *The rope stretched along the diagonal of a rectangle produces an area equal to that made by its vertical and horizontal sides*," or $l^2 = a^2 + b^2$, with *a* and *b* as side lengths.

The classical problem of squaring the circle is also addressed. The *sūtras* suggest that the side of the square equivalent in area to a circle is 13/15 times the diameter of the circle. That leads to the mathematical equation of $\pi r^2 = \left(4r^2\right)(13/15)^2$ yielding an approximation of π equal to 3.00444. Though different values of π appear across various *sūtras* (Gupta, 1988; Kulkarni, 1978).

This body of work shows that the *Śulbasūtras* were intended to produce practical and achievable constructions for ritual purposes. In contrast, Greek mathematicians sought ideal geometric constructions that were restricted to the use of a straight edge and compass. The challenge of finding the diagonal of a cube is also tackled. The diagonal of a cube of side 1 has a length equal to the square root of 3. The approximation given is of the square root of 2. The *Apastamba* and *Katyayana sūtras* describe: "*Increase a unit length by its third, then this third by its own fourth, less the thirty-fourth part of that fourth."* This translates to $1 + 1/3 + 1/\{(3)(4)\} - 1/\{(3)(4)(34)\}$ and yields a remarkably accurate approximation of $2^{1/2}$ equal to 1.414215686, correct to five decimal places—an impressive feat of ancient mathematics." To summarise, the Vedic geometry of the *Śulbasūtras* was driven by ritual and practical needs. It favoured approximations and constructions grounded in empirical results, unlike the Greek focus on axiomatic precision.

Fig. 2 Vedic falcon-shaped altar (first century BCE – second century CE) from Purola, Uttarakhand, India. Photo by Chandler Minh, via Wikimedia Commons, licensed under CC BY-SA 4.0 license https://creativecommons.org/licenses/by-sa/4.0/

1.1.1 Jaina mathematics (400 BCE – 200 CE): Cosmological geometry

The Jain tradition offered unique geometric insights, particularly through cosmological diagrams and infinite series. Though less famous for strict compass-and-straightedge constructions, they introduced:

i. *The concept of infinite numbers and infinite time cycles.*
ii. *Calculation of area of circles with* $\pi \sim 10^{1/2} \sim 3.162$.
iii. *Rules for volumes of solids like cones and cylinders.*
iv. *Use of combinatorics, early permutation formulas, and Pascal-type triangles.*

1.1.2 Āryabhaṭa (476 CE): Trigonometry and algebraic geometry

In his famous work, *Āryabhaṭīya*, Āryabhaṭa, advanced both astronomical computations and geometric reasoning. He introduced the concept of sine (*Jya*) tables for solving triangles and suggested the approximation of π as 3.1416. Although not constructing geometric figures in the Greek sense, he employed geometric concepts to calculate planetary motion and derive area formulas.

1.1.3 Brahmagupta (598–668 CE): Geometry of cyclic quadrilaterals

Brahmagupta's *Brāhmasphuṭasiddhānta* is one of the most sophisticated early works in mathematics. He introduced Brahmagupta's Formula for the area of cyclic quadrilaterals:

$A = (s-a)(s-b)(s-c)(s-d)$

where s is half the sum of sides a, b, c, d of the quadrilateral. He discussed interpolation formulas, volume formulas, and trapeziums. He handled negative numbers and zero with clarity unknown in Greek mathematics.

1.1.4 Bhāskara I (seventh century CE): Trigonometric Approximations

A follower of Āryabhaṭa, Bhāskara I is known for a remarkable approximation formula for

$$\sin(x) = 16x(\pi - x)/\left\{5\pi^2 - 4x(\pi - x)\right\}$$

He applied geometric reasoning to celestial motion (Shukla, 1976).

1.1.5 Bhāskara II (1114–1185 CE): Intersection of geometry and Algebra

In his masterpiece *Līlāvatī* and *Bījagaṇita*, Bhāskara II merged algebraic solutions with geometric intuition. He described the perpetual motion wheel, proposed methods to solve quadratic and cubic equations, and gave geometric interpretations of algebraic identities, anticipating the development of algebraic geometry (Ramasubramanian et al., 2019).

