A RATIONALE OF BHĀSKARA I'S METHOD FOR SOLVING $ax \pm c = by$

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Indian Scholar Bhāskara I (522 A. D.) perhaps used the method of continued fraction to find out the integral solution of the indeterminate equation of the type $by = ax - c$. The paper presents the original Sanskrit verses (in Roman Character) from Bhāskara I's *Mahā Bhāskaryia*, its English translation with modern interpretation.

INTRODUCTION

Bhāskara I (522 A. D.) gave a rule in his *Mahābhāskarīya* for obtaining the general solution of the linear indeterminate equation of the type $by = ax - c$. This form seems to have chosen by Bhāskara I deliberately so as to supplement the form of Āryabhaṭa I. Smith[1] following Kaye said that Āryabhaṭa I attempted at a general solution of the linear indeterminate equation by the method of continued fraction. In this paper we shall deduce the formula $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$ of the continued fraction from the Bhāskara I's method of solution of indeterminate equation of the first degree and then we may draw the conclusion that the formula $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$ of the continued fraction was implicitly involved in the Bhāskara I's method of solution of the indeterminate equation of first degree.

A FEW LINES ABOUT THE CONTINUED FRACTION

$$a/b = a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots \cdots$$

Let

$p_1/q_1, p_2/q_2; \cdots \cdots, p_n/q_n \ldots$ be the successive convergents of a/b then

$$p_1/q_1 = a_1 \qquad \ldots \qquad (i)$$

$$p_2/q_2 = \frac{a_1 a_2 + 1}{a_2} \qquad \ldots \qquad (ii)$$

$$p_3/q_3 = \frac{a_1(a_1 a_2+1)+a_3}{a_2 a_3+1} \quad \ldots \quad (iii)$$

$$p_4/q_4 = \frac{a_1\{a_2(a_3 a_4+1)+a_4\}+a_3 a_4+1}{a_2(a_3 a_4+1)+a_4} \quad \ldots \quad (iv)$$

$$p_5/q_5 = \frac{a_1a_2a_3a_4a_5+a_3a_4a_5+a_1a_4a_5+a_1a_2a_5+a_1a_2a_3+a_1a_2a_3+a_5+a_3+a_1}{a_2a_2a_4a_5+a_2a_5+a_2a_5+a_4a_5+1} \quad \ldots \quad (v)$$

and the following result will be easily obtained

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^n$$

BHĀSKARA I'S RULE

Bhāskara I (522 **A. D**) gave the following rule in his *Mahā Bhāskarīya*[2]

bhājyam nyasedupari hāramadhaśca tasya
khaṇḍayātparasparamadho binidhāya labdham ।
kenā hato' yamapanīya jathāsya śeṣaṃ
bhāgaṃ dadāti pariśudhamiti pracintyam ॥ 42 ॥

āptāṃ matiṃ tāṃ binidhāya ballāṃ
nityaṃ hyadho'dhaḥ kramaśaśca labdham ।
matyā hataṃ syāduparisthitaṃ ya
llabdhena yuktaṃ parataśca tadvat ॥ 43 ॥

hāreṇa bhājyo bidhino paristho
bhājyena nityaṃ tadadhaḥ' sthitaśca ।
aharganosmin bhāgaṇādayaśca
tadvā bhavedyasya samīhitaṃ yat ॥ 44 ॥

Datta and Singh[3] translate these *Ślokas* as follows :

"Set down the dividend above and the divisor below. Write down successively the quotients of their mutual division, one below the other, in the form of a chain. Now find by what number the last remainder should be multiplied, such that the product being subtracted by the (given) residue (of the revolution) will be exactly divisible (by the divisor corresponding to that remainder). Put down that optional number below the chain and then the (new) quotient underneath. Then multiply the optional number by that quantity which stands just above it and add to the product the (new) quotient (below). Proceed afterwards also in the same way. Divide the upper number (i.e. multiplier) obtaind by this process by the divisor and the lower one by the dividend ; the remainders will respectively be the desired *ahargaṇa* and the revolutions."

After translation Datta and Singh[4] further said

"The equation contemplated in this rule is

$$\frac{ax-c}{b} = \text{ a positive integer.}$$

This form of the equation seems to have been chosen by Bhāskara I deliberately so as to supplement the form Āryabhaṭa I in which the interpolator is always made posititive by necessary transposition. Further b is taken to be greater than a, as is evident from the following rule. So the first quotient of mutual divisions of a and b is always zero. This has not been taken into consideration. Also the number of quotients in the chain is taken to be even."

