

HIPPARCHUS'S 3600'-BASED CHORD TABLE AND ITS PLACE IN THE HISTORY OF ANCIENT GREEK AND INDIAN TRIGONOMETRY

BO KLINTBERG*

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With mathematical reconstructions and philosophical arguments it has been shown that Toomer's 1973 paper never contained any conclusive evidence for his claims that Hipparchus had a 3438'-based chord table, and that the Indians used that table to compute their sine tables. Recalculating Toomer's reconstructions with a 3600' radius -- i.e. the radius of the chord table in Ptolemy's *Almagest*, expressed in "minutes" instead of "degrees" -- generates Hipparchan-like ratios similar to those produced by a 3438' radius. It is therefore possible that the radius of Hipparchus's chord table was 3600', and that the Indians independently constructed their 3438'-based sine table (in English).

Je montre, avec des reconstructions mathématiques et des arguments philosophiques, que le papier de Toomers (1973) n'a jamais contenu aucune évidence concluante pour sa réclamation qu'il a prouvé que Hipparque avait une table de corde de base 3438', et que les Indiens utilisaient cette table pour calculer leur tables de sinus. En recalculent les reconstructions de Toomer avec un rayon de 3600' - c'est-à-dire le rayon de la table de corde dans l'*Almagest* de Ptolémée, exprimé en « minutes » au lieu d'en « degrés » -- génèrent des ratios genre « Hipparque » similaire à ceux produisent par un rayon de 3438' Il est donc possible que le rayon de la table de corde d'Hipparque était de 3600' et que les Indiens ont construit indépendamment leur tables de sinus de 3438' (in French).

Key Words: *Almagest*, *Āryabhaṭīya*, Chord tables, Hipparchus, Ptolemy, *Sūrya-siddhānta*.

Many famous western historians of astronomy and mathematics have actively been promoting the idea that the 3438'-based Indian sine tables, such as the ones found in the *Sūrya-siddhānta*,¹ the *Āryabhaṭīya*,² were derived

* Department of Philosophy, Emory University, 561 S., Kilgo Circle, Atlanta, Georgia 30322, USA;

E-mail: bklintb@emory.edu

from ancient Greek chord tables. Most of these historians, however, have not supported their claims with any real evidence. For example, in the 1890s P. Tannery wrote that the sine table in the *Sūrya-siddhānta* may have been derived from a pre-Hipparchan chord table³; in 1920s G.R. Kaye wrote that “The *Paulīśa Siddhānta* gives a table of twenty-four sines which appear to have been obtained from Ptolemy’s table of chords”⁴; in 1950s B.L. van der Waerden claimed that “As early as the fifth century AD, the Indian astronomers changed from the chords to the sines”⁵; in early 1960s A. Aaboe wrote that “to tabulate a new function, the half-chord of twice the arc...was, as a matter of fact, done by the Hindu astronomers (who were influenced both by Babylonian and early Greek theories)”⁶; and finally, in early 1970s, O. Neugebauer proclaimed that “In India the great step was made of changing the Greek ‘chords’ to ‘half-chords’.”^{7,8}

The situation changed somewhat in 1973 when G.R. Toomer published his paper “The Chord Table of Hipparchus and the Early History of Greek Trigonometry.” Unlike his forerunners, Toomer here claims that he has *proven* that Hipparchus had a chord table with base radius 3438', and that the Indians later copied it. Toomer’s line of reasoning goes like this. First, Toomer says that Hipparchus must have used trigonometry to solve certain problems in solar and lunar theory, and that he consequently also must have had a table of chords.⁹ But since Hipparchus’s table has not been found anywhere,¹⁰ Toomer starts with the assumption that Hipparchus arrived at the 3438'-base radius by “dividing 60. 360°, or 21600', by 2π ... and rounding the result to the nearest minute.”¹¹ Then, using the premise that some Greek mathematicians of the past allegedly have used a 48-division of the circle,¹² Toomer recreates an Hipparchan chord table by using the formula $\text{Crd } \alpha = (6,0,0/2\pi) \cdot 2 \sin (\alpha/2)$, “where α is tabulated at intervals of $7\frac{1}{2}^\circ$, and the results rounded to the nearest ‘minute’.”¹³ With this reconstruction of an Hipparchan chord table at hand, Toomer then sets out to confirm its correctness by trying to recreate the two “strange numbers”¹⁴ that Ptolemy attributes to Hipparchus in *Almagest* IV 11. After these two reconstructions (one for the epicycle model and another one for the eccentric model), which both generate values that deviate less than five percent from Hipparchus’s values, Toomer claims that he has “*proven* that Hipparchus used a chord table with base $R' = 3438'$ ”¹⁶. He also says that this is not the *only* 3438'-based chord table in the Greek world: “Besides *Almagest* IV 11, I know

of one other trace in surviving Greek literature of the use of a chord table with base $R' = 3438$. This is Ptolemy, *Geography* I 20.¹⁷ Toomer then goes on to say that the Indian sine tables (some of which were tabulated at intervals of 3.75 degrees) were derived simply by taking the values in Hipparchus's chord table (which allegedly tabulated its values at intervals of 7.5 degrees) and putting them to work in the equation $\frac{1}{2} \text{Chord } \alpha = \text{Sin } (\alpha/2)$.

Toomer's article led many historians of mathematics and astronomy to believe that the issue now, once and for all, was settled.¹⁸ In 1975 O. Neugebauer wrote: "The decisive step in *proving* that the Indian table of sines was derived from the Hipparchan table of chords was made by G.J. Toomer who showed that the two tables not only agree in the steps of the argument but also in basic radius of 3438 'minutes'"¹⁹; in 1994 H. Thurston wrote: "Toomer's *evidence* comes from early Indian tables, which were evidently based on early Greek tables"²⁰; in 1996 D. Pingree wrote: "Included with the Greek adaptations of Babylonian astronomy of the Seleucid period [that the Indians made] are many elements of Hellenistic astronomy, especially some derived from Hipparchus, including the Indian transformation of his table of chords into a table of sines";²¹ and in 1998 V.J. Katz wrote: "Because this latter [Hipparchan] radius [of 3438'] is used as the basis of the table in the *Paitāmaha-siddhānta*, we surmise that it was Hipparchus' trigonometry rather than Ptolemy's that first reached India."^{22,23}

The fact that Toomer's conclusions now are to be found in these scholarly texts is, however, no guarantee for that they are correct. As I will show in this essay, Toomer has failed to conclusively *prove* that Hipparchus's chord table was based on a radius of 3438'-a discovery that has several immediate and important consequences. If Hipparchus's chord table did not necessarily have a base radius of 3438', it not only follows that Hipparchus may have used another base radius (such as, for example, 3600'), but also that the 3438'-based Indian sine tables may not have been derived from Hipparchus's chord table.

Some readers, however, may question why I have bothered writing this paper. After all, in a footnote in his *Ptolemy's Almagest*,²⁴ Toomer has already presented some doubts that Hipparchus's chord table had a 3438' base radius. Why, then, repeat the same thing again?²⁵

My answer to this objection is simply that I am not just repeating the same thing again; instead, I will highlight four problematic aspects of Toomer's

1984 footnote that I hope will interest quite a few readers. The first problematic aspect is that Toomer does not explicitly withdraw his statement from 1973 in which he said that he had *proven* that Hipparchus had a chord table with base radius 3438'; he just says that there is a *possibility* that some scholars may interpret the facts in such a way as to doubt that Hipparchus had a chord table with base $R = 3438'$:

These [new, unpublished] calculations not only vindicate Hipparchus' computational abilities, but cast doubt on my claim that he was operating with a chord table with base $R = 3438'$.²⁶

My essay, however, not only "casts doubt" on Toomer's claims that Hipparchus had a chord table with base radius 3438', but shows that Toomer's 1973 essay never really contained any solid evidence for that theory.²⁷

The second problematic aspect of Toomer's 1984 footnote is that he here does not discuss what the *consequences* would be if Hipparchus²⁸ did not have a chord table with a base radius of 3438'. In particular, he does not present any alternative values for the base radius of Hipparchus's chord table. Also, he does not discuss, and certainly not withdraw, any of his 1973 statements about the Indian sine tables and their alleged relationship with Hipparchus's chord table. This essay, however, deals with all of these points.

