SOLUTION OF THE ASTRONOMICAL TRIANGLE AS FOUND IN THE *TANTRASAMGRAHA* (A.D. 1500)

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The spherical triangle formed on the celestial sphere by the positions of the Sun, north pole and the zenith on it is called an astronomical triangle. The three sides of this triangle are the co-latitude of the place of observation, the co-altitude of the Sun and its co-declination at the time of observation. The internal angles formed at the zenith and the north pole are the azimuth and the hour angle respectively. If any three of the above named five elements are known, the remaining two can be found out. This gives rise to ten cases to be considered.

A complete solution of the astronomical triangle dealing systematically with all the ten cases is found in the Sanskrit work *Tantra Saṃgraha* which was composed by Nīlakaṇṭha Somayāji in the year A.D. 1500. However, some material of the work belongs to an earlier period of Indian astronomy.

The present paper contains the translations (for the first time ?) of the various rules given in the above work for solving the astronomical triangle in the ten different cases. When these rules are expressed in modern forms (as done in the present paper), it is seen that they give results which are same as those obtained by using the current standard formulas of modern spherical trigonometry, such as the Sine, Cosine and Cotangent Rules, and applying the theory of quadratic equation in some cases. For example, in Case I, where latitude, declination and azimuth are given, the rule given in the *Tantra Saṃgraha* for finding out the altitude of the Sun, yields a result which is same as that obtained from the Cosine Rule

$$\sin\delta = (\sin\phi).\ (\sin\alpha) + (\cos\phi).(\cos\alpha).(\cos A)$$

when this relation is converted into a quadratic equation in $\sin\alpha$ and solved.

The said work, however, does not contain the rationales of the rules, and the present paper does not make any attempt to investigate as to how the rules were arrived at.

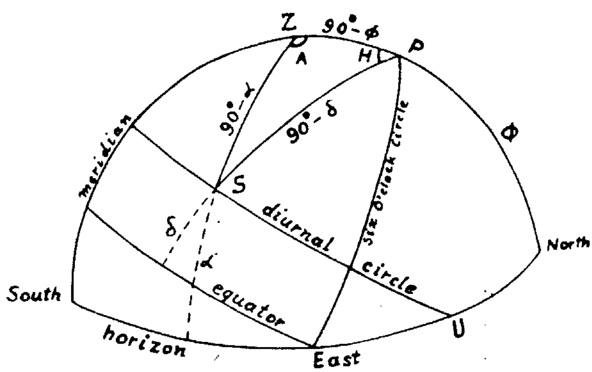
SYMBOLS AND SELECT GLOSSARY

A	Azimuth measured from the north.
B	*Bhā-bhuja* ('Shadow-arm') which is the distance of the Sun's projection on the plane of the celestial horizon from the east-west line.
C	Cosine of the local hour angle; $\sqrt{R^2-J^2}$.
D	Certain divisor (*s*).
'Day-sine'	Radius of the Sun's diurnal circle; $R\cos\delta$.
'Gnomon'	Sine of the altitude of the Sun.
H	Hour angle measured eastward.
J	*Svanata-jyā*, the Sine of the local hour angle defined by $J = (R\sin H).\ (R\cos\phi)/R$.

K	*Bhā-koṭi* ('Shadow-upright') which is the distance of the Sun's projection on the plane of the celestial horizon from the north-south line.
R	Radius, norm, *trijyā* or *Sinus Totus*.
'Shadow'	Cosine of the altitude of the Sun.
α	Altitude of the Sun or its co-zenith distance.
γ	*Digagrā* (directional amplitude), the (Indian) azimuth measured from the east-west line; so that we have $A = 90° \pm \gamma$.
δ	Declination of the Sun.
ϕ	Terrestrial latitude.

