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MULTIPLICATION AND DIVISIBILITY OF NUMBERS - THE SŪTRA WAY

S. DAS*

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Arithmetical computations of multiplication and divisibility of numbers are carried out more simply and faster using four algorithms of Vedic Mathematics. The algorithms are in the form of two *sūtras* (theorems) and two *upa* or *sub-sūtras* (corollaries). *Sūtras* as a form of literary composition is commented upon and the possible motivation for adopting this style of writing in India is given. The relevance of *sūtras* to modern times is pointed out. An etymologic analysis of the four formulae and sub-formulae is presented and their cryptic meanings are explained. The modus operandi of these *sūtras* algorithms when applied to the solution of problems on multiplication and divisibility of numbers are illustrated by means of examples. The study shows that in all cases, the number of mathematical operations and the corresponding computational time by *sūtras* methods do not exceed those by conventional methods. A number of 'comments' are appended to the end of each problem. Educational aspect of Vedic Mathematics is discussed and a few research problems are suggested.

Key Words : Algorithms, Bhārati Kṛṣṇa, Calculation Methods, Calculational Time, Divisibility of Numbers, Etymology, Mathematical Operations, Multiplication, Sūtras, Vedic Mathematics

INTRODUCTION

Śrī Bhāratī Kṛṣṇa Tīrthajī's Vedic Mathematics[1] constitutes an original and unique system of mathematics using *sūtra* style and forms a new category of Vedic Mathematics[2]. The system provides an alternative approach for solving a large variety of problems in classical mathematics and forms a new class of mathematical literature[3-7]. Vedic Mathematics comprises of 16 theorems *(sūtras)* and 13 corollaries *(upa or sub-sūtras)*, 45 words (single or compound) in all. The derivations of these *sūtras*, which are composed in *Sanskrit* language, show great insight and ability to synthesize on the part of the author. The salient features of this delightful mathematical system as well as the controversy over the source and the antiquity of the work arising from the word *Vedic* in the title of the book have been discussed by the author elsewhere[8] and possible reasons for the choice of such a title given. The educational and research aspects of this new theory and method of calculations have also been discussed[5,7-9] and the importance of the work to the history of twentieth-century science has been acknowledged[10]. The

* Theoretical Physics Division, Bhabha Atomic Research Centre, Trombay, Bombay-400 085.

book has been reviewed many times [3,6,7,11]. Prof. Bhatnagar[12], and more recently, Bhanu Murthy[13] have provided proofs to some of the results in Vedic Mathematics.

In our earlier paper[8], we have shown how the simple techniques of Vedic Mathematics for doing algebraic computations give quick results. We discussed therein, too, the many potentials of Vedic Mathematics as a mathematical system. The aims of this paper are: (i) To show that in Vedic Mathematics superior techniques are available using which the arithmetic problems of multiplication and divisibility of numbers can be solved in a simpler, faster and more straightforward manner compared to the conventional methods; (ii) To analyse, from etymological point of view, the four *sūtra* formulae used in the present investigation; (iii) To give possible reasons for the development of *sūtra* style of writing in India and point out its relevance to modern times; and (iv) To present certain educational and research aspects of Vedic Mathematics.

THE SŪTRA STYLE OF COMPOSITION

The idea of storing and transferring knowledge (information) in the highly condensed aphoristic or *sūtra* style occurred in India millenia ago. In the Vaidika age, several works called *Sūtras* pertaining to each *Veda* were composed such as: the Śrauta *Sūtras*, the *Gṛiha Sūtras*, the *Dharma Sūtras*, etc., which set forth detailed directions and rules for the performance of different rituals[14]. Subjects like phonetics *(śikṣā)*, grammar *(vyākaraṇa)*, etymology *(nirukta)*, metrics *(chhandas)* and astronomy *(jyotiṣa)* were also written in aphoristic style. The *sūtra* period in Vedic times, perhaps, marks the transition between the vedic and the classical age. The *sūtra* style of writing is noted for its avoidance of verbs, great economy of words, and an artificial form. On account of these, the *sūtras* always remained enigmatic excepting those who were initiated to them. The enigmatical character of the *sūtras* on one hand necessitated the *bhāṣyas* which aimed at elucidating the *sūtras*, and on the other paved the way for the development of the śloka form of composition. The *Parāśara-purāna* defines the *bhāṣyas* as follows:

सूत्रार्थो वर्ण्यते यत्र वाक्यैः सूत्रानुसारिभिः।
स्वपदानि च वर्ण्यन्ते भाष्यं भाष्यविदो विदुः।। इति

The original meaning of the *sūtra* was that of a whole collection of rules. But, later, *sūtra* came to denote both a whole collection of rules and a name for a single *sūtra*. We wish to point out here that the German word 'band' for book means a 'string (a tape, a ribbon, volume, etc.)'[15] which is the lateral translation of the word *sūtra*[8].

It is true that "the great objective of *sūtra* style of composition was shortness"[16]; but to go to the extent of saying that 'an author rejoiceth in the economising of half a short vowel as much as in the birth of a son' means something more than just trying to be short/terse. It indicates feelings of both obsession as well as of relief. Obsession may be justified from pure knowledge point of view as expression of an intense inner urge or intellectual curiosity. But there was no pressing necessity to have such obsession since there was no information explosion in those days. One may surmise that perhaps some of these exceptionally fine minds were inspired to develop a form of writing with which it would be possible to store information with the greatest economy of words. But relief comes from alleviation of pain, suffering or hardships. These considerations make me believe - contrary to what is commonly believed - that *sūtra* form of literature were not composed solely because they could easily be committed to memory. The more likely motivation for this form of composition, in my opinion, had a material basis. The writing material (palm-leaves) was not readily available everywhere and at all times, and it was difficult to store them for long period of time. Writing on palm-leaves was also not easy. These factors might have compelled the author in ancient times to be extremely concise.

