GAṆITA KAUMUDĪ AND THE CONTINUED FRACTION

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Indian scholar Nārāyaṇa (1350 A. D.) perhaps used the result $Nq_n q_{n-1} - Bp_n p_{n-1} = (-1)^n b_{n+1}$ and $\frac{p_c}{q_c} = \frac{p_n^2 + Nq_n^2}{2p_n q_n}$ of the continued fraction to find out the integral solution of the equation $Nx^2 + K^2 = y^2$. The paper presents the original Sanskrit verses (in Roman Character) with English translation from Nārāyaṇa's *Gaṇita Kaumudī.*

1. INTRODUCTION

Indian scholar Nārāyaṇa (1350 A. D.) composed two books, viz. (*i*) *Bījagaṇitam* and (*ii*) *Gaṇita Kaumudī.* He perhaps used the knowledge of simple recurring continued fraction in the solution of the indeterminate equation of type $Nx^2+K^2=y^2$. We shall show here how the following mathematical results of the continued fraction besides others are involved in the method of the type $Nx^2+K^2=y^2$.

Result I. If c be the number of elements in the cycle belonging to N then

$$\frac{p_c}{q_c} = \frac{p_n^2+Nq_n^2}{2p_nq_n} \qquad \dots \qquad (1)$$

Result II. $Aq_nq_{n-1} - Bp_np_{n-1} = (-i)^n b_{n+1}$. $\qquad \dots \qquad (2)$

Where p_n/q_n is the nth convergent of the continued fraction

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots\cdots\cdots .$$

Result I. (*Gaṇita Kaumudī, Varga prakṛti* Vss. $2-4\frac{1}{2}$) [1]

hrasvajyeṣṭhakṣepān
kramaśasteṣāmadho nyaset tānstu
anyānyeṣām nyāsa
stasya bhaved bhāvanā-nāma ॥ 2 ॥

vajrābhyāsau hrasva
jyeṣṭhakayoḥ saṃyutirbhaved hrasvam
laghughātaḥ prakṛtihato
jyeṣṭhavadhenānvito jyeṣṭham ॥ 3 ॥

kṣiptorghātaḥ kṣepaḥ
syād vajrābhyāsayorviśeṣo vā
hrasvam lavdhorghātaḥ
prakṛtighno jyeṣṭhyośca vadhaḥ ॥ 4 ॥

tadvivaraṃ jyeṣṭhapadaṃ
kṣepaḥ kṣiptyoḥ prajāyate ghātaḥ 4½

English translation :

"Set down successively (*kramaśaḥ*) the lesser (*hrasava*) root, greater (*jyeṣṭha*) root and interpolator (*kṣepa*) and below them set down in order the same or another (set of similar quantities). [From them by the principle of composition can be obtained numerous roots]. The principle of composition (*bhāvanā*) will be explained here. (2)

"(Find) the two cross products (*vajrābhyāso*) of the two lesser and two greater roots ; their sum is a lesser root. Add the product of the two lesser roots multiplied by *prakṛti* to the product of the two greater roots. The sum will be a greater root. (3).

In that (equation) the interpolator will be the product of the two previous interpolators. Again the difference of the two cross products is a lesser root. Subtract the product of the two lesser roots multiplied by *prakṛti* from the product of the greater roots ; (The difference) will be a greater root. Here also the interpolator is the product of the two (previous) interpolator (4, 4½)."

According to the above rule, if $x=\alpha$, $y=\beta$ be the solution of the equation $Nx^2+k=y^2$ and $x=\alpha'$, $y=\beta'$ be the solution of the equation $Nx^2+k'=y^2$, then $x=\alpha\beta'\pm\alpha'\beta$, $y=\beta\beta'\pm N\alpha\alpha'$ is the solution of the equation $Nx^2+kk'=y^2$. This is known as principle of composition.

We have the following relation

$$N(\alpha\beta'\pm\alpha'\beta)^2+kk' = (\beta\beta' N\pm\alpha\alpha')^2 \qquad \ldots \qquad (3)$$

where $\alpha\beta'\pm\alpha'\beta$=lesser root and $\beta\beta'\pm N\alpha\alpha'$=greater root.

When $\alpha=\alpha'$, $\beta=\beta'$ and $k=k'$ then (3) reduces to

$N(2\alpha\beta)^2+k^2=(\beta^2+N\alpha^2)^2$, (when upper sign is considered).
$x=2\alpha\beta$ and $y=\beta^2+N\alpha^2$.

Now $y/x=\beta^2+N\alpha^2/2\alpha\beta=p_n^2+Nq_n^2/2p_nq_n$ where $p_n=\beta$ and $q_n=\alpha$ and p_n/q_n has its usual meaning. This is same as result I.