The third problematic aspect of Toomer's 1984 footnote is related to his statement that his new (but unpublished) computations with Hipparchus's new time interval "cast doubt on my claim that he was operating with a chord table with base $R = 3438'$ "²⁹. In other words, he seems to be saying that the new Hipparchan time interval is generating worse results for the 3438' epicycle case than the old interval in his 1973 essay did. But this is only true if we would accept Toomer's implausible thesis that Hipparchus made a computing error; otherwise, Hipparchus's new time interval, in fact, only improves the results of the 3438' epicycle simulation, compared to the one Toomer published in 1973. As I show in Section III of this paper, a recalculation with Hipparchus's new time interval not only generates slightly better average values for the 3438' epicycle ratio, but it also delivers a better evaluated ratio. Thus, assessed on its own, the new Hipparchan time interval does not necessarily cast doubt on the thesis that Hipparchus used a 3438'-based chord table.³⁰

The fourth problematic aspect of Toomer's 1984 footnote is that it has not had a very great impact on the community of historians of mathematics.

Many influential historians still are saying, more than a decade after its publication, that Hipparchus had a chord table with base radius 3438', and that the Indians copied it.³¹ One possible reason for why these scholars still are publishing Toomer's 1973 conclusions could, of course, be that they have missed reading his 1984 footnote. If this is the case, then the present paper may help solve that problem. Another possible reason for why these scholars still are publishing Toomer's 1973 conclusions could be that they actually did read his 1984 footnote, but that they, for some reason, still decided to go with Toomer's 1973 conclusions.^{32,33} If this is the case, then the present paper will also be of importance, since it-unlike Toomer's 1984 footnote, shows that the main conclusions in Toomer's 1973 paper are invalid, and that they therefore should be discarded.

I have divided this paper into four sections. In the first section it is argued that 3438' may not be the only base radius alternative for an Hipparchan chord table, and that it therefore is not conclusively established that the Indians used a 3438'-based Hipparchan chord table to produce their 3438'-based sine tables. In the second section it will be shown that a 3600'-based chord table – using Toomer's 1973 data, and including his incorrect Hipparchan time interval – generates almost as good Hipparchan-like ratios as a 3438'-based chord table does. In the third section, I will show that – when using Toomer's corrected Hipparchan time interval published in his 1984 footnote – a 3600' reconstruction, on average, comes closer to the two Hipparchan ratios than a 3438'-based reconstruction does. Finally, in the fourth section, I will sum up my arguments and propose new investigations.

I. AN ANALYSIS OF TOOMER'S MAIN ARGUMENTS

In this section I will discuss four of Toomer's most important arguments, and show that none of them contain conclusive evidence for the points that he is trying to prove: first, that Hipparchus must have constructed his alleged 3438'-based chord table in the exact same way that Toomer suggests; second, that Hipparchus must have used a 3438'-based chord table; third, that Ptolemy, too, must have used a 3438'-based chord table; and fourth, that the Indians must have copied the values of Hipparchus's 3438'-based chord table to create their own sine tables. I will in this section also present my alternative thesis that it is possible that 3438'-based chord tables never may have been in use in the ancient Greek world.

1.1 Did Hipparchus Necessarily Construct His Chord Table with Toomer's Method?

Toomer's first unsuccessful argument is built on the idea that Hipparchus arrived at a 3438' base radius by first dividing 60.360° by 2π , and then rounded the result to the nearest minute. There are two problems with his argument. The first problem surfaces when Toomer discusses the possibility that Hipparchus may have used Archimedes's approximation for π (which was $3\frac{1}{7}$) in this computation, but concludes that this cannot have been the fact since such a value "would not be exact enough to produce a 3438' radius"³⁴. Toomer's argument, however, is circular and invalid. This is because no one has conclusively proved that Hipparchus ever used a 3438'-based chord table. Therefore, there is no need to assume, as Toomer does, that Hipparchus's computation must have resulted in a radius of 3438'. Thus, it is quite possible that Hipparchus could have used an Archimedean approximation of π in Toomer's computation scenario.³⁵ And even if it were proven that Hipparchus used a 3438' -based chord table, Hipparchus still may not have used Toomer's scenario ("dividing 60.360° , or 21600', by 2π ...and rounding the result to the nearest minute") when constructing his chord table; Hipparchus could very well have used some other way of determining the radius of his chord table, perhaps using Archimedes's approximation of π .

The second problem with Toomer's argument is that even though it is mathematically possible – with an accurate approximation of π – to generate a 3438' base radius by his method, this circumstance, taken on its own, in no way constitutes any proof for the thesis that Hipparchus ever used such a method; Hipparchus may, for instance, not have had a sufficiently accurate approximation of π at hand to be able to generate such a base radius. And even if Hipparchus would have had a sufficiently good approximation of π available, and fragments would have been available to verify that he indeed had such a value in his possession, still this would not be conclusive proof for that Hipparchus must have used the particular computation method that Toomer suggests. Since Toomer does not provide us with references to any surviving manuscript that confirms that Hipparchus (or any other ancient Greek mathematician or astronomer) ever used such a method³⁶, Hipparchus may very well have used a completely different thought process than Toomer's to construct his chord table.

In fact, if Toomer's suggested way of constructing a chord table is so "natural" as he asserts³⁷, then why did not for example, Ptolemy³⁸ construct his chord table with a base radius of 3438'?

1.2 Did Hipparchus Necessarily Use a 3438'-Based Chord Table?

A second unsuccessful Toomerian argument, and one which is also more important to highlight, is his idea that he has *proven* that Hipparchus used a chord table with a base radius³⁹ of 3438'. Toomer's main reason for claiming that Hipparchus had used such a chord table is that his 1973 reconstructions of the epicycle and eccenter cases both generate numbers that come somewhat close to the two "strange numbers" that Ptolemy ascribes to Hipparchus.⁴⁰ However, there are some serious problems with this argument. One serious problem is that Toomer seems to be of the opinion that (some) mathematical reconstructions may be powerful enough to, on their own, serve as conclusive evidence for how ancient mathematicians practically performed their calculations.⁴¹ But mathematical reconstructions cannot, on their own, function as conclusive proofs for historical events; they can, at best, only indicate a possible historical scenario. This is simply because any given mathematical result can potentially be generated in an almost unlimited number of ways.⁴² So even if some mathematically inclined historian (or, perhaps, some historically inclined mathematician) could provide a reconstruction that generated the exact same "strange numbers" that Ptolemy ascribes to Hipparchus, that historian (or mathematician) would still not be philosophically entitled to conclude – without providing other valid and supporting evidence (such as, for example, hitherto unknown fragments, containing new information about how Hipparchus derived his "strange numbers") – that he had found the exact procedure by which Hipparchus performed his calculations; the strongest conclusion that would have been permissible for him to draw, without providing additional evidence, is that the reconstructed procedure possibly may have been used by Hipparchus. Now, since Toomer not only does not succeed in generating, in any of his reconstructions, the exact same "strange numbers" as those that Ptolemy reports that Hipparchus had⁴³ but also does not have any real evidence that Hipparchus ever even used a chord table,⁴⁴ it is rather safe to conclude that Toomer never was in a position to conclusively prove anything about the existence of an alleged Hipparchan chord table with base radius 3438'.

Another serious problem with Toomer's 3438' hypothesis is that his presentation is far from complete. In order for his 3438'-based reconstruction to have been regarded as "very plausible" or, at least, "somewhat plausible," it would have been necessary for him to also provide reconstructions for other possible base radii, to show that they all generate significantly worse results than a reconstruction that is using a base radius of 3438'. But this is not what Toomer does. In fact, not only does he avoid to present detailed reconstructions that are based on other possible base radii than 3438', but he also does not provide even the simplest discussion about other possible values; the only base radius he talks about, and the only base radius he uses in his reconstructions, is 3438'. This circumstance makes his argument much less convincing. This is because if Toomer actually did not recalculate the epicycle and the eccenter cases with other values than a 3438' base radius, then how can he be so sure that other base radii would not have generated better, or at least just as good results? After all, Toomer, with his 3438' reconstructions, did not hit the exact same ratios as Hipparchus's; therefore, it is not impossible that some other base radius may be able to produce even better, or, at the very least, just as good "strange numbers" as Toomer's 1973 reconstructions did.⁴⁵ And even if a certain other base radius would have produced slightly worse results than those that Toomer generated, it is still possible that such a radius could have been the one that Hipparchus actually used.⁴⁵

1.3 Did Ptolemy Necessarily Use a 3438' - based Chord Table?

Still another unsuccessful argument of Toomer's is built on the idea that Ptolemy used a 3438'-based chord table in his *Geography*. Toomer says that "I know of one other trace in surviving Greek literature of the use of a chord table with base $R' = 3438$. This is Ptolemy, *Geography* I 20"⁴⁶. His idea is that the two Ptolemaic ratios of 93/115 and 52/115 are the result of Ptolemy using "the older chord table,"⁴⁷ i.e. one with an alleged 3438' base radius.