1.1.6 Kerala School (14th–sixteenth century CE): Infinite series and geometry

Mathematicians such as Mādhava of Saṅgamagrāma, Nīlakaṇṭha Somayāji, and Jyeṣṭhadeva laid the foundations of calculus centuries before Newton and Leibniz. He developed infinite series expansions for sine, cosine, and arctangent. He calculated π to 13 + decimal places using geometric series and blended geometric intuition with algebraic iteration, especially in astronomy and planetary models (Sarma et. al., 2008).

While Indian mathematicians did not adhere to Greek constraints of straight edge-and-compass constructions, they developed constructive geometry in a different philosophical mold—rooted in ritual, astronomy, and computation. The shift from geometry to symbolic algebra, the development of negative numbers, infinite series, and calculus-like reasoning, marked a major divergence from Greek methods. Nevertheless, the two traditions echo each other in depth, innovation, and intellectual ambition.

In modern times, tools like dynamic geometry software—as used in this paper—reunite these traditions by providing platforms that honour both geometric purity and algebraic flexibility, allowing us to reinterpret ancient problems through a new lens.

The Delian problem's emphasis on geometric constructions and its eventual algebraic formulations underscore the global nature of mathematical inquiry, reflecting shared challenges and diverse methodologies across cultures. Dynamic geometry, as explored in this paper, serves as a modern bridge connecting these rich mathematical heritages by providing new tools to reinterpret ancient problems.

In the year 1837, Geometer Pierre Wantzel proved its non-constructibility in the form of a geometric figure (Lützen, 2010; Wantzel, 1837). The trigonometric ratio cos x can be written by angle tripling identity,

$$\cos(x) = 4\cos^3(x/3) - 3\cos(x/3)$$

If x is π/3, then it reduces to an algebraic equation

$$4\cos^3(x/3) - 3\cos(x/3) = 1/2$$

By the rational root theorem, this equation can have rational roots,$\pm 1, \pm 1/2, \pm 1/4, \pm 1/8$, but out of these, no root satisfies the equation. Therefore, the equation can not be reduced to rational roots, and the minimum polynomial has a degree of three and the angle π/3 is not trisect-able (Yates, 1941). However, solutions to the problem using means other than a compass and straightedge have been found, including the one being explained in this paper. These methods are by way of intersection of two conic curves, *cissoid* of *Diocles*, the conchoid of *Nicomedes*, *Philo* line, *neusis*, tomahawk, likewise (Heath, 1931; Yates, 1941).

2 *Neusis* construction

Referring to Fig. 3, an equilateral triangle *ABC* is constructed with length of each side equal to unity, straight line *AB* is extended to point *D* such that *BD=AB=1,* and straight line BC is extended to point *E.* Point *D* is joined with point *C,* and straight line *DC* is extended to point *F*. At point *A,*

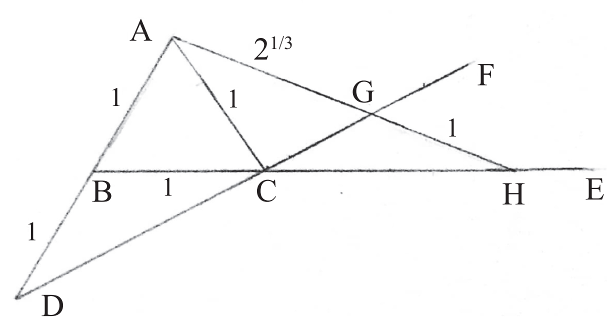

Fig. 3 *Neusis* construction for doubling a cube

the end of a ruler is placed and then rotated with centre at A so that length HG between the two points *H* and *G* falling upon the ruler measures unity and is extended to meet point A, then length AG will equal $2^{1/3}$ meaning thereby that a cube constructed on side *AG* will be double the volume of cube constructed on side *AC* (Heath, 1931; Yates, 1941). It is submitted that all the figures constructed in this paper are not drawn according to scale. Therefore, no inference needs to be drawn from the lengths, angles, areas, volumes displayed in these (Fig. 3).