As society evolves to higher and higher level of technology, storage of large volume of information becomes a matter of grave concern because of fear of piracy, sabotage, etc. Under such circumstances, storing information in mind in the form of *sūtras* seems to be the safest of all technologies. Since the *sūtra* style of composition has a scientific base where information is transferred logically with great economy on words, it can be well adapted to modern computers and possibility exists for developing machine or assembly language based on *sūtra*. These are two unforeseen practical utilities of *sūtra* style of writing which are of relevance to present as well as to future.

There are occasions when *upa-sūtras* are used concurrently with *sūtras*. In the word *upa-sūtra*, the prefix *upa* signifies its subordinate status to *sūtra;* otherwise, *upa-sūtra* has the same connotation as *sūtra*. For example, the word *upanisad* is formed from the *dhātu* or root षद् *(ṣad)* with kwip pratyaya using the *upasargas* or prepositions up and *ni*. Here *ṣad* means *'viśaraṇa* (dissolution), गति (motion, progress, procedure) and *avasadan* (disheartening)', *upa* means 'near *(समीप)*, *ni* means 'completeness or certainty' and in one interpretation, *Upaniṣad* means 'teaching given while sitting near' *(upaniṣadyate guroḥ samīpe sthitvā nischitatayā avabudhya prāpyata-ityupaniṣad)*.

ETYMOLOGY OF THE FOUR SŪTRA ALGORITHMS

For the solution of problems discussed in this paper, the four *sūtra* algorithms used are:

निखिल नवतश्चरमं दशत: nikhilam *navataścaramaṁ daśataḥ* which literally means "All from nine and the last from ten":

एक न्यूनेन पूर्वेण *ekanyūnena pūrveṇa whose cryptic meaning is:* "By one less than the previous one";

आनुरूप्येण *ānurūpyeṇa* which simply means "Proportionately"

वेष्टनम् veṣṭanam which means "Osculation".

Of these, the first two are *sūtras* and the last two are *sub-sūtras.*

The *sūtra nikhilaṃ navataścaramaṃ daśatah* consists of one single word *(daśataḥ)* and two compound words *(nikhilaṃ and navataścaramaṃ).* The word निखिल *(ni-khila)* is a noun and means 'complete, all, whole, entire' and issued in this sense in *Upaniṣads,* in Manu's *Law-Book* and in *Mahābhārata.* It is formed by the blending of two words : खिल *(khila)* and नि *(ni). Ni* is indeclinable and is always prefixed to verbs or to nouns; in the latter case, it has the meaning of `negation or privation'. The word *khilam* is a noun and means 'a space not filled up, gap', `that which serves to fill up a gap, supplement (of a book, etc.)'. In algebra, it may mean an `insolvable problem'. Hence nikhiLaṃ means "All". The compound *navataścaramaṃ* consists of two words : नवत : *(navataḥ)* and चरम *(caramaṃ). Nava,* in the present context, means 'nine', and, therefore, *navataḥ* means "From ueine". The word *caramaṃ* is indeclinable and means `last' *(Mahābhāratā, ii, iii),* at last' or `at the end' *(Rājataraṃgiṅi, v, 7).* In Manu's *Law-book* (ii, 194), it is used in genitive case to mean 'after any one'. दश *(Daśa)* means 'ten' and दशत: *(daśatah)* means 'from 10'. Hence, the *sūtra* is translated as: "All from nine and the last from ten." The words *nava, daśa* and *carama* occur in various forms at many places in our *śāstras* and *sanskṛit* literature. As examples: the indeclinable *naakṛitvas,* '9 times' *(in vedāntas); nava guṇita,* 'multipled by 9'; navatva, `an aggregate of 9'; the indeclinable *navadha,* '9 times, in 9 ways, into 9 parts' *(in Atharva-Veda, Upaniṣad); navati,* '99' *(but in Ṛg-veda* it denotes `any large number'); navarasika, `the rule of proportion with 9 terms comprising 4 proportions'; *navavidha,* `9-fold or consisting of 9 parts' *(in Kauśika-sūtra* and *Bhāgavata Purāṇa);* navasas, `by nines', or `9 by 9'; navaka, `consisting of 9' *(in Ṛg-veda prātisākhya); navata,* `the 90th' *(in Rājataraṅgiṇī); navati* for `90' in *Ṛg-veda;* navama, `the 9th' in *Atharva-veda* and `of 9 kinds, ninefold' in Hemādri's *caturvarga-cintāmaṇi.*

The word *carama* is used in *Ṛg-veda* (viii, 59, 3 and viii, 20, 14), *Taittirīya saṃhitā* (i, v) and *Bhāgavata purāṇa* to mean `last, ultimate, final'. *Carama* meaning `outermost', ie, `first or last, opposed to the middle one' is used in *Ṛg-veda* (viii, 61, 15). In Kapila's *Sāmkhya-pravacana* (i,72), it takes the meaning `later'. In Buddhist literature, it is used to denote a particular high number. *Caramatas* occurs in *Atharva-veda* (xix, 15, 3) and *Maitrāyani saṃhitā* (iii, 10, 1) as an indeclinable word to mean `at the outermost end'. The word *daśat* means `a decade' in *Maitrāyani Saṃhitā* (i), *Taittirīya saṃhitā* (vii), *Taittirīya brāhmaṇa* (i), *Śatapatha Brāhmaṇa* and *Tāṇḍya Brāhmaṇa*. *Daśama* has the meaning of 'the 10th' in *Atharva-veda* (v). *Daśamam* means 'for the 10th time' in *Ṛg-Veda*, and 'the 10th stage of human life' in *Atharva-veda* (iii, 4, 7), *Tāṇḍya Brāhmaṇa*, and *Gautama's Dharma-śāstra*. The word *deśin* occurs in *Śatapatha Brāhmaṇa* (xiii), *Aitareya Brāhmaṇa*, *Latyāyana* and *Maśaka Ṛg-veda Prātiśākhya* meaning 'having 10 parts'.

