INDIAN VALUES OF THE SINUS TOTUS*

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Unlike the modern trigonometric sine of an angle which is defined as the ratio of the side (facing that angle) to the hypotenuse in a right angled triangle, the ancient Sine of an arc was defined (apparently in India for the first time) as half the chord of double the arc in a circle of reference. The radius of this circle thus became the *Trijyā* (the Sine of three signs) or the *Sinus Totus* (the total or complete Sine).

It is curious as well as interesting to know that the Indians, through the ages, used a variety of values for the *Sinus Totus* such as, 43, 60, 120, 150, 200, 300, 500, 1000, 3270, and 3600 beside those typically Indian values which were based on the relation

$R=21600/2\pi$ minutes.

The value 3438 has been the most popular for Indian standard tables of Sines and 120 was frequently used for shorter tables.

Detailed discussions of the various values are presented in the paper along with full references. Terminology and some instances of transmission are also described. The value 150 which was used in India by Brahmagupta (seventh century A. D.) and Lalla (eighth century) has been found to be used later on in several foreign works obviously under Indian influences.

1. INTRODUCTION

The predecessor of the modern trigonometric function known as the sine of an angle was born, apparently, in India.[1] The Greek trigonometry had been based on the functional relationship between the chords of a circle and the central angles they subtend. The Indians, on the other hand, used half of a chord of a circle as their basic trigonometric function. The Indian (or Hindu) Sine (usually written with a capital letter to distinguish it from the modern sine) of an arc in a circle is defined as half the length of the chord of double the arc. Thus the (Indian) Sine of an arc α is equal to $R \sin \theta$ where R is the radius of the circle of reference and $\sin\theta$ is the modern sine of the angle θ subtended at the centre by the arc α.

The relations between (Indian) Sine, modern sine and the Greek chord (=crd) functions may be expressed as

$$\text{Sin } \alpha = R \sin \theta = \tfrac{1}{2} \text{ crd } 2\alpha \quad \ldots \quad (1)$$

$$\text{crd } \alpha = 2 R \sin (\theta/2) = 2 \text{ Sin } (\alpha/2) \quad \ldots \quad (2)$$

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In these relations the angular measure α of the arc is exactly equal to the angle θ.

The *Āryabhaṭīya* (*AB*) of *Āryabhaṭa* I (476 A.D.) is the earliest extant Indian work of the historical or dated type in which the Indian Sine is definitely used. However, there seems to be some evidence for earlier and possible use of Sine in India in some of the old *Siddhāntic* works which have been summarized later on by Varāhamihira (*c.* 550 A. D.) in his *Pañca-Siddhāntikā* (=*PS*)[2], Hereafter such abbreviations will be used for standard Sanskrit works ; all of them are listed in Appendix 1.

From the definition of the Sine, it is clear that its greatest value will be equal to R when the arc is equal to 90 degrees. That is why the norm R is called *Sinus totus** ('total or complete Sine'), the Sines corresponding to other arcs being regarded as parts or fractions of this.

The ancient length-definition (even with R equal to one) has thus at least one advantage over the modern definition of the sine, as the ratio of perpendicular to the hypotenuse in a right angled triangle, because in the case of 90 degress the former definition presents no difficulty while the latter can yield the sine of 90 degrees only by considering it as a limiting case.

For the parametrie norm R, a variety of values were used by the Indians during the ancient and medieval periods of their trigonometry. The purpose of this paper is to present and discuss those values along with some other related aspects.

2. Values of the Sinus Totus

The constancy of the ratio of the circumference of any circle to its diameter was known in the ancient world. So that when the cicumference C is known, the diameter D (=$2R$) can be witten down, their ratio being π.

The most typically Indian values of the *Sinus totus* R were obtained from the relation

$$R = C/2\pi \tag{3}$$

after having first chosen the value of C. Now C is a linear quantity and should be specified in linear units. But the Indians took a different attitude. They took the angular measure of the circumference (equivalent to the measure of 360 degrees angle subtended by it at the centre) itself to represent its linear measure.

However, this attitude is in itself not sufficient to fix the size of the

*The term sinus totus was introduced for the first time by Gerhard of Cremona (1114-1187) in his translation of *Al-Zurqālī's* astronomical tables. (information given by the referee.)

circle because we cannot associate any absolute length to an angular unit say a degree or a minute. For example, two arcs, one of a smaller and other of a bigger circle, can both be said to be of equal length (say 1800 miniutes) in angular units when each of them subtend the same angle (30 degrees in said case) at their centres although their linear sizes, that is lengths, are different.