INTRODUCTION

Let S be the position of the Sun on the celestial sphere, P the position of the north pole and Z the zenith. Then the astronomical triangle SPZ is a spherical triangle in which the arcual sides PZ, ZS and SP are equal to the co-latitude, co-altitude and co-declination respectively. The angles SPZ and



PZS are the hour angle and azimuth respectively. Knowing any three out of the above five elements of the triangle, the remaining two can be found. A nice exposition of the subject of determining the remaining two elements, when any three of the above named five elements are given, is found in the Sanskrit work *Tantra-Saṃgraha* (= *TS*) which was composed in A.D. 1500 by Nīlakaṇṭha Somayāji (1444-1545)[1]. *TS* is an important work of the late Āryabhaṭa I School of the Indian astronomy. The title of the work signifies that it is a "Compendium" or "Collection" of astronomical rules and no doubt it includes some earlier material on Indian astronomy. The work has been published[2] with the commentary (= *TSC*) *Laghuvivṛti* written in A.D. 1556 by Śaṅkara Vāriar (*circa* 1500-1560) who was a disciple of the author of *TS*.[3]

The above mentioned almost complete solution of the astronomical triangle has been dealt in the third chapter of the work. *TS*, III, 60 (p. 64) says:

Iha śaṅkunata-krānti-digagrā-akṣeṣu pañcasu /
Dvayordvayorānayanaṁ daśadhā syat paraistribhiḥ //60//

'The determination of any two elements at a time out of the five elements, altitude, hour angle, declination, (Indian) azimuth and latitude, from the other three (being given) is of ten types (that is, there are ten cases)' *.

The ten cases have been dealt systematically and one by one in *TS*, III, 62-87 (pp. 65-88) and each case is followed by numerical exercises (*uddeśakas*). In the following pages we shall describe these ten cases one by one giving each time the method of solution as stated in the *TS*. Most of the *TS* rules under these cases are included in the *Trigonometry in Ancient and Medieval India* by the author of the present paper.[4]

TABLE

Case	Given Elements	Elements to be found out	Reference
I	δ, A, ϕ	α, H	*TS*. III, 62-67
II	H, A, ϕ	α, δ	*TS*. III, 68-73
III	H, δ, ϕ	α, A	*TS*. III, 74-75
IV	$H. \delta, A$	α, ϕ	*TS*. III, 75-78
V	α, A, ϕ	H, δ	*TS*. III, 78-79
VI	α, δ, ϕ	H, A	*TS*. III, 80-81
VII	α, δ, A	H, ϕ	*TS*. III, 81-83
VIII	α, H, ϕ	δ, A	*TS*. III, 83-85
IX	α, H, A	δ, ϕ	*TS*. III, 86-87
X	α, H, δ	A, ϕ	*TS*. III, 86-87

CASE I : GIVEN δ, A, ϕ

In order to find the Sun's altitude, the *TS*, III, 62-65 (p. 65) states:

Āśāgrā lambakābhyastā trijyābhaktā ca koṭikā //62//

Bhujākṣajyā tayorvargayoga-mulaṁ śrutirharaḥ/
Krāntyakṣa-vargau tadvargāt-tyktvā koṭyau tayoḥ pade //63//

* Angle *ZSP* has not been considered here.

Kuryāt krāntyakṣayor-ghātaṁ koṭyor-ghātaṁ tathā param /
Saumye goḷe tayor-yogāt bhedād-yāmye tu ghātayoḥ //64//

Ādyaghāte 'dhike saumye yogabheda-dvayādapi/
Trijyā-ghnād-dhāra-vargāptaḥ śaṅkuriṣṭa-digudbhavaḥ //65//

'.......................The Sine of the directional amplitude multiplied by the Cosine of the latitude and divided by the radius is the upright (a small side of some right angled plane triangle). The Sine of the latitude is the base (the other small side of the same triangle). The square root of the sum of their squares is the hypotenuse which is the Divisor.

The square roots of the quantities got by subtracting (separately) the squares of the Sines of the declination and the latitude from the square of that (Divisor) are the two *koṭis*.