RATIONALE OF THE RULE

The equation is of the type $ax-c=by$ ... (1)

where a=dividend, b = divisor, x=multiplier, y=quotient, remembering that $a < b$.

Now according to *śloka* we have.

$$\begin{array}{llllllll}
\frac{a}{b}\,) & \frac{a}{0} & (0=a_1 & & & & & \\
 & a\,) & b & (a_2 & & & & \\
 & & aa_2 & & & & & \\
 & & r_1\,) & a & (a_3 & & & \\
 & & & r_1a_3 & & & & \\
 & & & r_2\,) & r_1 & (a_4 & & \\
 & & & & r_2a_4 & & & \\
 & & & & r_3\,) & r_2 & (a_5 & \\
 & & & & & r_3a_5 & & \\
 & & & & & r_4 & & \\
 & & & & & \cdots & \cdots & \cdots \quad \cdots \\
 & & & & & \cdots & \cdots & \cdots \quad \cdots
\end{array}$$

Here

$$\begin{aligned}
a &= a_1b+a \\
b &= a_2a+r_1 \\
a &= a_3r_1+r_2 \qquad \cdots \quad (2) \\
r_1 &= a_4r_2+r_3 \\
r_2 &= a_5r_3+r_4.
\end{aligned}$$

Consider the even number of (partial) quotients, say four Remember that Datta and Singh said "... . So the first quotient of mutual division of a by b is always zero. This has not been taken into consideration." Therefore a_5 is the even (partial) quotient.

Let t_1 = optional number.

Now $\dfrac{r_4 t_1 - c}{r_3} = k_1,$

$$t_1 = \frac{k_1 r_3 + c}{r_4}.$$

Consider the table

$$\begin{array}{r|l}
0 = a_1 & a_1 L + s_3 = U\ (=y) \\
a_2 & s_3 a_2 + s_2 = L\ (=x) \\
a_3 & s_2 a_3 + s_1 = s_3 \\
a_4 & a_4 s_1 + t_1 = s_2 \\
a_5 & a_5 t_1 + k_1 = s_1 \\
t_1 & \\
k_1 & t_1
\end{array}$$

Here
$$s_1 = a_5 t_1 + k_1$$
$$= a_5 \left(\frac{k_1 r_3 + c}{r_4}\right) + k_1 \quad \left[\because\ t_1 = \frac{k_1 r_3 + c}{r_4}\right]$$
$$= \frac{k_1(a_5 r_3 + r_4) + a_5 c}{r_4}$$
$$= \frac{k_1 r_2 + a_5 c}{r_4} \quad [\because\ r_2 = a_5 r_3 + r_4]$$

$$s_2 = a_4 s_1 + t_1$$
$$= a_4 \left(\frac{k_1 r_2 + a_5 c}{r_4}\right) + \frac{k_1 r_3 + c}{r_4}$$
$$= \frac{k_1(a_4 r_2 + r_3) + c(a_4 a_5 + 1)}{r_4}$$
$$\frac{k_1 r_1 + c(a_4 a_5 + 1)}{r_4} \quad [\because\ r_1 = a_4 r_2 + r_3,]$$

$$s_3 = a_3 s_2 + s_1$$
$$= a_3 \left\{\frac{k_1 r_1 + c(a_4 a_5 + 1)}{r_4}\right\} + \frac{k_1 r_2 + a_5 c}{r_4}$$
$$= \frac{k_1(a_3 r_1 + r_2) + c(a_3 a_4 a_5 + a_3 + a_5)}{r_4}$$
$$= \frac{k_1 a + c(a_3 a_4 a_5 + a_3 + a_5)}{r_4} \quad [\because\ a = a_3 r_1 + r_2]$$

$$L = a_2 s_3 + s_2$$

$$= a_2 \left\{ \frac{k_1 a + c\,(a_3 a_4 a_5 + a_3 + a_5)}{r_4} \right\} + \frac{k_1 r_1 + c\,(a_4 a_5 + 1)}{r_4}$$

$$= \frac{k_1\,(a_2 a + r_1) + c\,(a_2 a_3 a_4 a_5 + a_2 a_3 + a_2 a_5 + a_4 a_5 + 1)}{r_4}$$