Another difficulty created by first specifying C in ordinary angular units and then calculating R by (3), is that the latter cannot be found exactly since the number π is transcendental. Even if we use an approximate value of π, we may be compelled to involve another approximation in deciding the value of R from (3) in a nice form, for example, in whole number of minutes, seconds, and etc. depending on the accuracy desired.

Even with all these choices when we determine and accept a particular value of R specified in the angular units, we cannot still draw the circle of a definite size because the value of R is not in the absolute units of length (What is the size of one degree or one minute length ?).

Of course, this difficulty of drawing the circle will come even if R is specified directly in angular units (say minutes) or as an absolute number instead of determining it from (3). To overcome this theoretical difficulty, an angular unit was taken to be equivalent to some known unit of length (This practice is similar to what we do when we have to draw, for example, a force diagram where we decide that so many units of force are represented by so many units of length).

Thus in connection with the description of a 'shadow-instrument' (*chāyā-yantra*), the lost work of Āryabhaṭa I asks us to draw a circle of radius 57 *aṅgulas* (finger-breadth) to represent the 57 degrees of the *Sinus totus* (when the circumference is taken to be equal to 360 degrees)[3].

Again, while giving the details of the graphical method of finding the Sines as described in the *Brāhma-sphuṭa-siddhānta* (=*BSS*) of Brahmagupta (628 A.D.), the commentator Pṛthūdaka (*c.* 860) asks us to draw, by a pair of of compass (*karkata*), a circle of radius 3270 *aṅgulas* on a level ground, the number being the *Sinus totus* used in the above work[4]. If the *aṅgula* mentioned be taken to be equivalent to about three-fourths of an inch, then the radius will measure more than 200 feet. We may have a level ground to draw the said circle but where from such a big compass (of more than 100 feet arm) is to be obtained. Or we have to give some different explanations. Bhāskara II (1150) also, in a similar context, talks of drawing a circle with desired radius in *aṅgulas*[5].

Anyway, whatever be the practical difficulties or conventions in drawing the cir[illegible] reference, the values of the *Sinus totus* (and so of all other Sines) were not represented in absolute units of length. The same may be said of Ptolemy who took a diameter of 120 parts for his table of chords. All this shows that the ancient Sines were defined as lengths but not as absolute lengths.

In spite of all these defects, the Indians have been praised for their practice of taking the circumference and radius in the same angular units. Thus Otto Neugebauer remarked[6]

"......The Hindus took the reasonable attitude that the radial distances should be measured in the same units in which the length of the circumference is measured, an approach which would have led to the modern concept of radians. had they not retained the Babylonian sexagesimal division of a circle into 360 parts".

Now according to *AB*, II, 10 the circle of diameter 20000 is nearly equal to 62832 units[7]. This implies the approximation

$$\pi = 62832/20000 = 3.1416 \tag{4}$$

Using this and taking 360 degrees (or 21600 minutes) as the measure of *C*, the relation (3) gives

$$R = 75000/1309 = 57+387/1309 \text{ degrees} \tag{5}$$

$$= 4500{,}000/1309 = 3437+967/1309 \text{ minutes} \tag{6}$$

$$= 206264+424/1309 \text{ seconds} \tag{7}$$

$$= 1237{,}5859+569/1309 \text{ thirds} \tag{8}$$

$$= 57°17'44''19'''+569/1309 \tag{9}$$

$$= 57.2956{,}4553 \text{ nearly} \tag{10}$$

The value (6), inclusive of the fractional part as such. is mentioned or quoted in the Utpala's commentary (tenth century) on the *Bṛhat-Saṃhitā*[8]. Also we see that. depending on the degree of accuracy desired, the value of the *Sinus totus* can be taken to the nearest degree, minute, second, third, and etc. We discuss these individually.

(I) From (5) we get (to the nearest degree)

$$R = 57 \text{ degrees} \tag{11}$$

We have already pointed out that the lost work, called *Āryabhaṭa-Siddhānta* (= *AS*), of Āryabhaṭa I talks of drawing a circle with *trijyā* (*Sinus totus*) or radius 57 units which represent the 57 degrees (in a radian). However, we have not come across any evidence to show that this lost *AS* had used a Sine table with *R* equal to 57 degrees.

According to the *Hayata* (1764), a very late Sanskrit work on Arabic astronomy, some *Sūrya-siddhānta* (= *SS*) also used a radius of 57 units[9]. Which *SS* is referred to here is not clear, but it may be pointed out that the lost *AS* (which used the same radius) was based on the old *SS*[10].