But Toomer's thesis is far from conclusive. First of all, the number 3438 is not to be found anywhere in the text of Ptolemy's *Geography*; it is only due to Toomer's speculative reconstructions that it suddenly emerges there. Second, as Thompson has shown,⁴⁸ Ptolemy may have used another method to calculate these ratios. If Ptolemy followed the Greek tradition, or at least Hipparchus's tradition⁴⁹ of using a degree of latitude as a unit of distance, he

could have gotten the same ratios. According to such a system, the diameter could be calculated as $360/\pi$. If Ptolemy, for instance, used Archimedes's value of $3\frac{1}{7}$ to calculate this diameter, he could have produced a value such as 114.5454, which nicely rounds off to 115. "Using a compass, a ruler, and protractor," as Thompson puts it, "it is easy to construct a circle of this diameter, marked with the parallels of latitude at 36 degrees and 63 degrees. Their lengths turn out to be 93 and 52 in round numbers"⁵⁰ Third, it is rather implausible that Ptolemy used two completely different chord tables. Good mathematicians, like Ptolemy, normally develop good *general* mathematical tables and procedures that may be used in a variety of applications. So, since we know, with *certainty*, from the *Almagest*, that Ptolemy had a very accurate chord table with a base radius of 60, why would Ptolemy need to also have another, fundamentally different, chord table?

I therefore conclude that Toomer's "unveiling" of a second Greek chord table with the base radius of 3438' may not at all be as "inevitable"⁵¹ as he would like us to believe.

1.4 Did the Indians Necessarily Copy the Values of the Greek Chord Tables?

Toomer has not been able to conclusively establish his thesis that the Indians copied the values in Hipparchus's 3438'-based chord table to construct their own sine tables. For such a thesis to be *proven*, it also must be proven that Hipparchus really had a chord table with a 3438' base radius. But Toomer has not been able to prove, beyond all reasonable doubt, that Hipparchus ever used such a chord table.

But let us, for argument's sake, see what would have been possible for Toomer to deduce if he would have been able to conclusively establish that the Hipparchan chord table looked like the one that he presents in his 1973 essay. Would it not, then, also be possible for him to automatically conclude that the Indians must have copied the Hipparchan values? No, this would not necessarily follow. This is because the values in Toomer's reconstructed "Hipparchan" chord table do not produce the exact same values as those in the Indian tables; in some places there is, as Toomer expresses it, "a discrepancy of one 'minute'"⁵². This discrepancy can be explained in two rather different ways. One way to explain it would be to say, as Toomer does, that the Indians copied

the Hipparchan values, and that this discrepancy therefore must mean that Hipparchus's chord table in reality had slightly different values than the one Toomer reconstructed.⁵³ Another way to explain this discrepancy, however, would be to say that the Indians did not use the values in Hipparchus's alleged chord table as a basis for computing the entries in their sine tables, and that there therefore is no need to explain the "discrepancy." Such an account is supported not only by the fact that the Indians possessed rather advanced geometrical knowledge, as seen in the *Śulba-sūtras* (from ca. 700 BC and onward)⁵⁴, also by the particular method of generating a table of 24 values for every arc of $3^{\circ}45'$ of a circle of radius 3438' that is found in the *Sūrya-siddhānta*.

The objection can be made that even if chord tables with a base radius of 3438' never were in use in the Greek world, this does not conclusively prove that the Indians did not copy the values from some other Greek chord table and then converted them to their own 3438'-based sine tables.

My answer to this objection is that it is true that no one has conclusively proven that the Indians did not copy the values from some other Greek chord table. But it is also equally true that no one has conclusively proven that the Indians did copy the values from some other Greek chord table. For example, the values in Ptolemy's chord table (which is a good example of a chord table that does not use a base radius of 3438') do not produce the exact same values as those in the Indian sine tables.⁵⁶ Just as in the case of the alleged Hipparchan chord table with a 3438' base radius, Ptolemy's values produce multiple discrepancies of one "minute." We therefore end up in the same situation as in the case of the alleged 3438'-based Hipparchan chord table: we can either choose to adopt the thesis that the Indians did copy the Greek values (and then try to explain how this discrepancy possibly was produced), or we can choose to go with the thesis that the Indians did not copy the Greek values (which then, of course, frees us from the responsibility of explaining the "discrepancy").

1.5 Another scenario

Since as can be seen throughout this essay, Toomer's 1973 paper and his 1984 footnote contain so many invalid conclusions about Hipparchus's alleged 3438'-based chord table and its alleged connection to Indian trigonometry, there is ample room for other scenarios. I will therefore take the oppor-

tunity to present my alternative thesis that a chord table with a base radius of 3438' never may have been in use in the Greek world. I am personally of the opinion that Ptolemy was more indebted to Hipparchus's work than Toomer thinks he was, and I am particularly fond of the idea that there is a simple, but overlooked, relationship between Ptolemy's chord table and Hipparchus's. If Ptolemy constructed his chord table in a similar way to Hipparchus's⁵⁷, but chose to use a base radius expressed in "degrees" instead of, as Hipparchus may have done, in "minutes," then Ptolemy's base radius of 60 "degrees" would correspond to an Hipparchan base radius of 3600 "minutes." This hypothesis is, I propose, just as likely (if not more likely) than the one that Toomer presents. This is not only because, as we will see in the following sections, a 3600' base radius produces better Hipparchan-like ratios than a 3438' base radius does⁵⁸, but also because a 3600'-based Hipparchan chord table gives the history of ancient Greek trigonometry more continuity than a 3438'-based one does. It is, at least in my opinion, easier to accept the scenario that Ptolemy used a chord table that was very much like the one that Hipparchus used, than it is to believe that Hipparchus's chord table was fundamentally different from Ptolemy's, and that it, somehow or other, ended up in India, and that it simultaneously vanished from all Greek records.^{59,60}

2. RECALCULATING TOOMER'S 1973 EPICYCLE AND ECCENTER SIMULATIONS USING $R = 3600'$ INSTEAD OF $3438'$

There are three aspects of my reconstructions that are somewhat different from Toomer's. First of all, I do not follow Toomer's procedure of interpolating values from an "Hipparchan" chord table; instead, I derive my values with the sine function on my computer, using the identity $\text{Chord } \alpha = 2 \sin(\alpha/2)$. I have two reasons for doing so. The first reason is that the accuracy is good enough to be able to show that it is not impossible that Hipparchus could have used a chord table with radius 3600' to produce his two ratios, or as Toomer puts it, his "strange numbers"⁶¹. In fact, Toomer himself agrees that such a way of calculating is permissible in terms of the precision of the results: "...one gets very similar results (i.e. in neither significantly better nor significantly worse agreement with Hipparchus' ratios) using exact values (extracted from a modern sine table) instead of interpolation"⁶². The second reason is that I find

it programmatically more convenient to use standard sine functions than to use interpolation. After all, interpolation requires me not only to prepare the data in the form of a chord table, but also requires me to write (and test) special interpolation routines.

Another aspect of my computations that is different from Toomer's is that I, unlike Toomer, am using a particular technique to assess the accuracy of the reconstructed ratios in relation to the Hipparchan ratios⁶³. My evaluation technique depends on three factors: how close the numerator is to Hipparchus's numerator; how close the denominator is to Hipparchus's denominator; and how close the evaluated ratio is to Hipparchus's evaluated ratio. After I have determined these three deviation values, I also compute their average value, as an overall measure of how successful the reconstructed ratio is. One objection that can be made to my evaluating technique is that it is unnecessary to include anything else than the deviation of the evaluated ratio. This is, however, not a good objection, because even if a numerator and denominator are not at all close to Hipparchus's numerator and denominator, they still can produce an evaluated ratio that is Hipparchan-like⁶⁴. We therefore need to include also the deviation of the numerators and denominators in our complete assessment. Another objection that can be made is that it is unnecessary to include the deviation of the evaluated ratio, since an evaluated ratio always can be recalculated from its numerator and denominator. This is also not a very good objection. Although an evaluated ratio certainly always can be recalculated from its numerator and denominator, it does not follow that it is also true that we can recreate the deviation from Hipparchus's evaluated ratio by dividing the deviation of the numerator with deviation of the denominator. Therefore, we also need to include the deviation of the evaluated ratio to make our assessment complete.

A third aspect of my recomputations that is different from Toomer's is that I only display the *results* of each step in the calculations, not the equations themselves.⁶⁵ This makes my paper more compact, and hopefully also slightly less tedious to read. Readers who are concerned with how the values in my tables have been obtained are advised to consult Toomer's 1973 paper, which contains all the relevant equations.

My calculations in this paper are, if not explicitly stated otherwise, mostly carried out with approximately ten digits accuracy. In the following, however, I have chosen to display these calculations using only two decimals.

