SINES AND COSINES OF MULTIPLE ARCS AS GIVEN BY KAMALĀKARA

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The paper describes the expansion formulas for the Sines and the Cosines of the double, triple, quadruple, and quintuple arcs in terms of the Sine and the Cosine of a given arc as found in the *Siddhānta-Tattva-Viveka* (A.D. 1658) of the Indian astronomer Kamalākara and his own commentary on it. The formulas are mathematically correct. Their derivations, which are mostly based on the Addition and Subtraction Theorems for the Sine and the Cosine, are also discussed. A typical result is equivalent to

$$\sin 4A = 4(\sin A.\cos^3 A - \cos A.\sin^3 A)$$

1. INTRODUCTION

The Indian astronomer and mathematician Kamalākara, son of Nṛsiṃha, was a contemporary of the famous Newton in Europe. Kamalākara composed his famous work entitled *Siddhānta-Tattva-Viveka* (*STV*) in A.D. 1658 and he himself commented on it. The *STV*, composed according to the extant *Sūrya Siddhānta* (which is the most popular work of Hindu Astronomy), is regarded to be the best among the various *siddhāntas* (traditional astronomical works) of India[1]. The work along with the author's own commentary (*STVC*) on it has been printed[2].

We present below the trigonometrical formulas for the expansions of the Sines and Cosines of double, triple, quadruple, and quintuple arcs as found in the Third Chapter, called *Spaṣṭādhikāra*, of the *STV* and in the accompanying *STVC*. It may be recalled that the Indian Sine (written with a capital S to distinguish it from the modern sine) of an arc in any circle is defined to be half of the chord of double the arc and is equal to $R \sin A$, where R is the radius (norm or *Sinus totus*) of the circle and $\sin A$ is the the modern sine of the angle A, subtended by arc at the centre of the circle. Similarly the Indian Cosine is equal to $R \cos A$.

2. FORMULAS FOR THE DOUBLE ARC

The *STV*, III, 73, second half (p. 136) says :

दो:कोटिजीवाभिहतिर्द्विनिघ्नी
त्रिज्योद्धृता सा द्विगुणांश जीवा ॥ ७३ ॥

Doḥkoṭijīvābhihatirdvinighnī Trijyoddhṛtā sā dviguṇāṃśajīvā / /73/ /

'The product of the Sine and the Cosine of an arc multiplied by two and divided by the radius is the Sine of twice the degrees (of the arc)'.

That is,

$$R \sin 2A = 2(R \sin A).\,(R \cos A)/R \qquad \ldots (1)$$

or

$$\sin 2A = 2 \sin A.\, \cos A \qquad \ldots (2)$$

which is the modern form of the formula.

The *STV*, III, 90 (p. 144) says:

यद्बाहुकोटिज्यकयोश्च वर्ग-
वियोगमानं त्रिभजीवयाऽऽप्तम् ।
नूनं च तत्कोटिगुणस्य मानं
द्विसंगुणानां च तदंशकानाम् ॥ ९० ॥

Yadbāhukoṭijyakayośca varga-viyogamānaṃ tribhajīvayā''ptam;
Nūnaṃ ca tatkoṭiguṇasyamānaṃ dvisaṃguṇānāṃ ca tadaṃśakānām 90

'The difference of the squares of the Sine and the Cosine of an arc divided by the radius is certainly (equal to) the Cosine of two times the degrees (of the arc).'

That is,

$$R \cos 2A = [(R \sin A)^2 \sim (R \cos A)^2]/R \qquad \ldots (3)$$

which is the numerical equivalent of the modern formula

$$\cos 2A = \cos^2 A - \sin^2 A \qquad \ldots (4)$$

The *STVC* (p. 140) expresses the right hand sides of (1) and (3) in the old Indian mathematical symbols as (after restoring a dot)

भु. को २ / त्रि १		भुव १ कोव १ / त्रि १
bhu.ko 2 / tri 1	and	bhu-va 1 ko-va 1 / tri 1

respectively, where

भु	(*bhu*) stands for	*bhuja-jyā*	($R \sin A$);
को	(*ko*) ,, ,,	*koṭi-jyā*	($R \cos A$);
त्रि	(*tri*) ,, ,,	*trijyā*	(radius);
व	(*va*) ,, ,,	*varga*	(square).