Moreovers it may also be pointed out that the *Hayata* (p.16) roughly derives the radius 57 by using the approximation

$$\pi = 22/7 \tag{12}$$

instead of (4). Although this does not matter much, it should be noted that the value (12) is the first fractional approximation of the value (4) when the latter is expressed in continued fraction[11].

The *PS*, IV, 1 (see Section 3 below) gives the rule

$$\text{diameter} = \sqrt{(360)^2/10}$$

which implies the radius (11) to the nearest degree. cf. (22) below.

(II) The relation (6) shows that we shall have (to the nearest minute)

$$R = 3438 \text{ minutes} \tag{13}$$

By far this is the most commonly used value of the *Sinus totus* in Indian trigonometry. Sine-differences[12] stated in *AB*, I, 10 (pp. 16-17) imply it. These tabular differences have been referred[13] and used by Bhāskara I in his works (early seventh century). The resulting 24 tabular Sines are given[14] by Lalla (*c*. 748) who rightly calls them *Bhaṭoditā* (i. e. as computed by Āryabhaṭa), the last Sine being equal to the *Sinus totus* given by (13). Moreover, Lalla has also given the value[15]

$$R^2 = 1181,9844 \text{ (square minutes)} \tag{14}$$

which is obviously derived from (13).

The tabular Sines as given[16] in the extant *SS*, II, 17-22, also imply the value (13). The same is the case with the *Soma-siddhānta*, II, 4-8 (p. 7)[17a]. Sumati of Nepal (before 950 A. D.) also used the value (13) in his *Sumati Mahātantra* as well as in his *Sumati-Karaṇa*[17b].

Āryabhaṭa II (950) has employed the value (13) in his *Mahāsiddhānta* (= *MS*[18] but one of his tabular Sine is different from the corresponding value found in Lalla or *SS*. Bhāskara II has followed Āryabhaṭa II for his standard table of the Great Sines[19] although author's own accompanying commentary states that the same values are given in the *SS* (?) and the *Āryabhaṭa Tantra* (= *AB* ?).

The use of the value (13) for a radius by Pulisa is also attested[20].

(III) The relation (7) shows that, to the nearest second of the angular arc, we have

$$R = 206264 \text{ seconds}$$
$$= 3437' \, 44'' \qquad (15)$$

The value (15) for the *Sinus totus* (to the rearest second) is given by Vaṭeśvara (early tenth century) in his *Vaṭeśvara-siddhānta* (= *VS*)[21] just after stating the minutes and seconds of his tabular Sines and Versed Sines. Immediately after stating the value (15), the *VS* gives

$$R^2 = 1181,\ 8047'35'' \qquad (16)$$

Now (15) will give

$$R^2 = (3437+44/60)^2$$
$$= 1181,\ 8010+28/60+16/60^2 \qquad (17)$$

which does not agree with (16) to the nearest sixtieth part. However, the original relation (6) gives

$$R^2 = (4500{,}000)^2/(1309)^2$$
$$= 2025\times10^{10}/1713481$$
$$= 1181{,}8047+(35.3)/60 \text{ nearly} \qquad (18)$$

which agrees with the *VS* value (16) to the nearest second. Thus Rai[22] need not remark that there is an error in the *VS* value of the square of the radius nor his suggested emendation of the printed reading *jala* (= 4) to *jalada* (= 0), thereby getting

$$R^2 = 1181,\ 8007'\,35''$$

in place of the correct value (16), is necessary.

Parameśvara[23] in his commentary (*c*. 1408) on the *Laghu-bhāskarīya* (= *LB*) gives a table of Sines in which the last value (representing the Sine of 90 degrees) is same as (15).

(IV) From the relation (8) or (9) we have (to the nearest third)

$$R = 3437'\,44''\,19''' \qquad (19)$$
$$= 1237{,}5859 \text{ thirds} \qquad (20)$$

Govindasvāmin (*c*. 800-850) in his commentary on the *MB*, IV, 22 (pp. 199-201) has given fractional parts meant for improving the *AB* Sine-differences and the reasulting Sine-table[24] imply the radius (19).

For the value of the *Sinus totus* in the form (20), a set of tabular Sines and their differences is found in the *Sundarī* commentary[25] by Udaya Divākara (*c*. 1073) on the *LB*.

It is well-known that, for finding the circumference of a circle, the Indians often used the rough formula

$$C = \sqrt{10\, D^2} \tag{21}$$

which gives

$$D = \sqrt{C^2/10} \tag{22}$$

By comparing this with (3), we may say that (22) implies

$$\pi = \sqrt{10} \tag{23}$$

instead of (4) with C equal to 21600 minutes the relation (22) gives

$$D = \sqrt{4665,6000} = 6830.52 \text{ nearly.} \tag{24}$$

Thus (to the nearest minute)

$$R = 3415. \tag{25}$$

The value (25) has been worked out by Pṛthūdaka in his commentary on the *BSS*, XXI, 15 (Vol. IV, p. 1626) which contains the rule (22). For the square-root in (24), he gives 6830 which is correct if we disregard the fractional part though greater than half.