2.1 Calculating R/r with the Epicycle Model

This first reconstruction follows the same flow of computations that Toomer presents on pages 9-11 of his 1973 paper, but uses a chord table with a base radius of 3600' instead of 3438'. Now, according to Toomer's 1973 paper, the values ascribed to Hipparchus by Ptolemy are the following intervals between the eclipses (Δt = time; $\Delta \lambda$ = longitude)⁶⁶ :

<i>Interval</i>	Δt	$\Delta \lambda$
From eclipse I to II	178 days, 6 hours	180; 20°
From eclipse II to III	176 days, 1/3 hours	168; 33°

Toomer then derives the moon's motions in these time intervals by looking them up in Ptolemy's tables in *Almagest* IV 4⁶⁷:

<i>Interval</i>	$\Delta \alpha$	$\Delta \lambda_m$
From eclipse I to II	168 ; 50°	188; 42°
From eclipse II to III	139 ; 37°	159; 14°

Toomer therefore arrives at the following values, which he uses in his simulation of how Hipparchus calculated the R/r ratio for the epicycle model ($\delta = \Delta \lambda - \Delta \lambda_m$):

$\alpha_1 = 168; 50^\circ$	$\delta_1 = (-) 8; 22^\circ$
$\alpha_2 = 139; 37^\circ$	$\delta_2 = (+) 9; 19^\circ$
$\alpha_3 = 51; 33^\circ$	$\delta_3 = (-) 0; 57^\circ$

Now let us determine the ratio of the deferent R to the radius of the epicycle r :

Table 1. Calculating R/r with the epicycle model, using a chord table with base radius 3600'. Cf. Toomer's calculations in his 1973 paper, pp. 9-11.

$\angle OM_1B$	76;3°
$\angle OM_3B$	100;52.5°
M_1B	s.1047.65/6987.65
M_3B	s.1165.61/7070.70
M_3P	s.258.06/3600
M_1P	s.5.33/3600
M_1M_3	s.258/3600
r	s.258.12/3130.83
Chord ($\angle M_3CB$)	7198.47
$\angle M_3CB$	177; 38°
$\angle M_2CB$	38;1°
BM_2	s.(258.12·2345.21)/(3130.83·3600)

As Table 1 shows, a chord table with base radius 3600' generates a ratio of 3056.52/258.12. As seen below, these values are, on average, better than those that Toomer generated in 1973:

My results (base radius = 3600'):

Numerator:	3056.52	(-) 2.11% deviation from Hipparchus's 3122.5
Denominator:	258.12	(+) 4.29% deviation from Hipparchus's 247.5
Evaluated ratio:	11.84	(-) 6.18% deviation from Hipparchus's 12.62
Average deviation from Hipparchus's values: 4.19%		

Toomer's 1973 results (base radius = 3438'):

Numerator:	2913.5	(-) 6.69% deviation from Hipparchus's 3122.5
Denominator:	246.33	(-) 0.47% deviation from Hipparchus's 247.5
Evaluated ratio:	11.83	(-) 6.26% deviation from Hipparchus's 12.62
Average deviation from Hipparchus's values: 4.47%		

My recalculation shows an average deviation from Hipparchus's values of 4.19%, while Toomer's shows a greater deviation, 4.47%. Note also that my *evaluated* ratio is closer to Hipparchus's evaluated ratio than Toomer's is.

2.2 Calculating R/e with the Eccenter Model

This reconstruction follows the same flow of computations that Toomer presents on pages 12-15 of his 1973 paper, but uses a base radius of 3600' instead of 3438'. Now, according to Toomer's 1973 paper, the values ascribed to Hipparchus by Ptolemy are the following intervals between the eclipses (Δt = time; $\Delta \lambda$ = longitude):⁶⁸

Interval	Δt	$\Delta \lambda$
From eclipse I to II:	177 days, 13 3/4 hours	172;52.5°
From eclipse II to III:	177 days, 1 2/3 hours	175;7.5°

Toomer then derives the moon's motions in these time intervals by looking them up in Ptolemy's tables in *Almagest* IV 4:⁶⁹

Interval	Δt	$\Delta \lambda_m$
From eclipse I to II:	159; 59°	179;46°
From eclipse II to III:	153; 25°	173;8°

Toomer therefore arrives at the following values, which he uses in his simulation of how Hipparchus calculated the R/e ratio for the eccenter model ($\delta = \Delta \lambda - \Delta \lambda_m$):

$\alpha_1 = 159;59^\circ$	$\delta_1 = (-) 6;53.5^\circ$
$\alpha_2 = 153;25^\circ$	$\delta_2 = (+) 1;59.5^\circ$
$\alpha_3 = 46;36^\circ$	$\delta_3 = (+) 4;54^\circ$

and

$\zeta_1 = 153;5.5^\circ$
$\zeta_{II} = 155;24.5^\circ$
$\zeta_{III} = 51;30^\circ$

Now let us determine the ratio R/e:

Table 2. Calculating R/e with the eccenter model, using a chord table with base radius 3600'. Cf. Toomer's calculations in his 1973 paper, pp. 12-15.

$\angle OM_{II}B$	76;6°
$\angle OM_{III}B$	28;12°
$M_{II}B$	s.3258.46/6889.06
$M_{III}B$	s.5634.78/3402.36
$M_{III}P$	s.(5634.78.7007.13)/3402.36.2.3600)

$M_{II}P$	$s.((3258.46/6889.06)+(5634.78.1655.34/(3402.36.2.3600)))$
$M_{II}M_{III}$	$s.6566.13/3600$
R	$s.3283.06/3503.56$
Chord ($\angle M_{III}CB$)	6362.52
$\angle M_{III}CB$	124; 11°
$\angle M_I CB$	170; 47°
$M_I B$	$s.(3283.06.7176.71)/(3503.56.3600)$
$R^2 - e^2$	$s^2.(3283.06.7176.71) - (3503.56.3600)/(3503.56.3600)$
R/e	3283.06/350.78

As Table 2 shows, a chord table with base radius 3600' will generate a ratio of 3283.06/350.78. As seen below, my recalculation is not, on average, better than Toomer's:

My results (base radius = 3600'):

Numerator: 3283.06 (+) 4.42% deviation from Hipparchus's 3144.0
 Denominator: 350.78 (+) 7.06% deviation from Hipparchus's 327.66
 Evaluated ratio: 9.36 (-) 2.5% deviation from Hipparchus's 9.60
 Average deviation from Hipparchus's values: 4.66%

Toomer's 1973 results (base radius = 3438'):

Numerator: 3134.0 (-) 0.32% deviation from Hipparchus's 3144.0
 Denominator: 338.0 (+) 3.16% deviation from Hipparchus's 327.66
 Evaluated ratio: 9.27 (-) 3.44% deviation from Hipparchus's 9.60
 Average deviation from Hipparchus's values: 2.31%

My recalculation shows an average deviation from Hipparchus's values of 4.66%, while Toomer's shows a smaller deviation, 2.31%. Note, however, that my *evaluated* ratio is closer to Hipparchus's evaluated ratio than Toomer's is.

2.3 Comparing my Simulations with Toomer's 1973 Simulations

The reconstructions that I have presented in this section show that it would have been possible for Toomer, already in 1973, to discover that there are other base radii than 3438' such as 3600' that also can generate Hipparchan-

like ratios. Although Toomer's 1973 eccentric ratio certainly comes a bit closer to Hipparchus's ratio than mine does. It is, at the same time, also true that my reconstructed epicycle ratio comes somewhat closer to Hipparchus's ratio than Toomer's does. It is also important to observe that my reconstructions consistently, for both the epicycle and eccentric models, produce evaluated ratios that are closer to Hipparchus's evaluated ratios than Toomer's are. In conclusion, since the results obtained in Toomer's 3438'-based reconstructions are not far superior to those generated in my 3600'-based reconstructions, we cannot rule out the possibility that 3600' may have been the base radius of an Hipparchan chord table.

At this point the objection can be made that I have not presented a 3600' recalculation—or even discussed the possibility of such a recalculation—using Toomer's idea that Hipparchus made an error when he computed the R/r ratio for the epicycle case. If I would have presented such a reconstruction and compared it with Toomer's 1973 reconstruction, the reader would have found that Toomer's reconstruction actually produces a better Hipparchan-like R/r ratio than a base radius of 3600' does^{70,71}. Therefore, Toomer's 1973 reconstruction must be considered superior to my reconstruction in that it generates better Hipparchan-like ratios in both the epicycle and eccentric cases.