Thus

bhu-va $= (R \sin A)^2$; and *ko-va* $= (R \cos A)^2$.

The negativeness of a term is indicated by placing a dot *over* the number (or the coeffieient which is written on the right). Other similar symbols used in the *STVC*, following the usual Indian practice, are as follows :

gha stands for *ghana* (cube) :

va-va .. .. *varga-varga* (square-square).

Generally a dot or a small circle has been used between two terms to indicate their product, for example

कोवव. भु ५ (*ko-va-va.bhu* 5)

stands for

$5(R \cos A)^4.(R \sin A)$.

3. Formulas for the Triple Arc

The *STV*, III, 74 (p. 136) says :

दोर्ज्यैकराशिज्यकया विभक्ता
फलस्य वर्गेण विहीनितं च ।
त्रयं भुजज्यागुणितं त्रिनिघ्न-
भुजांशकानामिह शिञ्जिनी स्यात् ॥ ७४ ॥

Dorjyaika-rāśijyakayā vibhaktā phalaysya vargeṇa vihīnitaṃ ca
Trayaṃ bhujajyā-guṇitaṃ trinighna-bhujāṃśakānāmih siñjinī syāt (74)

'Divide the Sine of an arc by the Sine of one sign (that is, of 30 degrees). Three diminished by the square of the (above) quotient and (then) multiplied by the Sine of the arc becomes the Sine of thrice the degrees of the arc.'

That is,

$$R \sin 3A = (R \sin A)\,[3 - (R \sin A)^2/(R \sin 30°)^2\,] \quad .. \quad (5)$$

which is equivalent to the modern formula

$$\sin 3A = 3 \sin A - 4 \sin^3 A \quad .. \quad (6)$$

The *STVC* (p 142) gives the following symbolic expression for $R \sin 3A$ (also note the same given on p 140 of *STVC* but without the second dot)

भुघ ४̇ त्रिव. भु ३
त्रिव १

or

bhu-gha $\dot{4}$ *tri-va.bhu* 3/*tri-va* 1

That is,

$$R \sin 3A = [-4\,(R \sin A)^3 + 3R^2.(R \sin A)]/R^2 \quad .. \quad (7)$$

The corresponding expression for R cos $3A$ is given as (*STVC*, p. 140)

को. भुव ३̇ कोघ १
―――――――――
त्रिव १

or

ko.bhu-va 3̇ *ko-gha* 1/*tri-va* 1

That is,

$$R \cos 3A = [-3\ (R \cos A).(R \sin A)^2 + (R \cos A)^3]/R^2 \quad .. \quad (8)$$

The modern form of (8) will be

$$\cos 3A = \cos^3 A - 3 \cos A \,.\, \sin^2 A \quad .. \quad (9)$$

which is mathematically equivalent to the usual modern formula

$$\cos 3A = 4 \cos^3 A - 3 \cos A \quad .. \quad (10)$$

4. Formulas for the Quadruple Arc

The *STV*, III, 75 (p. 136) says :

एवं मिथः संगुणिते घनेन दोःकोटिजीवे विवरं तयोर्यत् ।
युगाहतं संविहृतं त्रिभज्याघनेन वेदघ्नभुजांशजीवा ॥ ७५ ॥

Evaṃ mithaḥ saṃguṇite ghanena doḥkoṭijīve vivaraṃ tayoryat
Yugāhataṃ saṃvihṛtaṃ tribhajyā-ghanena vedaghna-bhujāṃśajīvā (75)

'Multiply the Sine and the Cosine of an arc crossly by (their) cubes. Their (that is, of the two products obtained) difference multiplied by four and divided by the cube of the radius is the Sine of four times the degrees of the arc.'