Obtain the product of the Sines of the declination and the latitude and (also) the product of the two *koṭis*. Take their (of the above two products) sum (when the Sun's declination is) in north and difference (when it is) in south and both the sum as well as the difference (when it is) in north and the first product is greater (than the second product), multiply (the sum and/or the difference) by the radius and divide by the square of the Divisor. (The result is) the Gnomon (Sine of the altitude) in the desired direction.' *

That is,

$$\text{Divisor} = \sqrt{(R \sin\phi)^2 + \{(R \sin \gamma .\ R \cos \phi)/R\}^2} = D, \text{ say.}$$

Then

$$R \sin \alpha = (R/D^2).\ [R \sin\delta .\ R \sin\phi \pm^{**} \sqrt{D^2 - (R \sin\delta)^2}.\ \sqrt{D^2 - (R \sin\phi)^2}\,]$$

This solution can be easily seen to be equivalent to the result obtained by solving the quadratic equation

$$(\sin^2\phi + \cos^2\phi .\ \cos^2 A) \sin^2\alpha - 2 \sin\phi .\ \sin\delta .\ \sin\alpha + (\sin^2\delta - \cos^2\phi .\ \cos^2 A) = 0$$

which is derived from the relation $\sin\delta = \sin\phi .\ \sin\alpha + \cos\phi .\ \cos\alpha .\ \cos A$.

This last relation is written down by applying the modern cosine rule to the spherical triangle *ZSP*

Then *TS*, III, 66 (p. 65) includes a rule which can be expressed in modern symbols as follows

$$R \sin H = (R \cos\alpha .\ R \cos\gamma)/R \cos\delta.$$

* *TS*, III, 67 (p. 65) says that, in the case of the southern declination, the desired altitude of the Sun will not be attained if the first product is greater than the second (numerically); so also if $R \sin \delta$ is greater than the Divisor (whatever be the direction of δ).

**Here the minus sign denotes the positive difference of the quantities.

This relation for finding out the hour angle is equivalent to the sine formula for the spherical triangle *ZSP*.

CASE II : GIVEN *H. A.* ϕ

For finding the altitude, the *TS*, III, 68-71 (p. 70) says:

Natalambakayor-ghātāt trijyāptaṁ tatsvadeśajam/
Svadeśanata-koṭyāptaṁ natākṣajyā-vadhāttu yat //68//

Tadāśāgrāvadhe koṭyostayorghātaṁ kṣipedudak/
Śodhayed-dakṣiṇāgrāyaṁ trijyayā ca tato haret //69//
Labdhāt svanata-koṭighnāt pṛthak trijyāpta-vargitam /
Yutaṁ svanata-vargeṇa tanmūlena hṛtaṁ phalam //70//

Pṛthak-kṛtād bhavecchaṅkuḥ ..../

....//71//

'The product of the Sine of the hour angle and the Cosine of the latitude divided by the radius is the Sine of the local hour angle. Divide the product of the Sines of the hour angle and the latitude by the Cosine of the local hour angle and multiply by the Sine of the directional amplitude (the azimuthal angle measured from the east). The result should be added to, in case the directional amplitude is towards north, or subtracted from, in case the directional amplitude is towards north, or subtracted from, in case the directional amplitude is towards south, the product of their '*koṭis*' (that is, their uprights when they are taken as bases and the radius is taken as the hypotenuse in each case). The result (now obtained) be divided by the radius and the quotient (thus obtained) multiplied by the Cosine of the local hour angle be put separately (at two places).

At one place divide (the quantity) by the radius and add the square (of the result) to the square of the Sine of the local hour angle. By the square root of that (the sum of the squares just now obtained) divide the quantity (placed) separately. (The final result) becomes the Gnomon (the Sine of the altitude)....................,

That is,

Sine of the local hour angle

$$= (R \sin H.\ R \cos \phi)/R = J, \text{ say.}$$

Cosine of the local hour angle

$$C = \sqrt{R^2 - J^2}$$

Then we form the quantity

$$(C/R).\ [R \cos \gamma.\ \sqrt{R^2 - \{(R \sin H.\ R \sin \phi)/C\}^2} \pm^{*} R \sin \gamma.\ (R \sin H.\ R \sin \phi)/C]$$

$$= Q, \text{ say.}$$

The rule then gives

$$R \sin \alpha = Q\sqrt{J^2 + (Q/R)^2}$$

By combining the various above steps, the solution given in the *TS* can be seen to be equivalent to the relation.

$$\sin \alpha = \frac{\sin A.\ \cos H + \cos A.\ \sin H.\ \sin \phi}{\sqrt{(\sin H.\ \cos \phi)^2 + (\sin A.\ \cos H + \cos A.\ \sin H.\ \sin \phi)^2}}$$

which is a transformed form of the following result obtained by using the modern cotangent formula of the spherical trigonometry[5]

$$\tan \alpha.\ \cos \phi = \sin A.\ \cot H + \cos A.\ \sin \phi.$$

After finding the altitude, *TS*, III, 71 (second half) gives the equivalent of

$$R \text{ cose } \delta = (R \cos \alpha).\ (R \cos \gamma)/(R \sin H)$$

which completes the desired computations in the present case.