Although the theory that Hipparchus made a calculation error certainly produces a more Hipparchan-like R/r ratio for Toomer's 1973 epicycle reconstruction (which uses a 3438'-based chord table) than my reconstruction (which uses a 3600'-based chord table) does, this circumstance does not, in any way, prove that the "Hipparchan Computing Error" thesis is a more plausible thesis than that Hipparchus did not make a computing error. In fact, already in 1973 Toomer should have informed the reader that there may be several fundamental problems with the "Hipparchan Computing Error" thesis.^{72,73} One serious problem is that Toomer's recomputation with Hipparchus's alleged computing error does not produce the exact same epicycle ratio as Hipparchus's; it just comes a bit closer, compared to not using the "Hipparchan Computing Error" thesis. But for an Hipparchan computational error to be plausible—especially considering Hipparchus's reputation as one of the ancient Greek world's most brilliant astronomers, as well as being the founding father of Greek trigonometry⁷⁴—nothing else but an exact Hipparchan ratio would have had to be produced by Toomer for it to be plausible.⁷⁵ A second serious problem with Toomer's

"Hipparchan Computing Error" thesis is his argument that since he himself made a mistake when he reconstructed Hipparchus's alleged diagram, therefore Hipparchus must also have made exactly the same error⁷⁶. But this conclusion does not at all follow. For example, we do not know whether Hipparchus, at all, used a diagram. And even if he did, Hipparchus's diagram may have been radically different than Toomer's. Also, even if Hipparchus used an identical diagram, or one that was very similar to Toomer's, it is not necessary that Hipparchus must have made the same, clumsy error that Toomer did; Hipparchus could either have avoided computational mistakes altogether, or, if he at all made any, may have made them in other places than where Toomer is pointing. A third serious problem with the "Hipparchan Computing Error" thesis is that Toomer here significantly departs from one of his (and Neugebauer's) main premises, which is that Hipparchus used the same computation methods that Ptolemy presents in *Almagest* IV⁷⁷. Using his auxiliary hypothesis that Hipparchus made a computing error, Toomer now, in effect, is saying that Hipparchus did not use the same computation method for the epicycle case that can be reasonably inferred from Ptolemy's account in the *Almagest*. This assumption makes Toomer's total account weaker. If Hipparchus, in fact, made that clumsy computation error, and if the epicycle computation method that Ptolemy describes in the *Almagest*, in fact, is referring to the method that Hipparchus used, then why is it that Ptolemy does not explicitly mention that Hipparchus committed that computationally significant mistake?⁷⁸ 'Does Toomer, perhaps, suggest that Ptolemy himself did not know that Hipparchus made that particular error?'⁷⁹

With these three, easily identified problems with the "Hipparchan Computing Error" theory in mind, any reasonably critical scholar could, already in 1973 or shortly thereafter, have objected to Toomer's exaggerated claims that he had proven that Hipparchus made a computing error. But no one came forward at that time. Only Toomer himself, eleven years later, and for the wrong reason, finally published doubts about the "Hipparchan Computing Error" thesis. After finding that the newly corrected Hipparchan time interval (176 days and $1\frac{1}{3}$ hours)⁸⁰ generated worse ratios when using his "Hipparchan Computing Error" hypothesis⁸¹ he suddenly abandons that theory by saying that "These [new] calculations... vindicate Hipparchus' computational abilities"⁸². In other words, he seems to be claiming that he now has proven that Hipparchus did not

make a computational error. His conclusion is, however, once again, too strong. Although Toomer certainly may be correct in saying that his new reconstruction – which is using the “Hipparchan Computing Error” thesis and the corrected Hipparchan time interval—does not generate a very favorable epicycle ratio for a chord table with base radius 3438', this, in no way, proves that Hipparchus did not make any computational errors. Toomer's mistake is that he begs the question, assuming that Hipparchus must have had a chord table with a base radius of 3438'. But since no one has proven that the radius of Hipparchus's chord table was 3438', it is not permissible for him to use that radius as a premise in his argument. In fact, there may be several other base radii than 3438' that can produce—while still using the “Hipparchan Computing Error” thesis and the corrected Hipparchan time interval – a better epicycle ratio than the one that Toomer generated. However, even if we could generate such better epicycle ratios with certain other base radii, that circumstance, on its own, would still not count as conclusive evidence for that the “Hipparchan Computing Error” thesis is correct. It is still perfectly possible that Hipparchus did not make the particular computing error that Toomer suggested, and that Hipparchus instead chose a base radius such that it generated a similar epicycle ratio to the one which is produced by a 3438' reconstruction using the corrected Hipparchan time interval and the thesis that Hipparchus made a computational error. In conclusion, and as I have also pointed out elsewhere in this paper, these mathematical reconstructions only display possibilities; on their own, they cannot conclusively prove either that Hipparchus did make a certain computational mistake, or that he did not. All that we can say, without having more factual evidence at our disposal, is that no one has proven that Hipparchus made the particular computational mistake that Toomer has been referring to. For this reason, I will not in this paper further consider the idea that Hipparchus made that computing error.

3. RECALCULATING THE EPICYCLE CASE WITH THE HIPPARCHAN TIME INTERVAL CORRECTION FOUND IN TOOMER'S 1984 FOOTNOTE

As we have previously noted, Toomer recently changed his mind about his idea that Hipparchus had made a computational error. In his 1984 translation of the *Almagest* he admits that “I carelessly followed his [Manitius's] interpretation and emendation in Toomer [2]⁸³, in which I used Hipparchu's inter-

vals to recompute the ratios for the eccentric and epicyclic models.' Toomer's mistake was simply that he mistranslated Hipparchus's interval from eclipse II to III to be 176 days and $1/3$ of an hour instead of, as it should have been, 176 days and $1\frac{1}{3}$ hours, and then used the former (incorrect) value in his 1973 simulations. The problem is that this difference in interval is by no means negligible, as confirmed by Ptolemy: "These errors [in longitude and time] too can result in a considerable discrepancy in the ratio calculated for the (particular) hypothesis"⁸⁵. Let me therefore recalculate the epicycle case once again.

The following intervals between the eclipses are those that Ptolemy actually ascribed to Hipparchus (Δt = time; $\Delta\lambda$ = longitude):

Interval	Δt	$\Delta\lambda$
From eclipse I to II	178 days, 6 hours	180; 20°
From eclipse II to III	176 days, $1\frac{1}{3}$ hours	168; 33°

By using Ptolemy's lunar mean motion tables in *Almagest* IV 4⁸⁶ I arrive at these values for $\Delta\alpha$ and $\Delta\lambda_m$:

Interval	$\Delta\alpha$	$\Delta\lambda_m$
From eclipse I to II	168; 50°	188; 42°
From eclipse II to III	140; 10°	159; 47°

and the following values for α_1 , α_2 , α_3 and δ_1 , δ_2 , δ_3 :

$\alpha_1 = 168; 50^\circ$	$\delta_1 = (-) 8; 22^\circ$
$\alpha_2 = 140; 10^\circ$	$\delta_2 = (+) 8; 46^\circ$
$\alpha_3 = 51^\circ$	$\delta_3 = (-) 0; 24^\circ$

Now let us recalculate the ratio of the deferent R to the radius of the epicycle r :

Table 3. Calculating R/r with the epicycle model, according to Toomer's 1973 essay, pp. 9-11, but with the time interval correction that Toomer mentions in Toomer 1984, footnote 75. This calculation uses a chord table with a base radius of 3600'

$\angle OM_1B$	76;3°
$\angle OM_2B$	101;9°
M_1B	s.1047.65/6987.65
M_2B	s.1097.36/7064.10
M_1P	s.240.76/3600
M_2P	s.34.99/3600
M_1M_2	s.243.29/3600
r	s.243.29/3099.68
Chord ($\angle M_1CB$)	7125.15
$\angle M_1CB$	163; 28°
$\angle M_2CB$	23;18°
BM_2	s.(243.29.1453.66)/(3099.68·3600)
$R^2 - r^2$	s ² .(3099.68.3600) - (243.29. 1453.66)/(3099.68·3600)
R/r	3059.85/243.29

As Table 3 shows, a chord table with a base radius of 3600' will generate a ratio of 3059.85/243.29. This means that the average deviation from Hipparchus's values is 1.34%, as seen below:

My results (base radius = 3600'):

Numerator: 3059.85 (-) 2.00% deviation from Hipparchus's 3122.5

Denominator: 243.29 (-) 1.70% deviation from Hipparchus's 247.5

Evaluated ratio: 12.58 (-) 0.32% deviation from Hipparchus's 12.62

Average deviation from Hipparchus's values: 1.34%

It would now be interesting to know what ratio Toomer arrived at. Unfortunately, Toomer does not tell us; the only thing he says in his 1984 footnote is: ⁸⁷

Now, however, using the correct time interval of $1\frac{1}{3}$ h for II-III, I find much better agreement with the above ratio, as I shall show in detail elsewhere. (If the [Hipparchan] ratio were 3112 $\frac{1}{2}$: 247 $\frac{1}{2}$, agreement would be almost perfect, and this also provides a better fit with the equivalences given by Ptolemy.)