That is,

$$R \sin 4A = (4/R^3).\ [(R \sin A).(R \cos A)^3 \sim (R \cos A).(R \sin A)^3] \quad (11)$$

which is equivalent to

$$\sin 4A = 4 \sin A \,.\, \cos^3 A \sim 4 \cos A \,.\, \sin^3 A \quad .. \quad (12)$$

The *STVC* (pp. 140 and 143) gives the equivalent of the following symbolic form for R sin $4A$.

$$[-4(R \sin A)^3.(R \cos A) + 4(R \cos A)^3.(R \sin A)]/R^3 \quad .. \quad (13)$$

which is mathematically correct. The *STVC* (p. 143) also reduces (13) to the form

$$[-8\ (R \sin A)^3.(R \cos A) + 4R^2(R \sin A).(R \cos A)]/R^3$$

The corresponding expression for R cos $4A$ is given to be as (*STVC*, p. 140)

भुवव १ भुव. कोव ६̇ कोवव १
―――――――――
त्रिघ १

(where we have made slight corrections in the printed text, for example by taking the denominator as त्रिघ १ which otherwise appears to be त्रिव १) or

bhu-va-va 1 *bhu-va. ko-va* $\dot{6}$ *ko-va-va* 1/*tri-gha* 1

That is,

$$R \cos 4A = [(R \sin A)^4 - 6\,(R \sin A)^2.(R \cos A)^2 + (R \cos A)^4]/R^3 \quad .. \quad (14)$$

which is correct and is equivalent to

$$\cos 4A = \sin^4 A - 6 \sin^2 A.\cos^2 A + \cos^4 A \quad .. \quad (15)$$

5. Formulas for the Quintuple Arc

The *STV*, III, 76-77 (p. 136) says :

नृपाहतो दोर्गुणवर्गवर्गस्तथेषुनिघ्नी त्रिभमौर्विकायाः
कृतेः कृतिस्तद्युतितो विशोध्य नखैर्हतां वर्गसमाहतिं च ॥ ७६ ॥
दोर्ज्यात्रिमौर्व्योरवशेषनिघ्नी
दोर्ज्या त्रिभज्याकृतिवर्गभक्ता ।
लब्धं हि पञ्चघ्नभुजांशजीवा
.. .. ॥ ७७ ॥

Nṛpāhato dorguṇa-varga-vargastatheṣunighnī tribhamaurvikāyāḥ
Kṛteḥ kṛtistadyutito viśodhya nakhairhatāṃ varga-samāhatiṃ ca (76)
Dorjyā-trimaurvyoravaśeṣa-nighnī dorjyā tribhajyākṛtivargabhaktā
Ladbhaṃ hi pañcaghna-bhujāṃśa-jīvā
. . . . (77)

'(Take) sixteen times the square of the Sine-square and five times the square of the square of the radius. From their sum, subtract twenty times the square of the product of the Sine of the arc and the radius. The remainder (so obtained) be multiplied by the Sine of the arc and divided by the square of the radius-square. The result is verily the Sine of five times the arc........'

That is,

$$R \sin 5A = (R \sin A).[16\,(R \sin A)^4 + 5\,(R^2)^2 - 20(R.R \sin A)^2]/R^4 \quad .. \quad (16)$$

which is mathematically correct and whose modern equivalent is

$$\sin 5A = \sin A.(16 \sin^4 A - 20 \sin^2 A + 5) \quad .. \quad (17)$$

The *STVC* (p. 140) gives the following symbolic expression for $R \sin 5A$.

bhu-gha.ko-va $\dot{1}\dot{0}$ *ko-va-va. bhu* 5 *bhu-va-va. bhu* 1/*tri-va-va* 1

That is.