CASE III : GIVEN H, δ, ϕ

For finding the altitude, *TS*, III, 74-75 (p. 74 states:

Natakoṭyā hatā dyujyā vibhaktā tribhajīvayā/
Saumya-yāmyadiśor-bhūjyā-yutonā lambakāhatā//74//
Trijyāptā śaṅkur (ḥ)............................//75//

'Multiply the Cosine of the hour angle by the Day-sine (Cosine of the declination) and divide by the radius. (The quotient obtained be) increased or diminished, according as the direction (of the declination) is north or south, by the Earth-sine (the distance between the rising-setting line and the line joining the points of intersection of the diurnal circle and the six O'clock circle). (The result now obtained) multiplied by the Cosine of the latitude and divided by the radius is the Sine of the altitude.'

That is,

$$R \sin \alpha = [(R \cos H.\ R \cos \delta)/R \pm^{*} (\text{Earth-sine})].\ (R \cos \phi)/R.$$

* Here the minus sign denotes the positive difference of the quantities.

Now we know that (see *TS*, III, 59)[8]

$$\text{Earth-sine}=(R \sin \delta).\ (R \sin \phi)/R \cos \phi.$$

So that the rule is equivalent to the relation

$$\sin \alpha=\sin \delta.\ \sin \phi+\cos \delta.\ \cos \phi.\ \cos H$$

which can be directly written by using the cosine formula for a spherical triangle.

Juṣt after giving the above rule, *TS* text says that the Cosine of the directional amplitude should be found as before and the *TSC* (p. 74) gives the following usual rule for the purpose

$$R \cos \gamma=(R \sin H).\ (R \cos \delta)/R \cos \alpha.$$

Case IV : Given H, δ, A

TS, III, 75-78 (p. 76) states:

Chāyāṁ nītvātha tatkoṭidyujyā-vargāntarāt padam //75//

Tatchāyābāhughāto yaśśaṅku-krāntyorvadho'pi yaḥ /
Krāntyagrayos-tulyadiśos-tayorbhedo 'nyathā yutiḥ //76//

Unmaṇdala-kṣitijayor-antare'rke ca tadyutiḥ /
Taddhatāṁ vibhajet-trijyāṁ tacchāyā-koṭivargayoḥ //77//

Antareṇa bhavedakṣo (aḥ)........................ //78//

'After getting the Shadow (Cosine of the altitude), take the square root of the difference of the squares of its upright (east-west component) and the Day-sine (Cosine of the declination).

The product of that (square root) and the Shadow-arm and also the product of the Sine of the altitude and the Sine of the declination (are formed). Take their (of the above two products) difference, if the declination and the directional amplitude are in the same direction, otherwise sum, and (also take their) sum when the Sun is between the six o'clock circle and the horizon. That (the sum or difference) multiplied by the radius and divided by the difference of the squares of the radius and the Shadow-upright becomes the Sine of the latitude.........'

Explanation : For finding the latitude by the above rule, we should first determine the 'Shadow', Shadow-upright, and the Shadow-arm needed in the rule. For this purpose the *TSC* (p. 76) on the text of the rule gives the equivalent of the following formulas

'Shadow$=(R \sin H.\ R \cos \delta)/R \cos \gamma = R \cos \alpha$;

Shadow-upright$=(R \sin H.\ R \cos \delta)/R$,

that is, $K=(R \cos \alpha.\ R \cos \gamma)/R$;

and

Shadow-arm$=\sqrt{(R \cos \alpha)^2 - K^2}$

or

$$B=(R \cos \alpha.\ R \sin \gamma)/R.$$

Then the above rule gives

$$R \sin \phi = \frac{R.\ [R \sin \alpha.\ R \sin \delta) \pm^* B. \sqrt{(R \cos \delta)^2 - K^2}]}{R^2 - K^2}$$

This solution can be seen to be equivalent to the roots of the equation

$$(1-\cos^2 \alpha.\ \sin^2 A) \sin^2 \phi - \sin \alpha.\ \sin \delta.\ \sin \phi + (\sin^2 \delta - \cos^2 \alpha.\ \cos^2 A) = 0$$

which is derived from the relation

$$\sin \delta = \sin \alpha.\ \sin \phi + \cos \alpha.\ \cos \phi.\ \cos A$$

already mentioned in case I.