The Sine table given in the *Laghu Vaśiṣṭha Siddhānta*[26] implies the radius (25). Same is the case with the *Siddhānta-Śekhara* of *Śrīpati* (*c*. 1040) who adds that his *Sinus totus* is the radius of the circle whose circumference is equal to minutes in one revolution and also gives[27]

$$R^2 = 1166,2225 \tag{26}$$

which is exactly the square of the value (25).

The famous Mādhava's (*c*. 1340-1425) Sine table which is quoted by Nīlakaṇṭha Somasutvan (*c*. 1500) in his commentary (*NAB*) on the *AB* imply the radius[28].

$$R = 3437'\ 44''\ 48''' \tag{27}$$

The *NAB* (Part I, p. 55) states that the value (27) was obtained by Mādhava by using a (very accurate) relation, between the circumference and diameter of a circle, which is expressed by the rule *bibudhanetra* etc. The full Sanskrit stanza[29] giving this rule of Mādhava is quoted by *NAB* at another place (Part I, p. 42) and imply a value of π correct to 11 decimals. Thus we get (27) by using (3) with C equal to 21600 minutes and a more accurate value of π than (4).

The value (27) when rounded off to the nearest second, becomes

$$\begin{aligned} R &= 3437'\ 45'' = 3437.75 \\ &= 13751/4 \end{aligned} \tag{28}$$

which is implied in another rule (*NAB*, Part I, pp. 54-55), given by Mādhava, that contains the Taylor series expansions of the Sine and the Cosine upto the second order[30].

We note that, even with adoption of submultiple units of seconds and thirds, the various values of the *Sinus totus* considered above were obtained by neglecting smaller fractional parts or by rounding off. That is why Brahmagupta in his *BSS*, XXI, 16 (Vol. IV, p. 1626) states that by taking the circumference equal to 21600 minutes, we do not get the radius fully represented in terms of minutes or its (sexagesimal) parts ; hence the corresponding computed tabular Sines will not be accurate (or exact) and consequently he took a different radius. For his standard table of Sines. he took, *BSS*, II, 9 (Vol. II, p. 141)

$$R = 3270 \tag{29}$$

However, Brahmagupta has not explained as to how he selected the number 3270 for the radius. Whether this number was picked up at random or whether there was some basis for the choice, seems to be a difficult problem. Anyway, I may submit the following facts in this connection :

Brahmagupta's *BSS*, XII, 40 (Vol. III, p. 857) gives the rule (2) and *BSS*, XXII 15 (Vol. IV, p. 1626) gives the rule (22). If we use the very crude approximation

$$\sqrt{10} = 3.3 \tag{30}$$

for this implied value of π, then we get from (3),

$$R = 21600/6.6 = 36000/11$$
$$= 3273 \text{ nearly} \tag{31}$$

which is conveniently near (22). The crude value (30) can be obtained, roughly, by using the approximation

$$\sqrt{N} = \sqrt{a^2 + x} = a + (x/a) \tag{32}$$

so that

$$\sqrt{10} \quad \sqrt{3^2+1} = 3 + (1/3)$$
$$= 3.3 \text{ roughly}$$

The unusual gross rule (32) occurs as an intermediate step in the Babylonian-Heronian algorith[31] according to which, if a_1, equal to a, be the first approximation (in defect) to the square-root of N then

$$b_1 = N/a_1 = a + (x/a)$$

will be the second approximation (in excess) and etc.

The gross approximation (32) seems also to be implied in a verbal rule found stated in the *Arithmetic in Nine Sections*, an ancient Chinese work, for finding the fractional part (x/a) of the root[32].

However, these facts and our derivation of the value (31) are not meant here as any possible explanation for the true reason, if any, for Brahmagupta's supposition or imagination (***kalpanā***, according to Pṛthūdaka, his commentator, *BSS*, Vol. IV, p. 1626) of the value (29).

Mahendra Sūri (*c.* 1370), a court *Paṇḍita* of Firoz Shah Tugalaq) wrote the *Yantra-rāja* (= *YR*) which is based on foreign material and is commented upon by the author's own pupil. This work contains a Sine table based on the radius[33]

$$R = 3600 = 60^2 \tag{33}$$

Although this value can be easily derived from the relation (3) by using the simplest approximation

$$\pi = 3 \tag{34}$$

it is better to consider the choice of (33) as based on the convenience it provides for calculations with sexagesimal fractions which have been in common use throughout.