Having obtained one solution, an infinite number of other solutions can be found by the repeated application of the principle of composition. Nārāyaṇa

(1350)[2] says "By the principle of composition of equal as well as unequal sets of roots, (will be obtained) an infinite number of roots",

This result was already known to Brahmagupta[3], Bhāskara II[4] and Kamalākara[5].

Result II. (*Varga prakṛti* Vss 8—11)[6]

hrasvavṛhata prakṣepān
bhājyaprakṣepabhājakān kṛtvā
kalpyo guṇo yathā ta
dvargāt saṃśodhayet prakṛtim ‖ *8* ‖

prakṛterguṇavarge vā
viśodhite jāyate tu yaccheṣam
tata kṣepahṛtaṃ kṣepo
guṇavargaviśodhite vyastam ‖ *9* ‖

labdhiḥ kaniṣṭhamūlaṃ
tannijaguṇakāhataṃ viyuktam ca
purvālpapadaparaprakṣi
ptyorghātena jāyate jyeṣṭhaṃ ‖ *10* ‖

prakṣepaśodhaneṣva
pyekadvicatursvabhinnamūle staḥ
dvicatuḥ kṣepadābhyāṁ
rupakṣepāya bhāvanā kāryā ‖ *11* ‖

English translation :

"Making the lesser root (***hrasva mūla***), greater root (***vṛhata***) and interpolator (of a square nature=***varga prakṛti***) the dividend, addend and divisor (respectively of a pulverser), the (indeterminate) multiplier of it should be determined in the way doscribed before. The ***prakṛti*** being subtracted from the square of that or the square of the multiplier being subtracted from the ***prakṛti***, the remainder divided by the (original) interpolator is the interpolator (of a new square nature=***varga prakṛti***) ; and it will be reversed (***vyastam***) in sign in case of subtraction of the square of the multiplier. The quotient (corresponding to that value of the multiplier) is the lesser root (of a new square) ; and that multiplied by the multiplier and diminished by the product of the previous lesser root and (new) interpolator will be its greater root. By doing so repeatedly will be obtained two integral roots corresponding to the interpolator ± 1, ± 2 or ± 4. In order to derive integral roots for the additive unity from those answering to the interpolator ± 2 or ± 4, the principle of composition (should be adopted)".

After obtaining $Na^2+k=b^2$ and $N \cdot 1^2+(m^2-N)=m^2$ for a suitable k and m by the previous method, Principle of composition is applied between (a, b, k) and $(1, m, m^2-N)$ which gives rise to $Na_1^2+k_1=b_1^2$ and which when repeatedly applied by the principle of composition, the solution is obtained.

where

$$a_1=\frac{ax+b}{k},$$

$$b_1=\frac{bn+Na}{k}$$

$$k_1=\frac{n^2-N}{k}.$$

Changing the suffixes, we can write

$$a_{i+1}=\frac{a_i n+b_i}{k_i} \quad \ldots \quad (i)$$

$$b_{i+1}=\frac{b_i n+Na_i}{k_i} \quad \ldots \quad (ii)$$

$$k_{i+1}=\frac{n^2-N}{k_i}. \quad \ldots \quad (iii)$$

Now take $a_i=q_i$, $b_i=p_i$ then for every i, we have

$$\frac{b_{i+1}}{a_{i+1}}=\frac{b_i n+Na_i}{a_i n+b_i}$$

or, $$\frac{p_{i+1}}{q_{i+1}}=\frac{p_i n+Nq_i}{q_i n+p_i}$$

or, $$np_i q_{i+1}+Nq_i q_{i+1}=nq_i p_{i+1}+p_i p_{i+1}$$

or, $$Nq_i q_{i+1}-p_i p_{i+1}=n(q_i p_{i+1}-q_{i+1} p_i)$$

or, $$Nq_i q_{i+1}-p_i p_{i+1}=n(-1)^{i+1}. \quad \ldots \quad (iv)$$

But when we consider $\sqrt{N}$ as a simple continued fraction, then $\sqrt{N}=\sqrt{A/B}$. Therefore (iv) is transformed.

To $$Aq_i q_{i+1}-Bp_i p_{i+1}=(-1)^i\, b_{i+2}$$

where $n=b_{i+2}$.

This shows that results of Bhāskara II [7] has been discussed systematically in details by Nārāyaṇa by the knowledge of continued fraction.

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[2] *Bījagaṇitam* of Nārāyaṇa M. S. (incomplete) R—78.

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[5] Kamalākara—*Siddhānta tattva-viveka*—Edited with the *śeṣavāsanā* of the author, by Sudhakara Dvivedi, Benaras (1885) Ch. 13, P 210—214.

[6] Ibid.—P. 236—*ślokas* 8—11.

[7] Ibid. Page 26 ff