CASE V : GIVEN α, A, ϕ

For finding the declination of the Sun, the *TS*, III, 78-79 (p. 79) states:

Akṣaśaṅkvorvadho yaśca, yaśca bhā-bahu-lāmbayoḥ //78//

Saumya-yāmyasthite bhānau tayor-yogantarāttataḥ /
Krāntis-trijyāhṛtā........................ //79//

'....Take the product of the Sine of the latitude and the Sine of the altitude and also that of the Shadow-arm and the Cosine of the latitude. Their (of the two products) sum or difference, according as the Sun's position is north or south (of the prime vertical), divided by the radius is the Sine of the declination.......'

That is,

$$R \sin \delta = [(R \sin \phi.\ R \sin \alpha) \pm^* B.\ R \cos \phi]/R.$$

* Here the minus sign denotes the positive difference of the quantities.

Since

$$B = (R \cos \alpha).\ (R \cos A)/R,$$

the above rule gives the same result as obtained by using the Cosine formula which has already been mentioned in cases I and IV.

Just after stating the above rule, the *TS* says that the hour angle should be obtained as before. The *TSC* (p. 80) gives two methods for this. One of these is same as that contained in *TS*, III, 66 (p. 65) and which we have already mentioned under case I.

CASE VI : GIVEN α, δ, ϕ

To find the azimuth or the directional amplitude, *TS*, III, 80-81 (p. 81) says:

Trijyāpakrama-ghāto yo yaśca śaṅkvakṣa-yorvadhaḥ /
Tayoryogāntaraṁ yattu goḷayor-yāmya-saumyayoḥ //80//

Bhābāhur-lambakāpto'smāt-trijyāghnād bhāhṛteṣṭadik /
.... //81//

"Take the product of the radius and the Sine of the declination and also the product of the Sine of the altitude and the Sine of the latitude. Their (of the two products) sum or difference, (according as Sun's declination is) in southward or northward, divided by the Cosine of the latitude is the Shadow-arm. This multiplied by the radius and divided by the Cosine of the altitude is the desired Sine of the directional amplitude........................'

That is,

$$[(R.R \sin \delta) \pm^{*} (R \sin \alpha.\ R \sin \phi)]/R \cos \phi = B$$

Then

$$B.\ R/R \cos \alpha = R \sin \gamma$$

giving the Sine of the (Indian) azimuth which is measured from the east. The above result may be seen to be same as that obtained by solving the following cosine relation for getting A.

$$\sin \delta = (\sin \alpha.\ \sin \phi) + (\cos \alpha.\ \cos \phi.\ \cos A)$$

As explained in the *TSC* on the text of the above rule, we can then get the hour angle by using the relation

* Here the minus sign denotes the positive difference of the quantities.

$$R \sin H = K.R/(R \cos \delta)$$

where

$$K = \sqrt{(R \cos \alpha)^2 - B^2}$$

CASE VII : GIVEN α, δ, A

For finding the latitude, the *TS*, III, 81-83 (p. 83) says:

Vargāntarapadaṁ yatsyācchāyā-koṭidyujīvayoḥ //81//

Tacchāyābāhuyogo yaḥ śaṅku-krāntyaikya-vargataḥ /
Tenāptaṁ yat phalaṁ tasminneva tat svamṛṇaṁ pṛthak //82//

Tayoralpahatā trijyā mahatāptākṣamaurvikā /
.... //83//

'......Take the square root of the difference of the squares of the Shadow-upright and the Cosine of the declination and add it to the Shadow-arm. By the quantity so obtained, divide the square of the sum of the Sines of the altitude and the declination. The quotient should be separately added to or subtracted from that very quantity. When the radius is multiplied by the smaller result (of the above subtraction) and divided by the greater result (of the last addition), we get the Sine of the latitude...'