A short Sine table for the radius

$$R = 1000 \tag{35}$$

which is more suitable for the decimal rather than the sexagesimal system, is found [34] in the *Vṛddha Vaśiṣṭha Siddhānta* (= *VVS*), III. 9-10, whose date cannot be given with certainty.

Another conveniently or arbitrarily chosen value, namely

$$R = 500 \tag{36}$$

is implied in the tabular Sines and their differences which are found in the *Karaṇa Kaustubha* composed about 1650 by Kṛṣṇa Daivajña[35].

Still another such value, namely

$$R = 700 \tag{37}$$

was used in *Karaṇa Vaiṣṇava* of *Śaṅkara* (eighteenth century)[36].

3, SMALLER AND MISCELLANEOUS VALUES

For convenience, the Indians also usued some smaller values of Sinus totus which we take up now.

(I) The following radius was used as early as the sixth century A. D.

$$R = 120 \tag{38}$$

For this purpose we quote the *PS*, IV-1 which states[37]

पष्टिशतत्रयपरिधेर्वर्गदशांशात्पदं स विष्कम्भः ।
तदिहांशचतुष्कं सम्प्रकल्प्य राश्यष्टभागज्या ॥ १ ॥

Ṣaṣṭiśatatraya-paridhervarga-daśāṃśāi-padaṃ sa viṣkambhaḥ /
Tadihāṃśa-catuṣkaṃ samprakalpya rāśyaṣṭabhāgajyā //

'Take the square-root of the tenth part of the square of the circumference (which is) three hundred and sixty (degrees); it is the diameter. Here (in this work), assuming that (that is, the diameter) to be four degrees, the Sines are given at the (interval of) eighth part of a sign'.

The first part of the rule gives the relation (22) with C equal to 360 degrees and the second part implies a radius of 2 degrees or the value (38) in minutes.

The referred tabular Sines are given *PS*, IV, 6-15. But due to a wrong amendation by Thibaut and Dvivedi of the otherwise correet original Sanskrit reading, these two scholars were led to the erroneous value[38]

$$R \sin 90° = 120' \, 1'' \tag{39}$$

instead of (38).

The Sine differences given by Brahmadeva (1092) in his *Karaṇa-prakāśa*[39] imply the radius (38).

The same radius is used by Bhāskara II for his shorter table of Sines whose differences he has given in his *SSGG*, II, 13 (p. 41) and also in his *Karaṇa Kutūhala*[40] which was composed about 1183 A. D. The printed *Marīci* commentary (1638) on the *SSGG* gives a better Sine table for the same radius[41].

The *Yantra Śiromaṇi*[42] of Viśrāma (1615) contains a set of Sines, directly in a tabular form, for the radius (38). A similar table also appears in the *Marīci* commentary (p. 140) on the *SSGG*.

(II) Instead of (38), Brahmagupta took

$$R = 150 \tag{40}$$

for his short Sine table. The Sanskrit stanza containing the related tabular differences is given by him in his *Dhyānagrahopadeśa* (verse 16)[43] as well as in his *Khaṇḍa-Khādyaka* (=*KK*) III, 6 which[44] was composed about 37 years after his *BSS* (wherein is quoted the first of the above two works).[45]

Another short Sine table for the radius (40) is given by Lalla in his *SVGG*, XIII, 2-3, (p. 48).

(III) The use of the following two values for the *Sinus totus*, has come to light now.

$$R = 200 \tag{41}$$

$$R = 300 \tag{42}$$

According to Al-Bīrūnī (died 1048,[46] the first value was used by Vijayanandin in his *Karaṇatilaka* and the second by Vitteaśvara (= Viṭṭeśvara ?) in his *Karaṇasāra*.

(IV) It is surprising that the use of the simplest sexagesimal value

$$R=60 \tag{43}$$

which is used by the Greek astronomer Ptolemy (second century A. D.) for his table of chords and frequently by the medieval Arab authors, is found quite late in India. It is used by Kamalākara in his *Siddhānta-tattva-viveka* (=*STV*)[47] which contains sexagesimally five-figured Sine table (p.168).

The work *Samrāṭ-siddhānta* (1st half of the eighteenth century)[48] is a Sanskrit translation of Ptolemy's *Almagest* made by Jagannātha from an Arabic version. In addition to Ptolemy's table of chords (vol. I, pp. 30-40), the work contains a Sine table (pp. 55-57) for the same radius (43). The printed edition of the work contains some additional material on trigonometry (apparently by Jagannatha)which is also based on the the same value of rhe radius.