$$= \frac{k_1 b + c\,(a_2 a_3 a_4 a_5 + a_2 a_3 + a_2 a_5 + a_4 a_5 + 1)}{r_4} \quad [\therefore b = a_2 a + r_1]$$

$$= \frac{k_1 b + c q_5}{r_4} \quad [by\ (v)]$$

$$U = a_1 L + s_3$$

$$\frac{a_1 \{k_1 b + c\,(a_2 a_3 a_4 a_5 + a_2 a_3 + a_2 a_5 + a_4 a_5 + 1)\}}{r_4} + \frac{k_1 a + c\,(a_3 a_4 a_5 + a_5 + a_3)}{r_4}$$

$$= \frac{k_1\,(a_1 b + a) + c\,\{a_1 a_2 a_3 a_4 a_5 + a_1 a_2 a_3 + a_1 a_2 a_5 + a_1 a_4 a_5 + a_3 a_4 a_5 + a_1 + a_3 + a_5\}}{r_4}$$

$$= \frac{k_1 a + c p_5}{r_4} \quad [\therefore\ a = a_1 b + a \text{ and by } (v)].$$

Here

$$\frac{p_6}{q_6} = \frac{a}{b} \text{ and } \frac{L}{U} = \frac{k_1 b + c q_5}{k_1 a + c p_5}$$

Now

$$\begin{aligned} p_6 L - q_6 U &= p_6\,(k_1 b + c q_5) - q_6\,(k_1 a + e p_5) \\ &= a\,(k_1 b + c q_5) - b\,(k_1 a + c p_5) \\ &= k_1 ab + acq_5 - k_1 ab - bcp_5 \\ &= c\,(aq_5 - bp_5) \\ &= c\,(p_6 a_5 - q_6 p_5) \\ &= c\,(-1)^6 \\ &= c \end{aligned}$$

We have taken $L = x$, $U = y$

$$\because\ p_6 L - q_6 U = c$$

$$\therefore\ p_6 x - q_6 y = c$$

$$\text{or,} \quad ax - by = c$$

$$\text{or,} \quad ax - c = by$$

which is the original form $ax - c = by$.

Thus we see that the formula $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$ of the continued fraction is implicitly involved in the Bhāskara I's method of solution of the indeterminate eqution of the first degree.

EXAMPLE

Now let us take an example from the *Gaṇita Sāra Saṃgraha* [5] of Mahāvīra. Mahāvīra says

dṛṣtvāmrarāśin pathiko jathaika
triṃśatsamūhaṃ kurute trihīnam
śeṣe hṛte saptativistrimiśrai
rnarairviśudhaṃ kathayaikasaṃkhām

Rangacharya [6] translates this as follows :—

"A traveller sees heaps of mangoes (equal in numerical value) and makes 31 heaps less by 3 (fruits) ; and when the remainder (of these 31 heaps) is equally divided among 73 men, there is no remainder (of these 31 heaps) is equally divlded among 73 men, there is no remainder. Give out the numerical value of one (of these heaps)."

This gives us at once the following equation

$$73x = 31x - 3.$$

```
73 ) 31 ( 0=a1
      0
    ------
    31 ) 73 ( 2=a2
         62
         ----
         11 ) 31 ( 2=a3
              22
              ----
               9 ) 11 ( 1=a4
                    9
                   ---
                    2 ) 9 ( 4=a5
                        8
                       ---
                        1 ) 2 ( 2=a6
                            2
                           ---
                            0
```

Take the even number of partial quotients say 2. (Here a_3 = 2nd partial quotient as Datta and Singh said".... So the the first quotient of mutual division of a and b is always. This has not been taken into consideration).

Now according to Bhāskara I's rule we have

$\frac{9.t-3}{11}$ where t is the optional number

take $t = 4$, then $k_1 = 3$.

Consider the *Vallī* (table)

2	11.2+4 = 26
2	
4	4.2+3 = 11
3	

31) 26 (0
0
26

$\therefore x = 26$

Ans $x = 26$.

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[3] Datta, B. B. and Singh. A. N. *History of Mathematics* (a source book). Part II., 1938. Lahore p -99—100,

[4] Ibid

[5] *Gaṇita Sāra Saṃgraha by Mahāvīra*—Edited with English translation and notes by M. Rangacarya. Madras, 1912. *Śloka* 119½, Page 80.

[6] Ibid. Page 121.