That is,

$$B + \sqrt{(R \cos \delta)^2 - K^2} = D, \text{ say}$$

Then form the two quantities

$$D + (R \sin \alpha + R \sin \delta)^2/D = Q_1, \text{ say}$$

and

$$D \sim (R \sin \alpha + R \sin \delta)^2/D = Q_2, \text{ say.}$$

Finally we get

$$R \sin \phi = R.\ Q_2/Q_1$$

so that we have

$$R \sin \phi = R.\frac{\{D^2 \sim (R \sin \alpha + R \sin \delta)^2\}.}{\{D^2 + (R \sin \alpha + R \sin \delta)^2\}.}$$

This rule may be compared with that given under case IV for finding the latitude.

For finding the hour angle, the *TSC* (p. 83) gives the equivalent of the following rule before commenting on the above rule proper

$$R \sin H = (R \cos \alpha).\ (R \cos \gamma)/R \cos \delta$$

However, the same occurs in the text of the TS also and we have already given it (see under case I).

CASE VIII : GIVEN α, H, ϕ

For finding the declination, *TS*, III, 83-85 (p. 85) says :

Trijyāhatākṣaśaṅkū svanatakoṭyuddhṛtau pṛthak //83//
Ye tatkoṭyau ca tat trijyāvargabheda-padīkṛtau /
Mithaḥ koṭighnayor-yogādyāmye saumye'ntarāttayoḥ //84//
Trijyayā vihṛtā dyujyā.............................. //85//

'Multiply the Sine of the latitude and the Sine of the altitude (each) by the radius and divide (the results) separately by the Cosine of the local hour angle. The corresponding uprights (with the above two quotients as bases) are the square roots of the radius-square minus (each of) the quotients.

Multiply the (above) quotients crossly by their uprights. Their (of the two products just obtained) sum (when the Sun is) in the south (of the six O'clock circle), or difference in the north, divided by the radius is the Day-sine, (Cosine of the declination).....................'

That is,

$$R \cos \delta = (1/R).\ [(R \sin \phi).\ (R/C).\ \sqrt{R^2 - \{(R \sin \alpha).\ (R/C)\}^2}.\ \pm^{*} (R \sin \alpha).\ (R/C).\ \sqrt{R^2 - \{R \sin \phi).\ (R/C)^2\}}]$$

On simplification this will become

$$\cos \delta = \frac{\sin \phi.\ \sqrt{\sin^2 \phi + \cos^2 \phi.\ \cos^2 H - \sin^2 \alpha^{*}} \pm \cos \phi.\ \sin \alpha.\ \cos H}{(\sin^2 \phi + \cos^2 \phi.\ \cos^2 H)}$$

This solution is same as that obtained by solving the quadratic equation

$$(\sin^2 \phi + \cos^2 \phi.\ \cos^2 H) \cos^2 \delta \pm^{*} 2 \cos \phi.\ \sin \alpha.\ \cos H.\ \cos \delta + (\sin^2 \alpha - \sin^2 \phi) = 0$$

which itself is derived from the already mentioned (see case III) cosine relation, namely,

$$\sin \alpha = \sin \phi.\ \sin \delta + \cos \phi.\ \cos \delta.\ \cos H$$

* Here the minus sign denotes the positive difference of the quantities.

After giving the above rule the *TS* asks us to find the directional amplitude as before and the *TSC* (p. 86) lays down the equivalent of the following procedure for finding the azimuthal angle.

$$(R \cos \delta).\ (R \sin H)/R = K,$$

$$\sqrt{(R \cos \alpha)^2 - K^2} = B.$$

Finally,

$$R \sin \gamma = B.\ R/(R \cos \alpha).$$

CASE IX : GIVEN α, A. H.

For finding the declination, *TS*, III, 86 (p. 88) says:

Digagrāyāstu tatkoṭis-tacchāyā-ghātato hṛtā /
Natajyayā bhaved dyujyā....................... //86//

'From the (Sine of) the directional amplitude get its Cosine. The product of that (the above Cosine) and the Cosine of the altitude divided by the Sine of the hour angle becomes the Day-sine (the Cosine of the declination)...........'