I submit, one can appreciate that process involved in finding a point *G* on straight line *CE* so that *GH*=*1* is by way of exploration that warranted rotating the ruler around its one end placed at point *A*. It thus involved a dynamic process; hence, the determination of length AG was the result of dynamic geometry and did not satisfy the condition of using only an unmarked straightedge and a compass. Doubling the cube thus utilized dynamic geometry, making possible the Neusis construction. Dynamic geometry, as its name suggests, refers to varying either the value of an angle or a side of a figure so as to achieve a desired geometric shape that meets the given conditions. It does not employ a static figure, but the figure itself is varied to fulfill the given conditions, and then, measurement of a side or an angle, as the case may be, solves the problem. Dynamic geometry software is also available; however, in this paper, the calculations are performed manually.

3 Doubling the cube using dynamic geometry

Dynamic geometry is a method in which given conditions are fulfilled by varying the value of an angle or length, allowing solutions that are otherwise not achievable through static or fixed-figure geometry. Dynamic geometry, as described by King and Schattschneider, involves interactive software tools and environments. In such settings, objects in a geometric construction are not static: points or segments can be dragged, moved, or deformed with a cursor, while the dependent parts update automatically to preserve constraints. This allows for exploratory, interactive, visual, and even experimental geometry—tracing loci, observing how shapes change, discovering relationships, and motivating proofs. Although originally designed for classroom teaching, it has become an indispensable tool for researchers, mathematicians, and scientists (Schattschneider, 1997).

A little different, but related to the geometry of objects, Jay Hambidge describes dynamic geometry in terms of dynamic symmetry, which is a compositional and proportioning system in art and design. Here, proportions are alive and organic, based on irrational or “root rectangles,” the golden ratio, reciprocal rectangles, and growth patterns in nature. It is less about motion or software and more about growth, change, and movement in proportion—transitions of form that feel alive rather than rigid or purely static. Dynamic symmetry, in contrast with “static symmetry” (regular, passive, rational proportions), is meant to capture the vitality or life suggested by nature and classical Greek art. In Hambidge’s sense, geometric symmetry relates more to the proportional design of figures in art than to the lines, circles, and curves of mathematical geometry (Hambidge, 1920) [20].

With this brief definition of dynamic geometry and by resorting to the manual variation of angle C instead of using software, I refer to Fig. 4, where a triangle ABC is constructed with the length of the perpendicular *AB*=1, the base *BC*, and the hypotenuse *AC*. From point *B*, a perpendicular *BD* is drawn on hypotenuse *AC*, from point *D*, a perpendicular *DE* is drawn on base *BC*, from point *E,* a perpendicular *EF* is drawn on hypotenuse *AC* and, in this way, perpendiculars are drawn successively upon base and hypotenuse ad infinitum. By simple trigonometry, it is found

$$BD = AB\cos C = \cos C,\ DE = AB\cos^2 C = \cos^2 C,\ EF = AB\cos^3 C = \cos^3 C$$

$$FG = AB\cos^4 C = \cos^4 C,\ GH = AB\cos^5 C = \cos^5 C,\ HI = AB\cos^6 C = \cos^6 C$$

$$YX = AB\cos^n C = \cos^n C \tag{1}$$

assuming *YX* to be nth perpendicular, counting *BD* as 1st, *DE* 2nd, *EF* 3rd so on [6]. Equation (1) can also be written as

$$BD/AB = DE/BD = EF/DE = \ = \cos C \tag{2}$$

For doubling the unit cube, hypotenuse *AC* is rotated with centre at its end A keeping *AB*=*1,* thus varying angle *C* so that *EF* equals to$^1/_2$ Then $EF = 1/2 = \cos^3 C$,