The *sūtra ekanyūnena pūrveṇa* means "By one less than the previous one". It is composed of one compound word एकन्यूनेन *(ekanyūnena)* and one single word पूर्वेण *(pūrveṇa)*. *Ekanyūnena* is a blending of four words: *eka, ni, ūna*, and *enȧ*. एक *(ekā)* means 'one' and is derived from the root, 'i (इ)', with *Unādi-sūtra*. *With na preceding or following, it means 'no one, no body', e.g., the words ekayā na or ekān* na are used before decade numerals to lesson them by one as in *ekān* na *trinśat*, twenty nine *(Ṛg-veda)*. With and without *eva*, it means 'alone, solitary, single, happening only once, that one only' *(Ṛg-veda)*. It has the meaning of 'the same one and the same, identical' in *Śatapatha Brāhmaṇa* (v), *Kātyāyana Srauta-sūtra* and Manu's *Law-book*; 'one of two or many' in *Śatapatha Brāhmaṇa, Kātyāyana Srauta-sūtra, Mahābhārata* and *Hitopadeśa*. *Eka* repeated twice, either as a compound or uncompounded, may have the sense 'one and one', 'one by one', *(Ṛg Veda*, i, 20, 7; 123, 8; *Rāmāyaṇa, Bhāgavata Purāṇa)*. It is used in the sense of 'single of its kind, unique, singular, chief, pre-eminent, excellent' in *Raghuvaṃśa, Kathāsaritsāgara and Kumāra-saṃbhava*. Some of the words with *eka* as prefix are: *ekạ-citīka, eka-hansa, eka-pushkala, eka-nayana*, etc. The word न्यून *(ny-ūna)* is a compound noun and is formed from ni with *ūna*. It means 'less, diminished, defective, deficient (opposite to *ati-rikta, adhika, pūrna*)', 'destitute or deprived of (in instrumental case or in compound)', 'inferior to (in ablative case)' and is used in that sense in *Brāhmaṇas, Gṛihya* and *Srauta-Sūtra, Mahābhārata* and by Susruta. But, the word न्यून *(nyūnaṃ)* is indeclinable and means 'less'. The indeclinable *nyūnataram* means 'still less, lower or deeper'. *Nyūnatva* means 'want, deficiency, incompleteness'. *Yājñavalkya* uses the word *nyūna-vibhākta* to mean 'one who has received too little or too much at the division of an inheritance'. *Nyūna-vibhāga* means 'unequal partition'. In *Sāṅkhāyana Gṛihya-sūtra, nyūnādhika* means 'less or more, unequal, inequality, difference'. In *Bhaṭṭi-kāvya*, words like *nyūnayati* and *nyūnī* mean 'to lessen, to diminish'. The ending *ena* is a pronominal base meaning 'he, she, it', 'this, that'. *Ena* is used for certain cases of the third personal pronoun such as in the accusative (singular, dual, plural), in the instrumental (singular) and in genitive locative (dual). Therefore, *ekanyūnena* means 'one less'. The word पूर्व *(pūrva)* is a noun but is declined like a pronoun when implying relative position, whether in place or time. It means 'being before or in front, fore, first' *(Ṛg-veda)*. When used in ablative case, it means 'eastern, to the east of'. In ablative case or in compound, it means 'former, prior, preceding, previous to, earlier than' as in *gaja-pūrva (Ṛg Veda)* which means 'preceding the number eight, i.e., seven, the seventh' *(Śrutabodha)*. Here, *gaja* denotes the number 'eight'

(Sūryasiddhānta). In fine composition, it is often taken to mean `formerly`or before'. In the word *smita-pūrvā* (speech accompanied by smiles), it means 'preceded or accompanied by, attended with'. Sometimes it is not translatable as in *mṛidu-pūrvā-vāk* (kind speech). It has the meaning `ancient, old, customary, traditional' *(Ṛg-veda).* It also carries the meaning 'first (in a series), initial, lowest (opposite the uttara)' in Manu's *Law-book* (viii, I, 20). It can mean `an ancestor forefather' *(Ṛg-veda)* or `foregoing, aforesaid, mentioned before'. *Pūrva* can mean 'a fore part'. In Buddhist literature, it denotes a particular high number (applied to a period of years). The word *pūrvaṃ* is indeclinable and means 'before, formerly, hitherto, previously' *(Ṛg-veda)* and is often used in the beginning of a compound, e.g., *pūrva-kārin.* In fine composition, *pūrva* takes the meaning `with'. *Pūrveṇa* is indeclinable and means `in front, before, eastward, to the east of' *(śatapatha Brāhmaṇa).*`