That is,

$$R \cos \delta = (R \cos \gamma).\ (R \cos \alpha)/R \sin H$$

a rule which has already been given earlier in *TS*, III, 71 (see under case II above) and which is equivalent to the sine formula for the spherical triangle *SZP*.

For finding the latitude see under case X below.

CASE X : GIVEN γ, H, δ

For finding the azimuthal angle, *TS*, III, 87 (p. 88) says:

Dyujyā-natjyayor-ghātad-agrākoṭiḥ prabhā hṛtā /
.... //87//

'The product of the Day-sine (Cosine of the declination) and the Sine of the hour angle divided by the Shadow (Cosine of the altitude) becomes the Cosine of the directional amplitude...........'

That is,

$$R \cos \gamma = (R \cos \delta).\ (R \sin H)/R \cos \alpha$$

which is another form of the rule given under case IX above.

The result of finding one unknown element in each of the above two cases IX and X is that we know now the four elements α, H, A and δ in either of the cases. The problem in both these cases is therefore same, namely, to find out the remaining fifth element ϕ. For this the *TS* simply says.

'Latitude (should be found) as before'.

The *TSC* (p. 88) at this point asks us to use either the rule given under case IV or the rule given under case VII for determining the latitude.

Concluding Remarks

Just after giving the said solutions in the ten cases, *TS*, III, 87 (p. 88) says

'(Here) ends the description of the answers to the ten problems.'

However, the work contains several other rules which provide alternate methods of solution in some of the above general cases or their particular ones. For example, *TS*, III, 88-91 (pp. 89-90) gives an alternate rule for finding the 'Shadow' (Cosine of the altitude) from given azimuth, declination, and latitude (Cf. Case I dealt above).

From the present study, the readers must not conclude that Indian solution of the astronomical triangle in each case was given for the first time in the *TS*. In fact, solution in many general and particular cases were known in India much earlier than the date of the *TS*. It is outside the scope of the present paper to give the history and development of the Indian solutions in the various cases.

As is usual with most of the ancient Indian original texts, the *TS* does not state explicitly the methods through which the rules were arrived at. However, many of the ancient ways of deriving these rules can be known from the material found in the commentaries on various astronomical works. It is believed that most of the Indian rules were derived by working 'inside' the armillary sphere rather than 'on its surface'. For this purpose the Indians also employed the so-called latitudinal and declinational triangles. In this connection the following remark of Nīlakaṇṭha Somayāji is noteworthy[7].

'The whole of the planetary-mathematics is pervaded by the two theorems (namely) the *Bhujā-koṭi-karṇa Nyāya* (the so-called Pythagoras Theorem) and the Rule of Three (the proportionality of sides in similar triangles).'

Some other methods, like those based on the theory of successive approximations or of quadratic equations, were also employed by the Indians.

References and Notes

[1] Sarma, K.V. : *A History of the Kerala School of Hindu Astronomy (in perspective)*. Vishveshvarananda Institute, Hoshiarpur, 1972, pp. 55-56.

[2] *The Tantrasaṃgraha by Nīlakaṇṭha Somasutvan* ———. Edited by S. K. Pillai, Trivandrum Sanskrit Series No. 188, University of Kerala, Trivandrum, 1958. All references to *TS* and *TSC* in the article are according to this edition.

[3] Sarma, K.V. : Op. cit., p. 58.

[4] Gupta, R.C. : *Trigonometry in Ancient and Medieval India*. Doctoral Thesis, Ranchi University, Ranchi, 1970 (not yet published), Chapter VII, Part two (Solution of Triangles), pp. 238-259.

[5] Todhunter, I. : *Spherical Trigonometry*. Revised by G. Prasad, Pothishala, Allahabad, 1965, pp. 17-18.

[6] The expression for the Earth-sine was well known to the Indian astronomers. It is found even in early works like *Āryabhaṭīya* (IV, 26) and *Mahā-Bhāskarīya* (III, 6) etc.

[7] See his commentary on the *Āryabhaṭīya*, edited by K. Sambsiva Sastri, Trivandrum, 1930, Part I, p. 100.