(V) Of the various miscellaneous values, we first take

$$R=8^\circ\ 8'=8+(2/15)=488'. \tag{44}$$

This *Sinus totus* is arrived at by interpreting a rule[49] given by Muñjāla (*c.* 932) in his *Laghu-mānasa*, II, 12. However, according to another interpretation found[50] stated by Mukhopadhyayya, we have

$$R=8^\circ\ 11''=491' \tag{45}$$

instead of (44).

The set of six tabular Sine differences found in the *Vākya-karaṇa* (c. 1300)[51], III, 2-3, implies the value

$$R=43 \text{ parts} \tag{46}$$

which, according to the editors of the work (p. xxi), is obtained from (13) by dividing it by 80 for convenience.

Another peculiar Indian value of the *Sinus totus* is

$$R=191 \tag{47}$$

This was used by Gaṅgādhara (*c.* 1434) in his *Candramāna*[52]. The two sets of tabular Sines given by Munīśvara in his *Siddhānta Sārva-bhauma* (1946) are also based on the same radius[53]. Although awkward, the value (47) might have been obtained from (13) by removing the simple factors or divisisors 2, 3 and 3

Lastly, we mention that the value

$$R=24 \tag{48}$$

is stated to be used in the *Karaṇa-Vaiṣṇava* of Saṅkara (eighteenth century).[54]

We present the various Indian values of *Sinus totus* in a consolidated form in the accompanying table.

TABLE OF INDIAN SINUS TOTI

Sl. No. :	Sinus Totus :	Reference :
1.	2 parts or degrees	*Pañca-siddhāntikā* (*c.* 550), IV, 1 ; cf. No. 7 below.
2.	8°8′ (= 488′)	*Laghu-mānasa* of Muñjāla (932).
3.	24	*Karaṇa-vaiṣṇava* of Śaṅkara (1766).
4.	43.	*Vākya-karaṇa* (*c.* 1300).
5.	57 degrees	Old *Sūrya-siddhānta* (?) ; Āryabhaṭa I's lost work (*c.* 500) ; Some *Sūrya-siddhānta* according to *Hayata* (1764).
6.	60	*Siddhānta-tattva-viveka* of Kamalākara (1658) ; *Samrāṭ-siddhānta* of Jagannātha (*c.* 1730).
7	120 (Minutes)	*Pañca-siddhāntikā* of Varāhamihira (*c.* 550) ; *Karaṇa-prakāśa* of Brahmadeva (1092) ; *Siddhānta-śiromaṇi* (1150) and *Karaṇakutūhala* (1183) of Bhāskara II ; *Yantraśiromaṇi* of Viśrama (1615) ; *Karaṇendu-śekhara*.
8.	150	*Dhyāna-grahopadeśa* (*c.* 625) and *Khaṇḍa-khādyaka* (665) of Brahmagupta ; *Śiṣya-dhīvṛddhida* of Lalla (*c.* 748).
9.	191	*Cāndramāna* of Gaṅgādhara (1434) ; *Siddhānta-sārva-bhauma* of Munīśvara (1646).
10.	200	*Karaṇa-tilaka* of Vijayanandi (before *c.* 1000).
11.	300	*Karaṇa-sāra* of Vittcśvara (*c.* 900).
12.	491′	*Laghu-mānasa* (cf. No. 2 above) (according to D. N. Mukhopadhyaya).
13.	500	*Karaṇa-koustubha* of Kṛṣṇa-daivajña (*c.* 1650).
14.	700	*Karaṇa-vaiṣṇava* of Śankara (1766).
15.	1000	*Vṛddha-vaśiṣṭha-siddhānta* (undated ?).
16.	3270	*Brāhma-sphuṭa-siddhānta* of Brahmagupta (628).
17.	3415	*Laghu-vaśiṣṭha-siddhānta* (undated ?) ; *Siddhānta-śekhara* of Śrīpati (*c.* 1039).
18.	3437′ 44″	*Vaṭeśvara-siddhānta* of Vaṭeśvara (904) ; Parameśvara's commentary (1408) on the *Laghubhāskarīya*.
19.	3437′ 44″ 19‴	Govindasvāmin's commentary (*c.* 800-850) on the *Mahā-bhāskarīya* ; cf. No. 25 below.
20.	3437+967/1309	Utpala's commentary (*c.* 966) on the *Bṛhat-saṃhitā*.
21.	3437′44″ 48‴	Sine table Mādhava (*c.* 1400) quoted by Nīlakaṇṭha (*c.* 1500) and Śaṅkara Vāriar (1556).
22.	3437′ 45″	Implied in a rule of Mādhava which is quoted by Nīlakaṇṭha in his commentary on the *Āryabhaṭīya* (II, 12) and also in his *Tantra-saṃgraha* (II, 10-13).
23.	3438	*Āryabhaṭīya* of Āryabhaṭa I (born 476) ; Extant *Sūrya-siddhānta* ; *Mahābhaskarīya* of Bhāskara I (*c.* 625) ; works of Sumati (before 950) ; *Mahā-siddhānta* of Āryabhaṭa II (950) ; *Śiṣyadhīvṛddhida* of Lalla (*c.* 748) ; *Siddhānta-śiromaṇi* of Bhāskara II (1150) ; Puliśa or Pauliśa (?).
24.	3600	*Yantra-rāja* of Mahendra-sūri (*c.* 1370)
25.	12375859‴	Udaya-divākara's commentary (1073) on the *Laghu-bhāskarīya* (cf. No. 19 above).