This, indeed, sounds as if the ratio that he found was very close to the Hipparchan one. But if that is the case, why did he not show us the ratio, either in his translation of the *Almagest*, or in some other publication? Is it maybe because his values were not so good, after all? Just to get a feel for what kind of ratio Toomer may have ended up with, I performed a simulation using a base radius of 3438', with the new time interval that Toomer corrected. Although Toomer's evaluated ratio (12.58) is pretty close to Hipparchus's evaluated ratio (12.62), the numerator and denominator of Toomer's ratio are far away from Hipparchus's values. In fact, the average deviation from Hipparchus's values is as much as 4.30%, which is just marginally better than the ratio he arrived at in 1973, before he started to "improve" it with his thesis that Hipparchus made a gross calculation error:

Simulation of Toomer's 1984 results (base radius = 3438'):

Numerator:	2922.16	(-) 6.42% deviation from Hipparchus's 3122.5
Denominator:	232.24	(-) 6.17% deviation from Hipparchus's 247.5
Evaluated ratio:	12.58	(-) 0.32% deviation from Hipparchus's 12.62
Average deviation from Hipparchus's values: 4.30%		

Now, if these values are close to his actual simulation results, which I propose they are,⁸⁸ then we can conclude that a reconstruction of the epicycle case using a base radius of 3600' is far superior to one that is using a base radius of 3438'. While both reconstructions show an evaluated ratio that is identical (12.58), the 3600' reconstruction's numerator and denominator are much closer to Hipparchus's numerator and denominator than the 3438' reconstruction's are. In fact, the numerator and denominator in the 3600' reconstruction are so close that they, together with the evaluated ratio of 12.58, result in an average deviation from Hipparchus's values of only 1.24%. This should be compared to the 3438' reconstruction, which has a more than three times bigger average deviation from Hipparchus's values (4.30%).

4. CONCLUSION

As this essay has shown, it is virtually impossible—using factual evidence in extant Greek manuscripts, sound mathematical principles, and valid philosophical arguments—to draw any stronger conclusion than that Hipparchus may have had a chord table with a base radius of 3438'. In fact, if Toomer would have conducted a more thorough and unbiased scientific investigation—

with alternative reconstructions for other base radii than 3438', and without his too speculative "Hipparchan Computing Error" thesis – he should already in 1973 have been in a position to conclude the same thing that I am concluding in this essay, namely that 3438' is not the only base radius that can produce Hipparchan-like ratios.

The fact that Toomer has failed to conclusively prove that Hipparchus used a 3438'-based chord table has several important consequences. First of all, it opens up for the possibility that Hipparchus may have used some other base radius than 3438'. As I have shown in sections 2 and 3 of this paper, a chord table with base radius 3600' actually generates even better Hipparchan-like ratios than one with a base radius of 3438', and can therefore be regarded as a quite realistic alternative. However, I do not, of course, claim to have proven that Hipparchus used a base radius of 3600'; for that, my evidence is too weak.⁸⁹

Second, the fact that Toomer has failed to conclusively establish that Hipparchus had a 3438' -based chord table also means that it was never actually proven that the Indians must have derived their sine table values from an Hipparchan chord table with a base radius of 3438' although Neugebauer and others certainly gave us the impression that it was.⁹⁰ Therefore, it is still perfectly possible that the Indians not at all copied Hipparchus's values, but instead relied on their own, ancient mathematical tradition to independently produce their sine tables.

There are many possible future research projects that may provide (partial) answers to some of the questions that are raised in this essay. One project, for instance, may focus on how to, mathematically and philosophically, resolve the previously mentioned "one minute discrepancy" problem that arises when we try to create an Indian sine table by using the values in Toomer's reconstructed Hipparchan chord table. Another project may make a more complete survey of different base radii, to see how close they can get to the Hipparchan ratios. And yet another project may investigate other ways of calculating the two Hipparchan ratios than the method Toomer uses in his 1973 paper. Only after the successful implementation of such interesting investigations can we make more assertive statements about the true nature of the Hipparchan chord table and its alleged connection with the Indian sine tables.

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NOTES AND REFERENCES

1. Cf. E. Burgess 1860.
2. Cf. *Ġitikāpāda*, verse 10, *Gaṇitapāda*, verse 9,11-12,17. See K.S. Shukla 1976, pp.44-54, 59-60; W.E. Clark 1930, pp. 19-20,27-30,34.
3. P. Tannery 1893, pp. 60-68. Tannery's account is also discussed in Toomer 1973, p. 20, and in Heath 1981, p. 253.
4. G.R. Kaye 1924, pp. 54-55, 123. Note that Kaye does not provide any evidence for his claims, more than presenting Ptolemy's and Pauliṣa's tables, which are not identical.
5. B.L. van der Waerden 1954, p. 270. He does neither provide any evidence for his thesis that the Indians previously had Greek-like chord tables nor for his thesis that the Indians later changed to the sines.
6. A. Aaboe 1964, p. 111.
7. O. Neugebauer 1972, p. 251. Note, however, that Neugebauer already in the second edition of *The Exact Sciences in Antiquity*, originally published in 1957, tried to link Hipparchus, Ptolemy and the *Sūrya-siddhānta* together (Neugebauer 1970, p. 214).
8. I wish to thank Alexander Jones for comments on the historiography in this paragraph.
9. G.J. Toomer 1973, p. 6.
10. Hipparchus's chord table has not been found in any extant fragments. The only piece of recorded information that we have is (as Toomer also notes) that Theon of Alexandria, in his commentary on the *Almagest*, mentions that Hipparchus was the author of a twelve-book work on chords. Note, however, that Theon of Alexandria does not say that this work contained any chord tables; Theon only says that Hipparchus's book was a treatise on chords, and that it was in twelve books. This piece of information does not, on its own, conclusively prove that Hipparchus constructed (or used) a chord *table*.

11. G.J. Toomer 1973, p. 7. note that Toomer assumes that Hipparchus used a value for π such that " $21600/2 \pi$ equals 3438' to the nearest 'minute'."
12. Toomer does not give us references to any Greek texts that explicitly present a 48-division of the circle; Toomer only says (1973, p. 6) that Neugebauer's theory is based on "mentioned evidence in Greek texts for the existence of a 48-division of the circle into units of $7 \frac{1}{2}^\circ$." But when we check Neugebauer's 1972 paper, we only find that Neugebauer (1972, p. 250) also does not give any exact references: "we find in the geographical literature, e.g., in fragments from Eratosthenes, in Posidonius, and in Geminus, a 48-division of the circle, resulting in 'parts' ($\mu\epsilon\rho\eta$) of $7;30^\circ$. All these units are more or less interwoven in our sources and represent together a simple sequence of arcs: 30° , 15° , and $7;30^\circ$." Since Neugebauer has neither presented any direct quotes from, nor any exact references to, places in the works of Eratosthenes, Posidonius, and Geminus that contain explicit statements about a 48-division of the circle, it is quite possible that the number 48 (or $7 \frac{1}{2}^\circ$) is not *explicitly* mentioned in any of these fragments; perhaps a 48-division of the circle is only one of Neugebauer's innovative, but false, interpretations. It is interesting to note that H. Berger, in his *Die Geographischen Fragmente des Eratosthenes*, only attributes Eratosthenes with a 60-division and a 360-division of the circle; no other division is mentioned (1964, pp. 111). Also compare Neugebauer's invalid (and possibly incorrect) conclusion that Hipparchus had a 3438'-based chord table (1975, p. 299), which he drew without having found the number 3438' in any Hipparchan (or other Greek) fragment.
13. G.J. Toomer 1973, p. 7.
14. G.J. Toomer 1973, p. 7.
15. I am using a special technique to assess how close to Hipparchus's "strange numbers" the two regenerated ratios are. For a description of this technique, please see Section 2 of this paper.
16. G. J. Toomer 1973, p. 15.
17. G. J. Toomer 1973, p. 23. Also see p. 28, note 32. Note especially here that the number 3438 is not *explicitly* mentioned neither in the *Geography* nor in the *Almagest*; it is only due to Toomer's speculative reconstructions that this value "appears" in these works.
18. If Toomer's 1973 paper had not been accepted by the majority of scholars, would it not have generated rather immediate protests? I have not found a single published paper that thoroughly examines Toomer's claims; only Thompson 1990 has a short treatment on pp. 195-197.
19. O. Neugebauer 1975, p. 299.
20. H. Thurston 1994, p. 128.
21. D. Pingree 1996, pp. 126-127.