$[-10(R \sin A)^3.(R \cos A)^2+5(R \cos A)^4.(R \sin A)+(R \sin A)^4.(R \sin A)]/R^4$ (18)

The modern equivalent will be

$$\sin 5A = \sin^5 A - 10 \sin^3 A.\cos^2 A + 5 \sin A.\cos^4 A \quad .. \quad (19)$$

Further, the *STVC* (p. 144) uses the relation

$$(R \cos A)^2 = R^2 - (R \sin A)^2$$

to eliminate the Cosine terms from (18) and reduces the latter to a form which is exactly equivalent to the right hand side of (16).

The corresponding expression for R cos $5A$ is given to be as (*STVC*, p. 140)

भुवव. को ५ भुव. कोघ १० कोवव. को १
―――――――――――――
त्रिवव १

(where we have restored a dot in the last term of the numerator and dropped a superfluous unit multiplier in the middle term)

or *bhu-va-va.ko* 5 *bhu-va.ko-gha* 10 *ko-va-va. ko* 1/*tri-va-va* l

That is,

$$R \cos 5A = [5(R \sin A)^4.(R \cos A) - 10\ (R\sin A)^2.(R \cos A)^3 + (R \cos A)^4.(R \cos A)]/R^4 \quad .. \quad (20)$$

which is mathematically correct and whose modern equivalent is

$$\cos 5A = 5 \sin^4 A.\cos A - 10 \sin^2 A.\ \cos^3 A + \cos^5 A \quad .. \quad (21)$$

6. Derivations

A systematic derivation of all the formulas for the Sine and the Cosine of multiple arcs can be given by using the so-called Addition Theorems for the Sine and the Cosine, namely

$$R \sin (A+B) = (R \sin A).(R \cos B)/R + (R \cos A).(R \sin B)/R \quad .. \quad (22)$$

$$R \cos (A+B) = (R \cos A).(R \cos B)/R - (R \sin A).(R \sin B)/R \quad .. \quad (23)$$

In India, the Addition Theorem for the Sine is found explicitly stated by Bhāskara II (A.D. 1150) in his *Jyotpatti* (verse 21) which is given at the end of the *Golādhyāya* part of his famous astronomical work called the *Siddhānta Śiromaṇi*[3].

In his own commentary under *Jyotpatti*, 21-22, Bhāskara II has clearly indicated the method of deriving the Sines of multiple arcs in the following words[4]

तुल्यभावनया। प्रथमज्यार्धस्य प्रथमज्यार्धेन सह समासभावनया द्वितीयम् २; द्वितीयस्य द्वियीयेनैवं चतुर्थ ४ मित्यादि। अथातुल्यभावनया। द्वितीय तृतीययोः समासभावनया पञ्चमम् ५।
.. .. इत्यादि।

'The process of combining equal arcs : By applying the Addition Theorem to the first tabular Sine ($R \sin h$) with the first tabular Sine, we get the second tabular Sine ($R \sin 2h$) ; In the same manner, the second tabular Sine with the second tabular Sine gives the fourth tabular Sine ($R \sin 4h$) ; And so on.

Now the process of combining unequal arcs : By the Addition Theorem with the second (tabular Sine) and the third (tabular Sine) we get the fifth one ($R \sin 5h$) ;................Etc.'

The *STV*, III. 68-69 (p. 111) states the Addition and Subtraction Theorems for the Sine as well as for the Cosine, and in the next verse (No. 70), the author credits Bhāskara II for giving this method of trigonometrical computation in his (*Siddhānta*) *Śiromaṇi*. Following the method indicated by Bhāskara II, the *STVC* (p. 140) systematically gives the details of deriving the Sines and the Cosines of the multiple arc (upto five times) by using the Addition Theorems thus :

'From the known Sine and Cosine of an arc, we get the Sine and the Cosine of twice the degrees of the arc by applying the Addition Theorems with equal arcs. ....From these and the known Sine and Cosine, we obtain the Sine and the Cosine of thrice the degrees of the arc by using the Addition Theorems..... ....Similarly, from the Sine and Cosine of twice the degrees of the arc (as already found out above), we get the Sine and the Cosine of four times the degrees of the arc by using the Addition Theorems as applied to equal arcs............. ....From these and the originally known Sine and Cosine, we get the Sine and Cosine of five times the degrees of the arc.................'