4. Terminology

From definition it is clear that the Sine of three signs or of 90 degrees arc will be equal to the radius of the circle of reference. Therefore this value of the radius is commonly called *trijyā* which is a short form of the terms like *tri-rāśi-jyā* meaning 'Sine of three signs' literally. Most of the sanskrit terms are based on this interpretation. Few terms which literally mean '*Sinus totus*' (total or complete Sine) and 'greatest Sine' are also used for obvious reasons. All these terms are listed along with at least one reference of their use in Appendix 2.

Beside the listed one, many terms which mean radius or semi-diameter geometrically have been used synonymous to *trijyā*. Conversely *trijyā* has been frequently used for radius of any circle without confining its use in the sense of 'Sine of three signs'. However, expressions like

त्रिज्याव्यासार्धेन वृत्तं कृत्वा

' *Trijyā-vyāsārdhena vṛttaṃ kṛtvā* '

(Poona edition of *Marīci on Jyotpatti*, part I, p. 154)

clearly bring out the distinction between *trijyā* ('Sine of three signs') and *vyāsārdha* ('semi-diameter' or radius).

The use of such a large number of synonymous terms was partly necessited by the fact that mathematical rules were to be given in verses which involved definite number of syllables. Of course, the richness of the Sanskrit language easily provided them.

5. Transmisson of the Indian Values of the Sinus totus

Below we give a few cases of the use of some Indian values of Sinus totus in the works of foreign writers.

For example Yaqūb Ibn Tariq (2nd half of the 8th century) gives the following rules[55].

Radius of the Diurnal Circle ($R \cos \delta$)

$$=3438-\text{Vers}\ \delta$$

and

Sine of Ascensional Difference $=3438\ e(\sin \delta)/g(\cos \delta)$ where e is the equinoctial noon-day shadow and g is the length of the gnomon. These clearly imply a Sinus totus which was used in India since, at least, about A. D. 500. Indian table of 24 Sines for $R = 3438$ minutes was reproduced in the Chinese *Chiu Chih li* calender (A. D. 718)[56].

For the smaller *Sinus totus* 150, which was used in India by Brahmagupta (seventh century) and Lalla, (eighth century), the following instances may be noted :

(i) A radius of 150 minutes is associated with Al-Fazārī (*c.* 750) by Bīrūnī,[57a] and also with Yaqūb ibn Tāriq (eighth century.) and Abū Ma'shar' (*c.* 850)[57b].

(ii) The same radius is stated to be associated wirh the Shāh-Zīj (*c.* 790) according to a passage given by Bīrūnī[58].

(iii) It is stated[59] that the original *Zij* of Al-Khwārizmī (*c.* 840) had a Sine table for *R* equal to 150.

(iv) The Arab Az-Zarqālī (Arzachel of the Latins), a celebrated astronomer of the eleventh century Spain, also took a radius equal to 150 minutes[60a].

(v) R=150 is also used in an anonymous Byzantine treatise of (eleventh century)[60b].

(vi) An anonymous thirteenth century Latin manuscript also assumes the radius of 150 minutes for Sines[61].

(vii) The same radius also appears in a fifteenth century Newminister (England) manuscript[62].

On the otherhand, it may be pointed out that the Greek value 60 for the radius, which was used by Potelmy (150 A. D.) is found in India in the *STV* (1658) whose author was familiar with foreign material.

Similarly, the radius 3600, used in the *YR* (*c.* 1370), is obviously due to Islamic influence.

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17b See *AB* edited by K. S. Shukla and K. V. Śarma, INSA (New Delhi) 1976, p. 30

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60b Viator, Vol 7. (1970), p, 160. (Univ. of California).