22. V.J. Katz 1998, p. 212.
23. Many other scholars also acknowledge Toomer's 1973 article to be conclusive. B.L. van der Waerden, for example, writes: "Toomer has shown that the chord table of Hipparchos was a table of chords in a circle of radius $R=3438'$. This is just Āryabhaṭa's value. In my opinion, Toomer is justified in concluding that Āryabhaṭa's table of Sines was derived from Hipparchos' table of chords by halving the chords" (1983, p. 211). In a similar style, M. Yano writes: "...the first chord table ascribed to Hipparchus (fl. 150 BC) was successfully recovered by G.J. Toomer in 1973 from an Indian sine tableToomer showed that some numerical values ascribed to Hipparchus in the *Almagest* of Ptolemy (fl. AD 150) could only be explained by hypothesizing the use of this reconstructed table" (1997, p. 987).
24. G. J. Toomer 1984, p. 215, footnote 75.
25. I wish to thank Alexander Jones for this objection.
26. G. J. Toomer 1984, p. 215, footnote 75. Note that this is the last sentence of footnote 75; Toomer does not discuss Hipparchus's alleged 3438' based chord table (or its alleged relationship with the Indian sine tables) more in this footnote, or for that matter, anywhere else in his book.
27. I do not, however, say that it is impossible that Hipparchus may have had a chord table with base radius 3438', or that it is impossible that (some of) the Indians may have derived their sine tables from an Hipparchan one; I am only claiming that there is nothing in Toomer's 1973 paper (or in his 1984 footnote) that conclusively proves any such ideas.
28. One possible explanation for why Toomer did not discuss any of the consequences in his 1984 footnote could be that he did not fully subscribe to this doubt himself, and therefore did not think that there would be any consequences. Another (perhaps more likely) explanation is that he did subscribe to this doubt himself, but that he did not want to discuss the issue any further. One possible reason for that he did not want to continue the discussion could be that he then would have to withdraw many of his unwarranted speculations that were part of his 1973 essay, for examples his idea that Hipparchus had a chord table with a base radius of 3438', and the idea that the Indians used its values to produce their sine tables.
29. G.J. Toomer 1984, p. 215, footnote 75.
30. Note, however, as I point out in other places in this essay, that there are other arguments that can be made against a 3438' -based Hipparchan chord table.
31. See the beginning of this paper, where I have quoted H. Thurston 1994, D. Pingree 1996, and V.I. Katz 1998.

32. See, for example, van der Waerden 1988, who confirms (p. 27, footnote 4) that he has read Toomer's 1984 footnote, but still says (p. 27) that "G.I. Toomer has shown in his paper that the errors which Ptolemy ascribes to Hipparchos can be understood by assuming that Hipparchos used a table of chords in a circle of radius 3438'".
33. There may be many reasons for why a reader could decide not to fully embrace the doubt that Toomer puts forward in his 1984 footnote. One reason may be that the reader does not accept (some of) the new findings that Toomer presents in the beginning of footnote 75, and therefore decides to go with the old conclusions that Toomer presented in 1973, namely that Hipparchus was using a chord table with a base radius of 3438'. Another reason may be that the reader does accept (some of) the new findings that Toomer presents in the beginning of footnote 75, but does not accept that these new discoveries lead to any doubt as to whether Hipparchus had a chord table with base radius 3438'.
34. G.J. Toomer, p. 7.
35. This would have produced a base radius of 3436.36 "degrees" (or 3436 "degrees" and 22 "minutes"), which is not very close to the 3438' base radius that Toomer is looking for.
36. Since the particular computing method ("dividing 60.360° , or 21600' by 2π ... and rounding the result to the nearest minute") that Toomer is attributing to Hipparchus is not, to my knowledge, mentioned in any Hipparchan fragments (or in any other extant Greek manuscripts describing Hipparchus's methods), it is not so strange that Toomer does not provide any exact references.
37. Toomer does not hesitate to assert things that are not proven. One example of this is when he clearly says that Hipparchus derived his chord table in a certain way: "In such a [Hipparchan chord] table the chord of 180° (the diameter of the base circle) is found from $21600/\pi$ " (Toomer 1973, p. 7). But Toomer does not provide the slightest evidence for that Hipparchus must have used such a method.
38. See, for example, Toomer 1973 (pp. 16-17) and B.T. Brennan 1965 (*passim*) for accounts of how Ptolemy may have constructed his trigonometric tables. But note that even if some details in Toomer's and Brennan's expositions may not perfectly correspond to how Ptolemy *actually* constructed his chord table, it is still an unquestionable fact that Ptolemy did *not* use a base radius of 3438' in his chord table in the *Almagest*. And this is even though Ptolemy had a very good value of π available (3;8,30, according to Toomer 1973, p. 7). So, even if Ptolemy could have, in theory, used Toomer's suggested procedure to construct a chord table with a base radius of 3438', he, in reality, did not. Therefore, it is by no means necessary that Hipparchus *must* have used Toomer's suggested construction method to arrive at a specific chord table radius; Hipparchus could, for instance, just have used Ptolemy's way of deriving such a base radius, or used some other method.

39. "This is sufficiently close to Hipparchus' ratio to prove that Hipparchus did indeed use a chord table of the type posited in computing it" (G.J. Toomer 1973, p. 12).
40. G. J. Toomer 1973, p. 12.
41. There are at least two places in Toomer 1973 where such a stance may be found: "This is sufficiently close to Hipparchus' ratio to *prove* that Hipparchus did indeed use a chord table of the type posited in computing it" (p. 12); "Thus our calculations have *proven* that Hipparchus used a chord table with base $R' = 3438$ (p. 15).
42. Cf. the argument from underdetermination of theory by data—a well-known thesis in the philosophy of science. See, for example, D. Papineau 1995, pp. 152-154.
43. This is even after his attempt to come closer to Hipparchus's "strange numbers" by his speculative, unproven hypothesis that Hipparchus made a mistake in his computations.
44. "There is no explicit evidence about the nature of Hipparchus' chord table" (G.J. Toomer 1973, p. 6).
45. Since my aim is to produce alternative reconstructions that are directly comparable to Toomer's 1973 reconstructions, I am in this paper generally going with Toomer's assumption that Hipparchus followed Ptolemy's computational procedures. Toomer's assumption, however, need not be true. If Toomer's lineups of calculations for the two Hipparchan ratios (as presented in Toomer 1973, pp. 9-11 for the epicycle case, and pp. 12-15 for the eccentric case) do not *exactly* match those that Hipparchus used, then, even if Toomer's calculations for a certain base radius generate results that are close to Hipparchus's ratios, this does not guarantee that that particular base radius is the Hipparchan one.
46. G.J. Toomer 1973, p. 23.
47. G. J. Toomer 1973, p. 24.
48. R.L. Thompson 1990, pp. 196-197.
49. "Supposing anyone (like Hipparchus) divides the greatest circuit of the earth into three hundred and sixty divisions, each of the divisions will measure 700 stadia. This is the measure adopted by Hipparchus in fixing the distances..." (E.H. Warmington 1934, p. 244). Also cf. H. Berger 1964, 106f.
50. R. L. Thompson 1990, pp. 196-197.
51. G. J. Toomer 1973, p. 25.
52. G. J. Toomer 1973, page 26, note 8
53. Note that Toomer here, again, draws a much too strong conclusion. He claims that if the Indians copied the values from Hipparchus's chord table, then "the latter

[Hipparchus's chord table] must differ from our reconstruction in the corresponding places by at least one "minute" in order to explain this phenomenon (Toomer 1973, page 26, note 8). This, however, may not necessarily be true. If it is true that the Indians used the values from Hipparchus's chord table to produce their sine tables, then there are at least two possibilities of how to explain the "one-minute discrepancy" without assuming that the Hipparchan chord table must be different from the one that Toomer presents in his 1973 essay. One possibility is that the Indians may have converted the Hipparchan values slightly differently than what Toomer suggests; another possibility is that the Indians may have used some special (non-modern) algorithm for rounding (or truncating) their converted values.