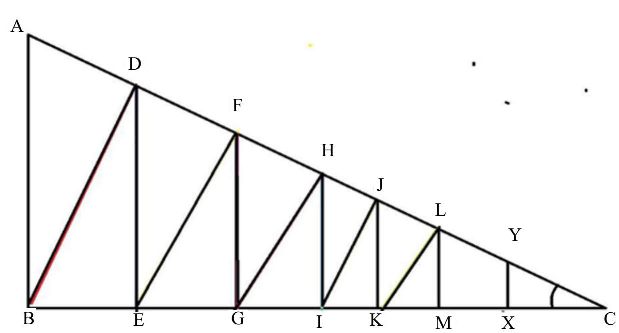

Fig. 4 Reducing to size half the unit cube and the unit hypercubes

or $\cos C = 1/2^{1/3}$ (Wadhawan, 2023). From Eq. (1), $BD = \cos C$, therefore, $BD = 1/2^{1/3}$.

Referring to Fig. 5, perpendicular BD is extended to D' so that a perpendicular AD' drawn from point A upon AB intersects BD' at point D', then in right angled triangle $D'AB$, $AB/BD' = \cos C$ but $AB = 1$, therefore,

$$BD' = 1/\cos C = 2^{1/3} \tag{3}$$

Hence, a cube constructed on side $BD' = 2^{1/3}$ will be double the volume of a unit cube. Length $EF = 1/2$ can be drawn by bisecting given unit with a compass. Hence the constructions of Figs. 4 and 5 were made using a compass and an unmarked straightedge. Figure 5 displays resultant volume and side on doubling a unit cube.

3.1 Other ways of doubling a cube

Referring to Fig. 6 and equations $EF = AB\cos^3 C, DE = AB\cos^2 C$ mentioned in Sect. 3, I construct triangle ABC and perpendiculars *BD, DE, EF*... but keeping the length of AB equal to 2 and rotating AC at point A, so that $EF = 1$, then $\cos C = 1/2^{1/3}$ and $DE = 2\left(1/2^{2/3}\right) = 2^{1/3}$ Thus a cube constructed on perpendicular DE will be double the unit cube constructed on EF, eliminating the need for the construction of Fig. 5. Kindly take care, AB must be equal 2 units in case of doubling the cube using this method.

Using the broomsticks that were readily available in my drawing room, I could construct geometric Fig. 4, but keeping $AB = 2$ centimetre (cm) and rotating AC at point A so that $EF = 1$ cm. I took precautions so that BD and *EF,* on visual inspection, are found perpendicular upon AC and side DE perpendicular upon BC. The photograph of the construction made with broomsticks is pasted in Fig. 7, which is a replica of Fig. 4, except that the alphabets marked in Fig. 4 are omitted. Upon measurement, I found that DE approximates 1.25 cm, whereas it should actually be 1.26 cm. This error cropped up, keeping in view the thickness of sticks, visual rough judgement of the perpendicularity of sticks, and rough measurements made by me. Needless to say, precise result would have been achieved, had the length of AB increased to 2 Metres (M) from 2 cm, length of EF to *1 M* and DE measured in metres with scale graduated up to millimetres.