In the sub-*sūtra* *ānurūpyeṇa* आनुरूप्य *(ānurūpya)* or *ānurūpyam* is a noun and comes from the word *anu-rūpa* and means 'conformity, suitableness' *(Sāhitya-darpaṇa).* The word अनुरूप *(anu-rūpa)* means `following the form, comformable, corresponding, like, fit, suitable, adapted to, according to'. *Anurūpam* and *anurūpeṇa* are indeclinable words and mean 'conformably, according'. अनु *(anu)* is indeclinable and when prefixed to verbs and nouns, expresses `after, along, alongside, lengthwise, near to, under, subordinate to, with'. As prefix to nouns, especially in adverbial compounds, it means 'according to, severally, each by each, orderly, methodically, one after another, repeatedly.' As a separable preposition (with accusative), it means `after, along, over, near to, through, to, towards, at, according to, in order, agreeably to, in regard to, inferior to'. As a separable adverb, it takes the meaning `after, afterwards, thereupon, again, further, then, next.' The noun *rūpa* means `any outermost appearance or phenomenon or colour, form, shape, figure' *(Ṛg-veda).* In fine composition or in compounds, *rūpa* is used in the form *rūpeṇa.* Under such circumstances, it often has the meaning `having the form or appearance or colour of,' 'formed or composed of', 'consisting of', 'like to'. Connected with the verb, it implies 'very well'. In *Vājasaneyi Saṃhitā,* it means 'nature, character, peculiarity, feature, mark, sign, symptom'. In Manu's *Law-book* and in *Kathāsaritsāgara,* it means 'likeness, image, reflection, circumstances (opposite to 'time' and 'place')'. In Varāhamihira's *Bṛihat saṃhitā* and *Gaṇitādhyāya,* it denotes `a term for the number one, probably a rupee coil'. In algebra, it is `the arithmetical unit'. In plural, it can denote `an integer number, a known or absolute number, a known quantity as having specific form'. In grammar *(Pāṇini,* i, I, 68*),* it means 'any form of a noun or verb (as inflected by declension or conjugation)'. In *Atharva-veda* (xviii, 3, 40), it is a word of unknown meaning. Hence, the general meaning of the *sūtra* is given as: "proportionately".

The sub-*sūtra* *veṣṭanaṃ* (neuter gender) is a single word, *veṣṭana* being the noun form of the verb वेष्ट *(veṣt).* *Veṣṭana* means 'the act of surrounding or encompassing or enclosing or encircling' *(Gṛihya* and *śrauta-sūtra, Kāvya* literature, *Kathāsaritsāgara),* 'anything that surrounds or wraps, a bandage, band, girdle' *(Mahābhārata, Pañcatantra),* 'an enclosure, a wall, a fence' *(Meghaduta).* It also refers to 'a particular attitude in dancing' or 'a rope round the sacrificial post'. *Sāhitya-darpaṇa* gives the meaning of *veṣt* as 'to wind or twist around'. In *Atharva-veda,* it has the meaning of 'to adhere or cling *to' (in locative case).* In *Taittirīya Brāhmaṇa,* Manu's *Law-Book* and *Mahābhārata,* it means 'to wrap up, envelop, enclose, surround, cover, invest, beset'. In *Śvetāśvatara upaniṣad, it means 'to cause to shrink up.' In Śatapatha Brāhmaṇa,* the word *veṣṭita* means 'enveloped, bound round, wrapped up, enclosed, surrounded, invested, beset'. In *Amarakosaḥ*[17], सूत्रवेष्टनम् *(sūtra-veṣṭanaṃ)* is a name for 'कपड़ा बुनने के लिए जुलाहा आदिको सूत लपेटने'. Therefore, the *sūtra* is translated in a general way as: "osculation".

MULTIPLICATION OF NUMBERS

For performing the arithmetical computation of multiplication, Vedic Mathematics provides two *sūtras: nikhileaṃ navataścaramaṃ daśataḥ* and *ūrdhva-tiryagbhām.* Of the two, the second is the general formula applicable to all cases of multiplication. The first is a special formula whose applicability is restricted to problems whose multplicand and multiplier contain the same number of digits and the numbers are such that one can choose a base (in powers of ten) that is nearest to the numbers to be multiplied. In this article, we shall discuss the applications of the special *sūtra.*

PROBLEM 1

Current Method

74 x 92 = 74—26
92—8
66 | 08 = 6808
2

74
92
148
666
6808

The modus operandi is as follows:

Since the given numbers 74 and 92 consist of two digits, we take our base to be 10^2 = 100 which is nearest to them.

Put the two numbers 74 and 92 above and below on the left-hand-side as shown.

Subtract each of these two numbers from the base 100 and write the differences 26 and 8 on the right-hand-side with a connecting minus sign (-) between them to indicate that the numbers to be multiplied are less than the base.

The product of the given numbers will consist of two parts, the left and the right.

We demarcate this by drawing a vertical dividing line as shown.

The left part is obtained by cross-subtracting either the deficiency 8 on the second row from the original number 74 in the first row or the deficiency 26 on the first row from the original number 92 in the second row. In either case, we get the same remainder 66.

The right-part of the answer is obtained by multiplying the two deficit figures 26 and 8 vertically. The product is 208 which is a 3-digit number. But our base being 100, we can have only a two-digit number on the right-part. We, therefore, retain only the last two digits 08 of this 208 on the right-side and "carry" the surplus portion 2 to the left and add. The left-side gets augmented to 68 and the result of multiplication is 6808.

Problem 2

9999999996 x 9999999996 = ?

Take 10000000000 as the base and proceed as in Problem 1 to obtain:

<table>
<tr><td></td><td></td><td><u>Current Method</u></td></tr>
<tr><td>9999999996 -0000000004</td><td></td><td>9999999996 x 9999999996</td></tr>
<tr><td>9999999996 - 0000000004</td><td>=</td><td>(10000000000 - 4) (10000000000 - 4)</td></tr>
<tr><td>————————————</td><td>=</td><td>100000000000000000000 + 16</td></tr>
<tr><td>9999999992 | 0000000016</td><td></td><td>- 80000000000</td></tr>
<tr><td>99999999920000000016</td><td>=</td><td>99999999920000000000 + 16</td></tr>
<tr><td></td><td>=</td><td>99999999920000000016</td></tr>
</table>

It is seen that the process of vertical multiplication on the right-portion yields the two-digit number 16, where as our entitlement is for ten digits. We, therefore, fill the 8 vacancies with 8 zeros. The right-part, then becomes 0000000016 and the final result is the twenty-digit number 99999999920000000016.