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Appendix 1.

The following abbreviations are used in the paper for some works.

AB	:	*Āryabhaṭīya*, see reference 7.
AS	:	*Āryabhaṭa Siddnānta* (lost). see reference 3.
BSS	:	*Brāhma-sphuṭa-siddhānta*, see ref. 4.
KK	:	*Khaṇḍa-khādyaka*, see ref. 44.
LB	:	*Laghu-bhāskarīya*, see ref. 23.
MB	:	*Mahā-bhāskarīya*, see ref. 13.
MS	:	*Mahā-siddhānta*, see ref. 18.
NAB	:	*Nīlakaṇṭha's commentary on the AB*, see ref. 28.
PS	:	*Pañca-siddhāntikā*, see ref. 37.
SS	:	*Sūrya-siddhānta*, see ref. 16.
SSGG	:	*Siddhānta-śiromaṇi Graha-gaṇita*, see ref. 5 and 19.
SSK	:	*Siddhānta-śekhara*, see ref. 27.
STV	:	*Siddhānta-tattva-viveka*, see ref. 47.
SVGG	:	*Siṣyadhīvṛddhida* (*tantra*) *Graha-gaṇita*, see ref. 14,
TS	:	*Tantra-saṃgraha*, see ref. 28.
VS	:	*Vaṭeśvara Siddhānta*, see ref. 21.
VVS	:	*Vṛddha-vasiṣṭha-siddhānta*, see ref. 17 and 34.
YR	:	*Yantra-rāja*, see ref. 33.

Appendix 2.

List of Sanskrit Terms for Sinus totus

1. *Antyā* or *antya-jyā* (last Sine) : *Soma-siddhānta*, II, 3 ; *YR*, I, 42 and its commentary (p. 30).
2. *gagṛha-maurvī* : *MS*, III 36. (Note *ga*=3),
3. *gajyā* : *MS*, III, 1 etc.
4. *gabha-maurvī* : *MS*, IV, 21.
5. *triguṇa* :*VVS*, II, 16.
6. *trigṛha-guṇa* : *MB*, III, 27.
7. *trigṛha-jyā* : *BBS*, XVI, 11, *KK*, III, 9.
8. *trigṛha-maurvī* : *Siṣyadhīvṛddhida*, *Golā*, *IX*, 37.
9. *trigṛha-śiñjinī* *SSK*, III, 50.

10. *trijaka* : *Siddhānta-sārva-bhauma*. II, 57.
11. *trijīvā* : *SS*, II, 28.
12. *trijyā* : the most common and popular term.
13. *tribha-guṇa* or *bhatraya-guṇa* : *SVGG*, IV, 5 and II, 37.
14. *tribha-jīvā* : *VS*, II, iii, 2.
15. *tribha-jyā* : *PS*, *IV*, 5.
16. *tribha-maurvī* : *VS*, II, i, 69.
17. *tribhavana-guṇa* : *VS*, II, iv, 4.
18. *tribhavana-jyā* or *bhavanatraya-jyā* ; *SVGG*, II, 18 and 20 ; *Karaṇa-prakāśa*, III, 9 (p. 30) uses *bhavana-tritayottha-jīvā*.
19. *tribhavanasya-guṇapratānam* : *MB*, III, 5.
20. *tribhavanasya-jīvā* : *MB*, III, 19.
21. *tribha-śiñjinī* : *SSK*, III, 50.
22. *tri-maurvī* : *MB*, III, 39 etc., *LB*, III. 12 etc.
23. *trirāśi guṇa* : *SSK*, IV, 119.
24. *trirāśi-jīvā* : *LB*, III, 29.
25. *trirāśi-jyā* : *MB*, III 16.
26. *tri-śiñjinī* : *SSK*, III, 14.
27. *padasamuttha-jīvā* (Sine for one quadrant) : *SSK*, III, 63.
28. *parama-jyā* (greatest Sine) : *VVS*, II, 8 and II, 41, etc.
29. *parama-śiñjinī* (greatest Sine) : *MS*, III, 2.
30. *bhatraya-guṇa* : *see* serial No. 13. above.
31. *bhavana-traya-jyā* etc. : *see* serial No. 18 above.
32. *vyāsa* (=*trijyā*) : *MB*, III 20 and 38.
33. *vyāsa-khaṇḍa* : *MB*, III, 7.
34. *vyāsa-khaṇḍa-nicaya* : *MB*, III, 20.
35. *sakala-guṇa* (total Sine or Sinus totus) : *MB*, II, 10 and III, 27.