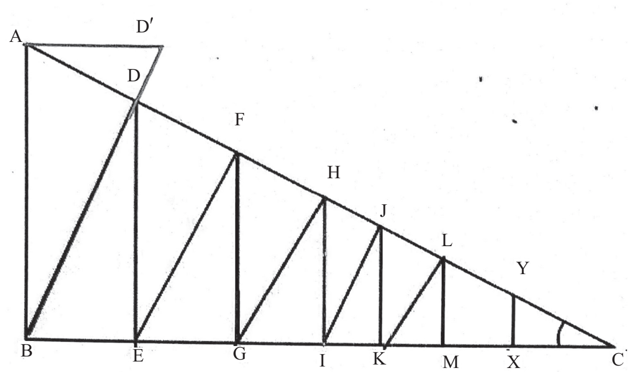

Fig. 5 Doubling the cube and hypercubes

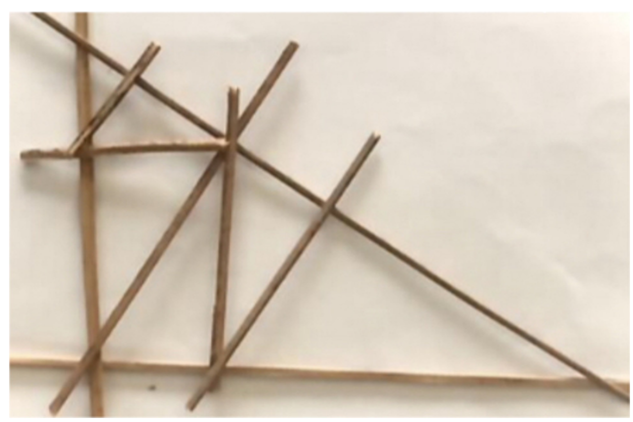

Fig. 7 Photograph of a replica of Fig. 4 constructed with broomsticks

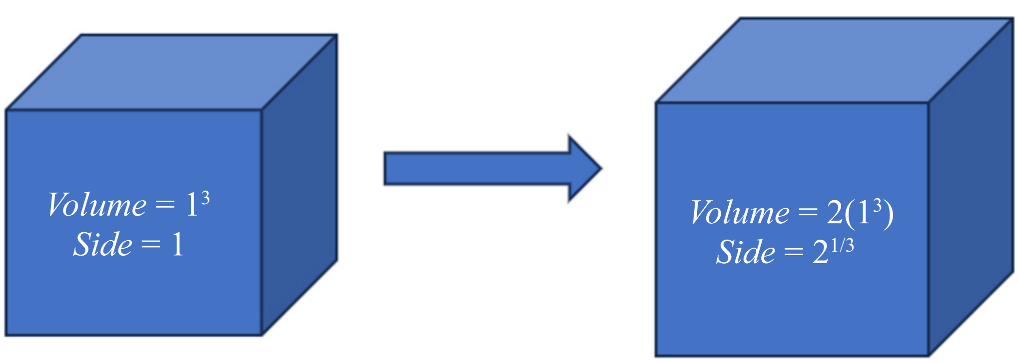

Fig. 6 Depicting the sizes of cubes on doubling

4 n-Dimensional cubes or n-cube or hypercubes

A cube of three dimensions has length, breadth, and height, whereas a hypercube generalises these three dimensions to n-dimensions and is thus an n- dimensional analogue of a square (of dimensions$n = 2$) and a cube of dimension ($n = 3$). An n-dimensional cube contains k-dimensional $\left[n!/\{k!(n-k)!\}\right]2^{n-k}$ cubes in it. An n-dimensional cube has 2^n corners or nodes,$\left\{n2^{n-1}\right\}$ edges,$2^{n-2}n(n-1)/2!\}$ { squares and $\{2^{n-3}n(n-1)(n-2)/3!\}$ cubes. It has volume also called n-volume as s^n, where s is the length of its side or an edge (Coxeter, 1973, p.123).

4.1 Doubling 4-dimensional cubes or tesseract using dynamic geometry

First, I will examine the case of doubling a 4-dimensional cube, known as a tesseract, and utilize Figs. 4 and 5, which are relevant to doubling all n-dimensional cubes. The process described for doubling a 3-dimensional cube will be applicable, except that in this case, hypotenuse AC is rotated, keeping its center at end A, and $AB = 1$, thus varying angle C so that the length of perpendicular FG is equal to 1/2 (Wadhawan, 2023). Rest procedure will be same, and BD' in this case will equal $2^{1/4}$. Hence, a 4-dimensional cube constructed on side $BD' = 2^{1/4}$ will have volume (V) double the unit 4-dimensional cube constructed on side (s)$AB = 1$ Fig. 8 displays resultant volume and side on doubling a tesseract.