Comments

(i) The basic procedure as described in the above two problems is capable of infinite application.

(ii) The cross-subtraction process may be the reason for the acceptance of the `x' mark as the sign for multiplication.

(iii) To obtain the deficit figures on the right-hand-side, it is not necessary to subtract the original numbers from the base. We can get them by using the original *sūtra.* The rule is to subtract the last digit of the given number from 10 and all the other digits of the number from 9. This process helps to perform on-sight subtraction and accelerate the calculation.

PROBLEM 3

In the foregoing two problems, both the multiplicand and the multiplier were a little less than the working base. We now take an example where the numbers are a little more than the base. The procedure is identical except that instead of putting down the differences on the right-hand-side and cross-subtracting, we put down the surplus of the given numbers from the base and cross-add.

To multiply 1018 by 1007 ?

Choose the base to be 1000 and proceed as described above. We obtain:

			Current Method
1018 x 1007	=	1018 + 18	1018
		1007 + 7	1007
		1025 + 126 = 1025126	7126
			0000
			0000
			1018
			1025126

PROBLEM 4

The problem here is of mixed type, i.e., one of the numbers to be multiplied is a little above the working base and the other a little below. For example:

10005 x 9997 = ?

We proceed as before cross-subtracting where there is 'minus' sign and cross-adding where there is 'plus' sign to obtain the left-part of the answer. However, since the vertical multiplication produces a negative product, the right-part is subtracted from the left-part to arrive at the final result. To facilitate the subtraction process, *Bhāratī Kṛṣṇa* has introduced the *'vinculum'* (a simple 'Bar') symbol to indicate that the product of the vertical multiplication is a negative number.

In the present problem, we take 10000 as the base since it is nearest to the numbers to be multiplied and write

	Current Method
10005 + 5	10005
9997 - 3	9997
100020015	70035
= 10019985	90045
	90045
	90045
	100019985

COMMENTS

(i) The subtraction of the *vinculum* portion (right-hand-portion) can be simplified by using the original *sūtra*, i.e., all from 9 and the last from 10.

(ii) In *vinculum* notation, 32 = (30-2) = 28 and so on.

(iii) Compare the *'vinculum'* symbol with the 'Bar' symbol used in log-arithmic calculation.

PROBLEM 5

In all the problems so far, the numbers to be multiplied were such that at least one of them was near to the base selected. This resulted in a small difference or surplus, thereby rendering the vertical multiplication easy. We now consider an example where none of the numbers in the multiplication is near a convenient base. Problems of this type can be solved with the aid of the *upa-sūtra ānurūpyeṇa*. Expanded, what this sub-formula tells us is: "Where there is a rational ratio-wise relationship, this ratio should be taken into account and a proportionate multiplication or division should be performed depending on the situation." Its application to problems of multiplication consists in choosing a "working base" which is a convenient multiple or sub-multiple of the theoretical base, performing the necessary operations and at the end multiplying or dividing the result in the same proportion as the original base is to the working base.

To find 635 x 502 = ?

We take 1000 as the theoretical base and its sub-multiple 1000/2 = 500 as the working base since 500 is conveniently near to the numbers to be multiplied. Proceeding as before, we obtain 637/270. As 500 is one-half of 1000, we divide 637 proportionately, i.e., by 2 and obtain the fractional quotient 318 (1/2) as the left-hand portion. The right-hand portion 270 remains unaltered. Now the fraction 1/2 is carried over to the right-hand as 500 and added to 270 to give the final value of the right-portion as 770.

$$635 \times 502 = \frac{635 + 135}{502 + 2}$$

$$637 \mid 270$$

$$= 318\ (1/2) \mid 270 \qquad \textit{(By sub-sūtra Ānurūpyeṇa)}$$

$$= 318 \mid 270$$

$$\quad 500$$

$$318 \mid 770 = 318770$$

Problem 6

In this last problem on multiplication, we discuss a very special type wherein the multiplier-digits consist entirely of nines. This type of multiplication occurs in advanced mathematical astronomy and can be solved by the *sūtra ekanyūnena pūrveṇa*. Here, the word *pūrva* is interpreted to mean "multiplicand" and the sÛtra means: 'Decrease the multiplicand by unity'. There are three cases under this category. First, the multiplicand and the multiplier consist of the same number of digits. Second, the multiplicand consists of a smaller number of digits than the multiplier. Third, the multiplier contains a smaller number of digits than the multiplicand.

(a) Multiplicand and multiplier have same number of digits :

The procedure is to decrease the multiplicand by 1. This gives the left-part of the answer. The right-part is obtained by taking the complement of the digits on the left-part of 9. For example:

$$5671349201 \times 9999999999 = 5671349200 + 4328650799$$

$$= 56713492004328650799$$

Current Method

$$5671349201 \times 9999999999 = 5671349201 \times (10000000000-1)$$

$$= 56713492010000000000 - 5671349201 = 56713492004328650799$$

(b) Digits in multiplicand less than digits in multiplier :

In this case, the basic procedure remains the same. The only difference is that we fill the blanks in the multiplicand on the left-hand-side with the required number of zeros, perform the calculation exactly as before and then omit the zeros. For example:

64 x 9999999 = 0000064 x 9999999 = 0000063 | 9999936 = 639999936

Comment

In the above two cases, it is essential that the multiplicand and the multiplier consist of the same number of digits.