As far as the Sine of twice the arc is concerned, the *STVC* (pp. 138-140) also gives a geometrical proof for it before giving the above general analytical proof. This geometrical proof may be outlined as follows :

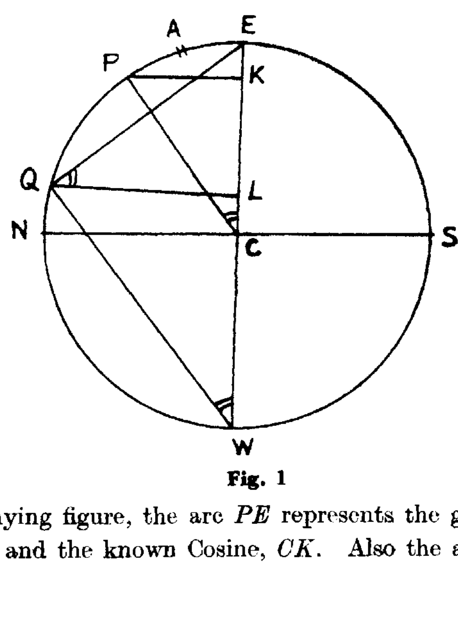

Fig. 1

In the accompanying figure, the arc *PE* represents the given arc ($= A$) with its known Sine, *PK*, and the known Cosine, *CK*. Also the arc *PQ* is taken equal to the arc PE, so that the arc QE will represent twice the given arc whose Sine, QL, is to be found out.

The *STVC* names the four right angled triangles, EWQ, PCK, QLW, and ELQ which are all similar to each other. Since the diameter EW is double the radius PC, the side QW will also be double of the side CK (because the triangles EWQ and PCK are similar). So that QW is equal to twice the Cosine of the given arc.

For finding the required Sine of double the arc, the *STVC* says that we take the proportionality of sides of the third and the second similar right angled triangles named above. That is, from the triangles QLW and PCK, we have

$$QL/QW = PK/PC$$

or

$$QL/(2\,R\cos A) = (R\sin A)/R$$

giving the required length QL which represents the desired Sine of twice the arc.

7. Concluding Remarks

By using the respective Addition Theorems, the expansion formulas for the Sines and Cosines of the multiple arcs can be easily derived as is done in modern text books. In India, Bhāskara II (A.D. 1150) had given the Addition and Subtraction Theorems for the Sine and had also indicated the method of deriving Sines of the multiple arcs by using the Theorems No earlier Indian statement of these Theorems has come to light yet.

However, after Bhāskara II, the subject is found to be dealt in more and more details We have presented such details as given by Kamalākara (about 500 years after Bhāskara) but we do not claim that such details do not exist in the works of earlier authors.

In Europe, Vieta (about 1600 A.D.) had already found[5] the general expansions of cos nA and sin nA is terms of sin A and cos A Among the Arabs, Abu al-Wafa (about A.D. 980) had obtained the Addition and Subtraction Theorems of the Sine[6].

References

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[3] The *Siddhānta Śiromaṇi* (with author's own commentary). Edited by Bapu Deva Sastri, Chowkhamba Saskrit Series Office, Benares City, 1929, p. 283. (Kashi Sanskrit Series No. 72).

[4] Ibid., p. 285.

[5] Boyer, Carl B.: *A. History of Mathematics.* John Wiley, New York, 1968, pp. 340-341.

[6] Bond, J. D.: "The Development of Trigonometrical Methods etc." *Isis*, Vol.4 (1921-22), p.308.