4.2 Doubling 5-dimensional Cubes or penteract using dynamic geometry

The method described for doubling a three-dimensional cube will also be applicable here, except that in this case, hypotenuse AC is rotated, keeping its center at end A, and $AB = 1$, thus varying angle C so that the length of perpendicular GH is equal to$^1/_2$ The length of AB will be maintained at unity. The rest procedure will be the same, and BD' in this case will equal $2^{1/5}$. Fig. 9 displays the resultant volume and side on doubling a penetract.

4.3 Doubling n-dimensional cubes using dynamic geometry

The process described for doubling the cube will also be applicable here, except that in this case, hypotenuse AC is rotated while keeping its end A fixed, and $AB = 1$, thus varying angle C so that perpendicular XY equals 1/2, where XY is the nth perpendicular by counting DE as the first perpendicular. The length of AB will be maintained at unity. The rest procedure will be same and BD' in this case will equal $2^{1/n}$.

An n-dimensional unit cube can also be doubled using only and only Fig. 4 by keeping $AB = 2$ units and rotating AC at point A so that the nth perpendicular $XY = 1$ and measuring the length of (n-1)th perpendicular, which will be the side of the doubled n-dimensional cube.

5 Enlarging *m* times a unit cube or an n-dimensional unit cube using dynamic geometry

Procedure for enlarging m times a cube or an n-dimensional cube (hypercube) will be the same as already described, except that the length of perpendicular EF for a cube, FG for a four-dimensional cube, or XY for an n-dimensional cube will be equated with $1/m$ instead of $^1/_2$ and the result in such cases will also be the lengthBD'.