(c) Digits in multiplier less than digits in multiplicand :

To find 1067345 x 99999 ?

The procedure here is to first divide the multiplicand (1067345) off with a vertical line as 10:67345 so that the right-hand portion consists of as many digits as the multiplier (here 5 digits). Then take the *ekanyūna* (one less) of the multiplicand which is 1067344 and from this subtract the previous portion 10 on the left of the multiplicand to get 1067334 as the left-part of the answer. The right-part of the answer is obtained by subtracting the right-hand-side part 67345 of the multiplicand using the nikhilam rule "All from nine and the last from ten". This gives 32655. Therefore, the result of multiplication is 106733432655.

The entire process can be written as

```
1067345 x 99999   =   10:67345:
                      10:67344:
                         -10:67345
                      ---------------
                      10:67334:32655   =   106733432655
```

Divisibility of Numbers

Given a number (dividend), no matter how large, and another number called the divisor D, the problem of divisbility is to determine whether the dividend is exactly divisible by the divisor. For small divisors like 2, 3, 4, 5, 6, 9, 11, 13 etc., this is done by the various divisibility tests. But for larger divisors, the usual procedure is to perform an actual division. This is cumbrous and time consuming.

Vedic Mathematics, however, provides an easy way of tackling this problem of divisibility using the *sub-sūtra veṣṭanaṃ* or "osculation". In *veṣṭana*, there are two processes: the P-process or positive osculator method, and the Q-process or negative osculator method. In both the processes, osculation is performed using osculators or *veṣṭanas* ("वेष्टन"). The procedure is to first determine the osculator and then do the osculation. In practice, the process that gives a smaller osculator is used.

The P-process or positive *veṣṭana* is applicable to divisors that end in 9. If not, then by suitable multiplication, the divisor is made to yield a product that ends in 9. The osculator P in this process is the *ekādhika* ('one more') of the divisor and is obtained by dropping out the last digit, i.e., 9 and increasing the penultimate digit by 1. As examples:

Divisor	=	9	Osculator = 0+1 = 1
Divisor	=	19	Osculator = 1+1 = 2
Divisor	=	23	Multiplication by 3 gives 69 and P = 6+1 = 7
Divisor	=	37	Multiplying by 7, we get 259 and P = 25+1 = 26

The Q-process or negative *veṣṭana* is applicable to divisors that end in 1. If not, then by suitable multiplication, the divisor is made to yield a product ending in 1. The osculator Q is obtained by simply dropping 1. As examples:

Divisor	=	21	Dropping 1, we get Q = 2
Divisor	=	17	Multiplying by 3, we get 51. Therefore, Q = 5
Divisor	=	53	Multiplying by 7, we get 371. Therefore, Q = 37

COMMENTS

(i) The sum of the two osculators is always the divisor, i.e., P+Q = D for all divisors, D. Therefore, if one of them is known, the other is automatically known.

(ii) Osculation of any number N by its own *ekādhika* P is either the number N itself or a multiple of it. This is similar to the concept of identity element in modern algebra.

(iii) The positive osculator is an application of *ekādhika pūrva sūtra* and is a process of addition. The negative osculator is an application of the *parāvartya sūtra* is a process of subtraction (leftward). The latter actually means an alternation of plus and minus.

(iv) If the last digit of a divisor is 3, $P < Q$.

If the last digit of the divisor is 7, $Q < P$.

(v) When divisors are small, the osculators are small and individual digits are osculated. For bigger number divisors, groups of digits are osculated. This is known as multiplex *veṣṭana.*

We now proceed to demonstrate the method of *veṣṭana* for finding the divisibility of numbers. The positive *veṣṭana* method consists in multiplying the last digit of the dividend by the positive osculator and adding the product to the previous digit. The process is continued till all the digits of the dividend are exhausted. If the final result of osculation is the divisor itself or a multiple of the divisor or a repetition of a previous result or zero, the original dividend is divisible. For the negative osculators, the procedure *is* same except that the osculated multiplies are subtracted instead of being added. As a consequence, in the Q-process, the result is an alternation of positives and negatives. Therefore, as a safeguard against committing any error, the dividend is marked beforehand alternatively by means of a *vinculum* from the right to left on all the even-place digits.

PROBLEM 7

To find if 2775 is divisible by 19 ?

Since the divisor is 19, the positive osculator is 2.

Put down the digits of the dividend 2775 as shown below.

Multiply the last digit 5 of the dividend by 2, add the product 10 to the previous digit 7 and put down the sum 17 under the second right-hand digit 7.

Now osculate this 17 with 2 and obtain 15. Add this to the digit 7 to the left-hand on the top row to get 22. Put this 22 under 7.

We then osculate this 22 to get 6, add this 6 to the left on the top row and get 8 as the final osculated result. Put this 8 under 2.

As 8 is neither the divisor nor a multiple of the divisor nor zero nor repetition of a previous result, we conclude that 2775 is not divisible by 19.

	STEPS
2 7 7 5	(2x5) + 7 = 17
8 22 17	(2x7+1) + 7 = 22
	(2x2 + 2) + 2 = 8

Alternatively, we can perform a continuous series of osculations of the dividend 2775 by the osculator 2. We get

$2775 \rightarrow 2 \times 5 + 277 = 287 \rightarrow 2 \times 7 + 28 = 42 \rightarrow 2 \times 2 + 4 = 8$

Answer : No

Problem 8

Determine whether 5526687885 is divisible by 149 ?

Here, divisor = 149. Therefore, positive osculator P = 15.