54. Cf. Kulkarni 1983, Grover 1994, pp. 15-37, Amma 1979, pp. 14-60.
55. Datta and Singh 1983, p. 75-78. Cf. Gupta 1971, 1972, 1974, 1976; Shukla 1976, pp. 45-51.
56. Let us, for example, take Ptolemy's chord value for 52.5 degrees, which is $53^{\circ}4'29''$, or 53.075, and rescale it to a radius of 3438' instead of 3600, producing 50.686. Expressing this value in "minutes" instead yields 3041.181', after which halving it yields 1520.591'. After proper (modern) rounding principles, this value of 1521 should, according to Neugebauer and Toomer, be identical to the corresponding entry for 26.25 degrees in a typical Indian sine table. The fact is, however, that it is not. The *Sūrya-siddhānta* (ca. 300 AD) reports 1520'; Āryabhaṭa's *Āryabhaṭīya* (499 AD) reports 1520'; Brahmagupta's *Brāhmasphuṭa-siddhānta* (620 AD) reports, 1446'; Śrīpati's *Siddhānta-sekhara* reports 1510'; Āryabhaṭa's II's *Mahāsiddhānta* (950 AD) reports 1520'; and Bhāskara II's *Siddhānta-śiromaṇi* (1150 AD) reports 1520' [A.K. Bag 1969, p. 80]. And this is certainly not the only example of when the Greek values fail to generate corresponding Indian values. There are exactly three other angles for which a direct conversion of Ptolemy's values also do not generate the values as seen in Indian sine tables: the angle of 120 degrees generates 2977.396, but both the *Sūrya-siddhānta* and the *Āryabhaṭīya* report 2978' the angle of 127.5 degrees generates 3083.456 but both the *Sūrya-siddhānta* and the *Āryabhaṭīya* report 3084'; and the angle of 135 degrees generates 3176.306', but both the *Sūrya-siddhānta* and the *Āryabhaṭīya* report 3177'. Note that also other Indian tables (*Brāhmasphuṭa-siddhānta*, *Siddhānta-sekhara*, *Mahāsiddhānta*, *Siddhānta-śiromaṇi*) do not have any "Greek-compatible" values for these three angles; I just did not list them. See A.K. Bag 1969, p. 80; G.R. Kaye 1924, p. 123.
57. Or, at the very least, arrived at a base radius of 3600 "minutes" (or 60 "degrees") in some other way.
58. See sections 2 and 3 of this paper.
59. Note that other historians also have previously thought that Hipparchus's chord table was more or less like the one Ptolemy presents in *Almagest* I 11. See G.J. Toomer 1973, p. 6, and p. 25, note 2.

60. If it were a fact that Hipparchus had a chord table which was fundamentally different from Ptolemy's own, then we can reasonably assume that Ptolemy would have been informed in this matter. In such a case, it seems natural that Ptolemy would have mentioned this circumstance somewhere in the *Almagest*. In other words, the fact the Ptolemy does not mention any specific details about Hipparchus's chord table may be interpreted as a sign that Ptolemy's chord table was so similar to Hipparchus's that it did not need any special attention.
61. G. J. Toomer 1973, p.7.
62. G. J. Toomer 1973, p. 15.
63. Consider the following quotes from Toomer 1973: "This is rather far from Hipparchus' ratio" (p. 11); "one gets a result considerably smaller than this" (p. 11); and "This is sufficiently close to Hipparchus' ratio" (p. 12). These quotes raise, I believe, the following question: Which evaluation technique does Toomer use to conclusively and consistently determine that the ratios produced by his reconstructions are sufficiently close to Hipparchus's ratios?
64. For example, Hipparchus's ratio $3122.5 / 247.5$ is, when evaluated, equal to 12.62. But an evaluated ratio of 12.62 can also be produced by ratios such as $31,225 / 2,475$ and $201.92 / 16$, both of which contain numerators and denominators that are not at all Hipparchan-like.
65. I wish to thank David Fowler for this idea.
66. G. J. Toomer 1973, p. 9. Also cf. K. Manitius 1963, pp. 252-253; GJ. Toomer 1984, p. 215.
67. See G. J. Toomer 1973, p.26, note 11 for Toomer's justification of this procedure.
68. G. J. Toomer 1973, p. 9. Also cf. K. Manitius 1963, pp. 252-253; GJ. Toomer 1984, p. 215.
69. See GJ. Toomer 1973, p. 26, note 11 fo' Toomer's, justification of this procedure.
70. These are the results I get if I use Toomer's "Hipparchan Computing Error" scenario in a reconstruction with a base radius of 3600': the numerator is 3224.16, (+) 3.15% deviation from Hipparchus's 3122.5; the denominator is 258.12, (+)4.11 % deviation from Hipparchus's 247.5; and the evaluated ratio is 12.49, (-) 1.04% deviation from Hipparchus's 12.62. Therefore, my average deviation from Hipparchus's values is 2.77%. In comparison, here are Toomer's 1973 results using his "Hipparchan Computing Error" scenario: the numerator is 3082.66, (-) 1.29% deviation from Hipparchus's 3122.5; the denominator is 246.33, (-) 0.47% deviation from Hipparchus's 247.5; and the evaluated ratio is 12.51, (-)0.88 deviation from Hipparchus's 12.62. Therefore, Toomer's average deviation from Hipparchus's values is 0.88%.

71. Note that my reconstruction used the same data as Toomer did in his 1973 paper (except, of course, for the base radius, which was different), including Toomer's incorrect value for one of Hipparchus's time intervals.
72. Toomer did not inform the reader of his 1973 paper about these potential problems with his theory.
73. On several occasions, Toomer, when his reconstructions fail to generate the exact results that he is looking for, explains such discrepancies not with faults in his own theory, but with someone else's mistakes. So, when his reconstruction did not generate the exact ratio for the epicycle case, then *Hipparchus* must have made a computing error. Another example is when he says that "if the ratio were $3112\frac{1}{2} : 247\frac{1}{2}$ [instead of $3122\frac{1}{2} : 247\frac{1}{2}$], agreement [with Toomer's own recalculation] would be almost perfect" (GJ. Toomer 1984, p. 215, footnote 75), suggesting that the number $3122\frac{1}{2}$ was a scribal error—perhaps produced by Hipparchus, Ptolemy, or some scribe copying one of their manuscripts and that Hipparchus's original value was $3122\frac{1}{2}$. However, another possible explanation for these discrepancies is that Toomer's own reconstruction scenario is wrong. If, for example, Hipparchus did not use a 3438'-based chord table as Toomer assumes he did, or if Hipparchus did not compute his two ratios in the same manner as Toomer assumed he did, then it is only natural that Toomer's reconstructions do not produce the same ratios as those that are found in the Greek manuscripts.
74. Thomas 1993, p. 406, footnote *a*.
75. And even if an exact ratio could be produced this way, such a circumstance, on its own, would still not be, enough to conclusively prove Toomer's "Hipparchan Computing Error" thesis.
76. G. J. Toomer 1973, pp. 11-12.
77. G. J. Toomer 1973, pp. 8-9. Cf. O. Neugebauer 1970, pp. 210-214.
78. This error produces, as can be seen both in this paper and in Toomer's 1973 paper, a significantly different epicycle ratio, compared to not making this error.
79. If Hipparchus made the particular computing error that Toomer refers to, and if Ptolemy did not know about it, then we can conclude that Ptolemy cannot have had access to Hipparchus's complete set of calculations, and, in particular, to the part where Hipparchus makes the computing error.
80. For more information on the Hipparchan time interval correction, see Section 3 of this paper.
81. Toomer did not publish his new calculations, or provide any elaborate discussions of them, in his 1984 footnote. And although he did not explicitly state that his recalculation with corrected Hipparchan time interval, using his "Hipparchan Computing Error" thesis, generated worse results than his 1973 reconstruction with the old, uncorrected Hipparchan time interval did, this is how I understand his account.

82. G. J. Toomer 1984, p. 215, footnote 75.
83. G. J. Toomer 1984, p. 215, footnote 75.
84. G. J. Toomer 1984, p. 215, footnote 75.
85. Ptolemy, *Almagest* IV 11: G. J. Toomer 1998, p. 215; K. Manitius 1963, pp. 253-254.
86. See *Almagest* IV 4: K. Manitius 1963, pp. 206-211; G. J. Toomer 1984, pp. 182-187.
87. G. J. Toomer 1984, p. 215, footnote 75.
88. 'I have two reasons for thinking that this simulation of Toomer's "1984 epicycle case" may be rather good. My first reason is that I have been able to regenerate the ratios of other Toomer simulations with a very good accuracy. For the original 1973 epicycle case I generated a Toomer ratio of 2918.97/246.50, to be compared with Toomer's officially reported ratio of 2913/246.33 (1973, p. 11); for the original 1973 eccentric case I generated a Toomer ratio of 3135.33/334.99, to be compared to Toomer's officially reported ratio of 3134/338 (1973, p. 15). My second reason is linked with Toomer's statement that "If the [Hipparchan] ratio were $3112\frac{1}{2} : 247\frac{1}{2}$, agreement would be almost perfect." This tells us that his own ratio, when evaluated, should be very close to what $3112\frac{1}{2} / 247\frac{1}{2}$ is, when evaluated. Since an evaluation of $3112\frac{1}{2} / 247\frac{1}{2}$ yields 12.5758, and an evaluation of my regeneration of Toomer's ratio yields 12.58, I am confident that Toomer's numerator was close to 2922, and that his denominator was close to 232.
89. Just as Toomer's 1973 evidence was too weak to say that it was proven that 3438' was the base radius of the Hipparchan chord table.
90. O. Neugebauer 1975, p. 299; H. Thurston 1994, p. 128; D. Pingree 1996, pp. 126-127; V. J. Katz 1998, p. 212.

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