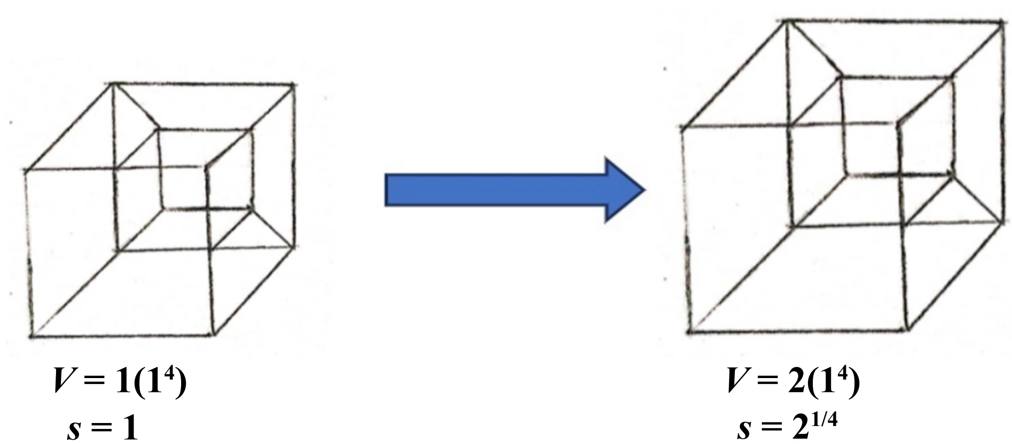

Fig. 8 Depicting the size of doubling the tesseract

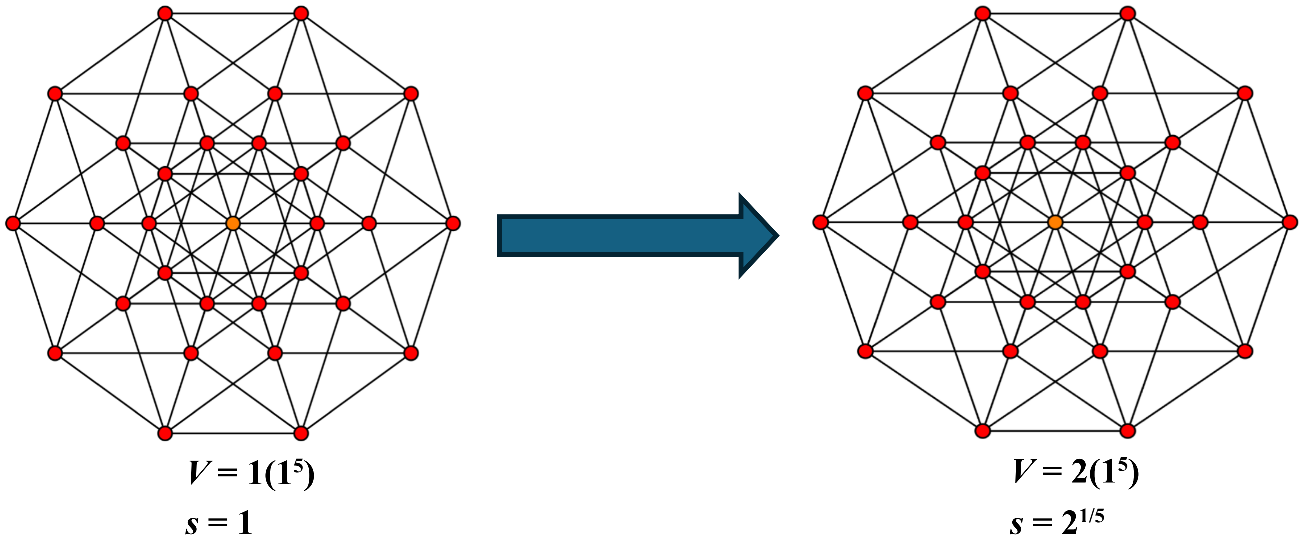

Fig. 9 Depicting the size on doubling the 5-Cube penetract

6 Reducing *m* times a unit cube or an n-dimensional unit cube using dynamic geometry

Procedure for reducing m times the cube or hypercubes will be the same as described in Sect. 5, except that the result in such cases will be length BD, and Fig. 4 will be used.

7 Precision and error

Geometric computations are error-free provided that the measuring instruments (in this case, length measuring instruments) are precise and error-free. Thus, the results obtained by enlarging or reducing the cubes in the present case are error-free. It will be observed that when angle C is very small in the vicinity of zero, the lengths of the perpendiculars also tend to zero, resulting in congestion that leads to difficulty in measuring lengths. Similarly, when angle C is in the vicinity of π/2 radians, again congestion of perpendiculars is observed. These difficulties can be avoided if the length of perpendicular AB is increased, say to l units from unity. This will spread the geometric drawing, giving adequate room to accommodate perpendiculars, thus ensuring their better measurability. In such a situation, the following equations will be used instead of Eqs. (1).

$$BD = AB\cos C = l\cos C,\ DE = AB\cos^2 C = l\cos^2 C,\ EF = AB\cos^3 C = l\cos^3 C$$

$$FG = AB\cos^4 C = l\cos^4 C,\ GH = AB\cos^5 C = l\cos^5 C,\ HI = AB\cos^6 C = l\cos^6 C,$$

$$YX = AB\cos^n C = l\cos^n C \tag{4}$$

For doubling a cube, EF/l will have to be equated with 2, and for doubling an n-dimensional cube, YX/l will have to be equated with 2. The rest procedure will be the same as already explained, and the result of enlarging a unit cube or a unit n-dimensional cube will be BD’/l, referring to Fig. 4. On reduction, these would be BD/l referring to Fig. 4.