We follow exactly the same procedure as in Problem 1 and write

5	5	2	6	6	8	7	7	8	5
149	99	46	142	19	130	28	61	83	

The steps are:

(15x5) + 8 = 83	(150x0+13)+8 = 19	(15x9+9) + 5 = 149
(15x3+8) + 8 = 61	(15x9+1) + 6 = 142	
(15x1+6) + 7 = 28	(15x2+14) + 2 = 46	
(15x8+2) + 8 = 130	(15x6+4) + 5 = 99	

Since the final result of osculation is 149 which is the divisor itself, the given number is divisible by 149.

Answer : Yes

The current method, which is an actual division, is shown below for comparison.

```
       3 70 9 1 8 6 5
      ________________
149) 5 5 2 6 6 8 7 8 8 5
     4 4 7
       1 0 5 6
       1 0 4 3
      ---------
           1 3 6 8
           1 3 4 1
          ---------
               2 7 7
               1 4 9
              -------
               1 2 8 8
               1 1 9 2
              ---------
                   9 6 8
                   8 9 4
                  -------
                   7 4 5
                   7 4 5
                  -------
                   0 0 0
```

Quotient (Q) = 37091865
Remainder (R) = 0

Problem 9

To find the divisibility of 4898857 by 47?

Since the divisor is 47, we multiply it by 3 to obtain 141. Dropping 1, the negative osculator Q =14.

Now put down the digits of the dividend with vinculum marks above all the even-place digits from right to left as shown. Multiply the last digit 7 of the dividend by 14, subtract from this product 98 the previous digit 5 on the top row and put down the difference 93 under the second right-hand digit 5. Now osculate this 93 with 14 by multiplying 14 with 3 and subtracting 9 from the product 42. This gives 33. Add this to the digit 8 to the left-hand on the top row to get 41. Put this 41 under 8. Now osculate this 41 with 14 by taking the product of 14 and 1, subtracting 4 from the result 14 and 1, subtracting 4 from the result 14 and obtain 10. Subtract the digit 8 on the left-hand on the top row from this 10 to get 2. Put this 2 under 8. Continuing this process as long as the digits of the dividend last, we get 94 as the final osculated result. Since 94 (=47x2) is a multiple of the divisor 47, we conclude that the given number 4898857 is divisible by 47.

Steps

4 8 9 8 8 5 7	(14x7-0)-5 = 93	(14x2-0)+9 = 37
94 87 37 2 41 93	(14x3-9)+8 = 41	(14x7-3)-8 = 87
	(14x1-4)-8 = 2	(14x7-8)+4 = 94

Comments

(i) In finding the divisibility of the number in the second problem above by the current method, we had to perform 58 operations. In addition, the chances of committing numerical error during computations are more, plus a lot of time is spent in writing the steps and checking and rechecking of results. In contrast, the number of operations by the method of osculation is only 45. But in the third problem, the number of operations by both the methods is about the same, i.e. 22. In all cases, however, simplification of calculation results by replacing the comparatively bigger divisor by a smaller osculator and avoiding multiplication by larger-digit numbers.

(ii) *Veṣṭana* process can only tell if a number is divisible by another number. It can not give the quotient and the remainder.

(iii) True to the root meaning of the word "*Veṣṭana*", the process really looks like "twisting around" as in the calculations.

CONCLUDING REMARKS

To summarize, the *sūtra* approach to doing multiplication and divisibility of numbers using Vedic Mathematics has been presented. The essential idea in this approach is to simplify and accelerate calculation by reducing the number of mathematical operations, replacing operations with larger-digit numbers by smaller digit numbers and by performing, at some places, addition and subtractions where one would normally do multiplication. In the specific applications considered in this paper, nine problems are solved, six on multiplication and three on divisibility of numbers. In all cases, the number of mathematical operations and the calculation time are no more by *sūtra* methods than by current methods (see Table 1). In Problem 2, though *sūtra* method reduces the number of operations by only 35% this accelerates the calculation 4 times. Similar behaviour is seen, to a lesser degree, in Problems 6(a), 6(c), 7 and 8. But in problems 3 and 4, the reduction in calculational time is less than proportionate to the corresponding reduction in the number of operations. In problem 1 and 5 on multiplication, the ratio of the number of operations (N_{cm}/N_{sm}) and the ratio of the computational time (t_{cm}/t_{sm}) are equal by both the methods. Same is the case with Problem 9 on divisibility of numbers. This behaviour in Problems 1, 5 and 9 could either be due to lack of practice on the part of the author to use the *sūtra* method or it is inherent in the method itself. These require further investigations. It is important to point out here that the nikhilaṃ *sūtra* works equally well for problems where the difference (deficit or surplus) is zero, e.g., 77x50 with 50 as the working base and 100 as the theoretical base.

Table 1. Comparisons of The Number of Operations and The Calculational Time Between the Current Methods and The *Sūtra* Methods (SM = *Sūtra* Method, CM = Current Method)

Reference problem number	Number of operations		Ncm/Nsm	Calculational time (appropriate)		
	SM (Nsm)	CM (Ncm)		SM (tsm)	CM (tcm)	tcm/tsm
Problem 1	10	10	1.0	10 s	12 s	1.2
Problem 2	22	30	1.36	30 s	120 s	4.0

Problem 3	7	25	3.6	20 s	30 s	1.5
Problem 4	7	19	2.7	20 s	30 s	1.5
Problem 5	17	17	1.0	22 s	22 s	1.0
Problem 6(a)	11	22	2.0	20 s	70 s	3.5
Problem 6(b)	8	13	1.6	15 s	30 s	2.0
Problem 6(c)	8	10	1.25	25 s	50 s	2.0
Problem 7	9	14	1.55	12 s	30 s	2.5
Problem 8	45	58	1.3	60 s	150 s	2.5
Problem 9	22	22	1.00	40 s	45 s	1.1