8 Results, conclusions, and discussions

Doubling the cube has been a classical Greek problem, known as the Delian problem, and has remained unsolved despite efforts by the Greeks and others. While Indian geometry dates back to the Harappan period and advanced during the Vedic era through the construction of geometric shapes for Vedic purposes, it continued to develop through the medieval period and beyond. The Indian approach remained largely practical, focusing on achieving functional goals rather than pursuing the abstract purity of geometry. Moot question that arises in geometric construction of a cube double the volume of a given cube is whether Euclidean geometric construction is possible to have length segments $\mathrm{BD},\ \mathrm{DE},\ \mathrm{EF}\ldots$ such that $\mathrm{BD} = \mathrm{DE}/\mathrm{BD} = \mathrm{EF}/\mathrm{DE} = \cos \mathrm{C}$. Hippocrates of Chios discovered that solution to the problem is possible, if in given segments of lengths a and b, segments of lengths x and y could be found so that

$$a/x = x/y = y/b = k \quad (5)$$

From Eq. (5), it can be found, $a^3/x^3 = k^3 = a/b$ If b is taken equal to $2a$ and a equal to 1, then $x = 2^{1/3}$. The length x can be calculated, provided it is possible to construct the given line segments. The ratios given by Eq. (5) is same as are written in Eq. (2), if I presume $AB = 1$ and $EF = 1/2$ From Eq. (5), *Menaechmus* derived equations $xy = ab$, $y^2 = bx$, and $x^2 = ay$ In this way, value of x and y can be calculated from the point of intersection of the parabola $y^2 = ab$ with rectangular hyperbola $xy = ab$, and also from points of intersection of two parabolas $y^2 = bx$ and $x^2 = ay$.[2]

Determining the values of x and y from points of intersection of a parabola with a hyperbola and also those of two parabolas involves geometric construction of a hyperbola and parabolas. But parabolas and hyperbolas are not constructible geometrically with the use of an unmarked straightedge and a compass. Thus, points x and y can not be determined geometrically, and therefore, geometric construction of doubling the cube is an impossibility, which has also now been proved by Pierre Wantzel. It may be noted that there are other ways, not adhering to Euclidean geometry, available to solve the problem. *Neusis* construction is one of them. Kindly refer to Fig. 3. On examination of *Neusis* construction, one finds that it is possible to construct this figure without the use of an unmarked straightedge and a ruler, except that part of the construction related to drawing of line AGH. For drawing this line, one will have to keep one end of the ruler at point A and then rotate it about point A so that the ruler intersects line CE at H and CF at G in such a way that GH equals unity. Obviously, the method calls for exploring the position of the ruler while rotating it at its end A so as to satisfy the length $GH = 1$. Neusis construction thus utilizes exploration to satisfy a condition, and such exploration is an essential ingredient of dynamic geometry. It is thus not a straightforward Euclidean geometric construction, thereby fortifying Pierre Wantzel's dictum on the impossibility of its construction using an unmarked straightedge and a compass.

Coming to the method resorted to by me in this paper, I have also rotated the hypotenuse AC at point A, keeping length $AB = 1$ and varying angle C so that length of perpendicular EF exactly equals half the length of AB, and that resulted in $BD = \cos C = 1/2^{1/3}$ [6]. Once $cosC = BD$ is determined, using trigonometry, length $BD' = 2^{1/3} = 1/\cos C$ was determined referring to Fig. 4. Since I also varied angle C for exploring EF equal to $1/2$, I took refuge in dynamic geometry, the way *Neusis* construction did. If, in stead of equating EF with $\frac{1}{2}$ I had equated nth perpendicular say XY with $1/m$, referring to Fig. 4, then $BD = cosC$ would have equalled $1/m^{1/n}$ And referring to Fig. 4, $BD' = 1/\cos C$ would have equalled $m^{1/n}$ A unit cube thus would have been doubled in the former case and enlarged m times in the latter case using dynamic geometry when a cube or an n-dimensional cube, as the case may be, is constructed on side BD'.

I used a compass and an unmarked straightedge, but resorted to the use of varying angle C, keeping AB equal to unity and achieving $EF = 1/2$, and determined length BD' as a side of the resultant doubled cube. *Neusis* construction also made use of rotating an unmarked ruler to have $GH = 1$ and found length AG, which was the resultant side of the doubled cube.

[2] J. J. O'Connor and E F Robertson, Doubling the cube, Mac Tutor, History of mathematics at. https://mathshistory.st-andrews.ac.uk/HistTopics/Doubling_the_cube/.

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