Etymological analysis of the four *sūtra* formulae has shown that the words used in composing these formulae not only occur in *vedās, vedāṅgas, epics, purāṇas* and other *sanskṛit* literature, but also they take different meanings in different situations. This multiplicity of meanings make the *sūtras* amenable to multiple interpretations, thereby making it possible for the very same *sūtra* to have applications in more than one area of mathematics. As example, the word *pūrva* is used in *Ṛg-veda* to mean 'in front, fore, first'. In *Śatapatha Brāhmaṇa, pūrveṇa* means 'in front'. Therefore, in the *sūtra ekanyūnena pūrveṇa, pūrva* could be interpreted to mean "multiplicand" to solve Problem 6. Ena in this problem takes its normal meaning 'this, that'. But in the *sūtra ekādhikena pūrveṇa,* which is used to convert vulgar fractions into their equivalent decimal form, *ena* meaning 'by' (instrumental) is interpreted to denote 'multiplication or division'[18]. The rationale, as stated in the above citation, of the second interpretation of ena is that for addition and subtraction, 'to' and 'from' would have been the appropriate prepositions to use and not 'by'. As another example, consider the *sūtra veṣṭanam.* Since the root meaning of *veṣṭana* is related to the act of encircling or winding or twisting around, wherever the mathematical processes look like this, they can be categorised under this *sūtra.*

There seems to exist a close parallel between Vedic Mathematics and the grammar of *Pāṇini*. Just as *Pāṇini* built up the *sanskṛit* language from a few roots which embody certain general concepts, defined different classes of words and then prescribed rules to construct more and more words, in a similar manner, *Bhāratī Kṛṣṇa* formulated the *sūtras* using words which embody certain general principles or concepts and then prescribed suitable problem-solving procedures by appropriate interpretations of the words in the *sūtras.*

Vedic Mathematics may appear difficult at first sight, since it is written in *sanskṛit* language. But once understood, it is a simple system so much so that even children can learn and use the methods. A course on Vedic Mathematics would require about 1000 man-hours compared to the 10,000 man-hours or so that is spent in the present academic curriculum. In order to do multiplication using Vedic Mathematics, it is adequate if multiplication tables are learnt upto 5 times 5 and simple addition and subtraction of single-digit numbers are known. The rest can be done using the *sūtra* formulae.

Swāmī Bhāratī Kṛṣṇa has said that the *sūtras* apply to all mathematical problems[1]. But, since only one volume on the subject has survived, there is incentive to delve deeper into the subject and see how the area of applicability of the *sūtra* formulae can be widened and more results are derived using the existing formulae. For example, Vedic Mathematics provides rules for doing multiplication and divisibility of numbers. But, there is no provision for doing subtraction. This aspect should be looked into. The *veṣṭana* method gives a quick way of finding the divisibility of numbers; but it does not give the quotient and the remainder. One should therefore explore the possibility of finding the quotient and the remainder using the *veṣṭana sūtra*. Similarly, a way should be found to extend the method of osculation to problems where the divisors end in 5 or 2.

REFERENCES

1. *Bhāratī Kṛṣṇa Tīrthajī Mahārāja. Jagadguru Swāmī Śrī. Vedic Mathematics or Sixteen Simple Mathematical Formulae from the Vedas For One-Line Answers to all Mathematical Problems. edited* by V.S. Agrawala. *Hindu Vishwavidyālaya Nepal Rājya Granthamālā* Series. Volume 10. *Banāras* Hindu University. *Vārāṇasi*. 1965.

2. *Gaṇitānand.* In the name of Vedic Mathematics. *Gaṇita-Bhāratī*. 10. (1-4) 75-78. 1988.

3. Bag. A.K.. Review of the book *Vedic Mathematics or Sixteen Simple Mathematical Formulae from the Vedas*. Indian Journal of History of Science. **3**(2). 59-60. 1968.

4. Pickles. J AND NICHOLAS *A.P. Lectures Delivered at the University of London Union*. 1980.

5. Das S. Computational algorithms and Vedic Mathematics. *Society and Science*. **3**. 21-37. 1980

6. Gupta R.C. Review of the book *Vedic Mathematics or Sixteen Simple Mathematical Formulae from the Vedas*. Indian Journal of History of Science. 18. 222-225. 1983.

7. Kapur J.N. The So-called Vedic Mathematics. *Mathematical Education*. 5(4). 200-204. April-June 1989

8. Das S. Solutions of Linear Algebraic Equations and Sums of Fraction - Additions Using Sutra Method. *Indian Journal of History of Science*. **29**(4). 529-564. 1994.

9. Das S, *Abstracts XVIIth International Congress of History of Science*. Acts. Volume 1, Ma 1. P, 31 July-8 August 1985, University of California, Berkeley, USA.

10. Nicholas A P. Review of the book *A Peep into Vedic Mathematics* by B.R. Baliga, Ganita-Bhāratī 6 (1-4), 37-38, 1984.

11. Das. S. Review of the book *Vedic Mathematics or Sixteen Simple Mathematical Formulae from the Vedas For One-Line Answers to all Mathematical Problems*. Physics News. **26**(2), 93-102, 1995.

12. Bhatnagar P L. *Mathematics Teacher*, Vol. X, 83-111, 1976.

13. Bhanu Murthy, T.S. *A Modern Introduction to Ancient Indian Mathematics*, Wiley Eastern Limited, New Delhi, 1992, pp. 59-101.

14. Bose D M, Sen S N and Subbarayappa, B V (Ed). *A Concise History of Science in India*. Indian National Science Academy, New Delhi, 1971, pp. 23.