Madhyamānayanaprakāraḥ: a hitherto unknown manuscript ascribed to Mādhava

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Abstract

It is generally believed that only two works of Mādhava, namely *Veṇvāroha* and *Sphuṭacandrāpti* are extant today. However, it seems a few of his small tracts have also survived. The present article deals with one such small tract entitled '*Madhyamānayanaprakāraḥ*' ('the procedure for obtaining the mean') found among the K. V. Sarma collection of manuscripts. Starting with a graphic description of the geometrical construction involved in the *manda-saṃskāra*, the manuscript proceeds to describe in great detail the concept of *viparītakarṇa*. Along with the formula for the *viparītakarṇa*, it also presents a detailed discussion on its application to bring about a simplification—that enables an astronomer to avoid the strenuous and persevering task of performing an iterative process—in the computation of *aviśiṣṭa-manda-karṇa*. Finally it concludes with the description of the procedure for the computation of the *madhyamagraha* (mean longitude of a planet) from the *manda-sphuṭa-graha* (longitude of the planet corrected for eccentricity), which indeed explains the choice of the title *Madhyamānayanaprakāra*.

Keywords: *madhyama, manda-sphuṭa, viparīta-karṇa, aviśiṣṭa-manda-karṇa*

1 Introduction

The Kerala school of astronomy pioneered by Mādhava (c. 1340–1420) of *Saṅgamagrāma* seems to have blazed a trial in the study of a branch of mathematics

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that goes by the name of analysis today. His major contributions include not only the discovery of the infinite series for π, but also several fast convergent approximations to it. Unfortunately, we do not have any manuscript attributed to Mādhava that contains verses presenting the above series. It is only through the quotations and citations made the successors of Mādhava, we come to know that it was Mādhava who discovered the infinite series for π as well as the power series for the sine and cosine functions, which were re-invented in the West three centuries later.

Mādhava's contributions to astronomy are not as well known as his contributions to mathematics. It is generally believed that only two works of him, namely *Veṇvāroha* and *Sphuṭacandrāpti*[1]—both related to the computation of Moon's positions—are extant today. However, K. V. Sarma ascribes one more work on astronomy entitled *Agaṇitagrahacāra* to Mādhava, which is yet to see the light of the day. Besides editing and publishing several manuscripts, the renowned indologist Sarma, has also painstakingly collected paper transcripts of a plenty of palmleaf manuscripts, that are in possession of various oriental manuscript libraries.

All these transcripts, as well as his personal collection of thousands of books are currently preserved in Sree Sarada Education Society and Research Centre.[2] Among this collection, recently we came across a couple of short tracts in Sanskrit prose that have been identified by Sarma as due to Mādhava. One such tract entitled '*Madhyamānayanaprakāraḥ*' which literally means 'the procedure for obtaining the mean', has been chosen for study in the present article. Here, besides explaining the technical content of the manuscript in the light of modern mathematical language, we also present an edited version of the text along with English translation.

The article is divided into six sections including the present one on introduction. Sections 2 and 3 present the details of the manuscript material employed and the scheme adopted in preparing an edited version of the text. While section 4 is devoted to explain the content of the *mūla-ślokas* (the original verses on which a commentary is being authored) using modern mathematical notations, section 5 presents the edited text along with English translation. In the last and the final section we give detailed explanatory notes on the manuscript material

[1] These two texts have been crtitically edited and published by K. V. Sarma.

[2] An institution that was founded by Sarma himself, by way of donating his own savings as well as invaluable collection of books and manuscripts, to foster indological studies. The address of this institution, currently renamed as K. V. Sarma Research Foundation is: 2, East Coast Flat, 32, II Main Road, III Cross, Gandhi Nagar, Adyar, Chennai 600 020.

by dividing it into various subsections that seemed convenient and appropriate.

2 The Manuscript material

The edition of the text *Madhyamānayanaprakāra* presented below is based on the lonely paper transcript of the orginial palmleaf manuscript in the possession of Indian Office Library, London. The paper transcript, in four pages (19–22) found among the K. V. Sarma manuscript collection and bears the number **KVS 354-C**. It may be noted on the first page of the transcript (see Figure 1), that in the top-left-corner there is a marking that mentions the number **IO 6301, Ms. # 16–18**.

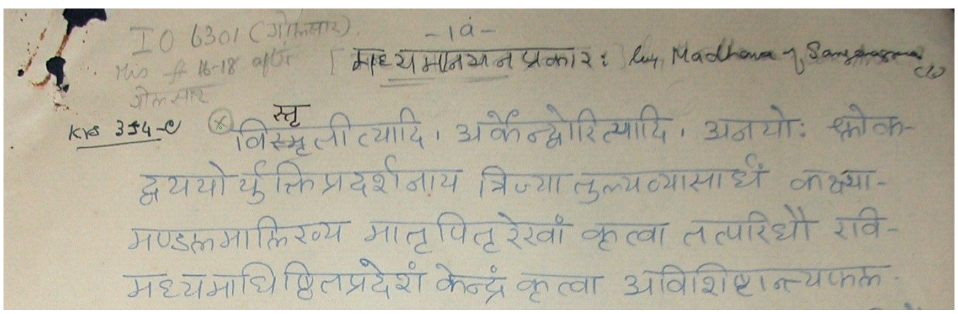


Figure 1: The first few lines of the first page of the manuscript

From this noting (in pencil) made by Sarma, we understand that the original manuscript is with the Library of India Office, London, and that this material is in three folios (16–18). More details about this manuscript have been provided by A. B. Keith in the catalogue prepared by him[3] wherein it has been mentioned that the manuscript is on $8\frac{3}{8}'' \times 1\frac{5}{8}''$ *talipat* (palmleaves) and that it in Malayalam script.

3 Editorial note

While studying the manuscript (**KVS 354-C**), which is in Devanāgarī script as seen from Figure 1, it was quite evident to us that the readings in certain

[3] A. B. Keith, *Catalogue of the Sanskrit and Prakrit manuscripts in the Library of the India Office*, vol. 2, Oxford 1935, pp. 774–5, (no. 6301).

places were definitely fautly. As an illustration we present a couple of examples. In one instance, a string of words was simply repeating (see Figure 2).

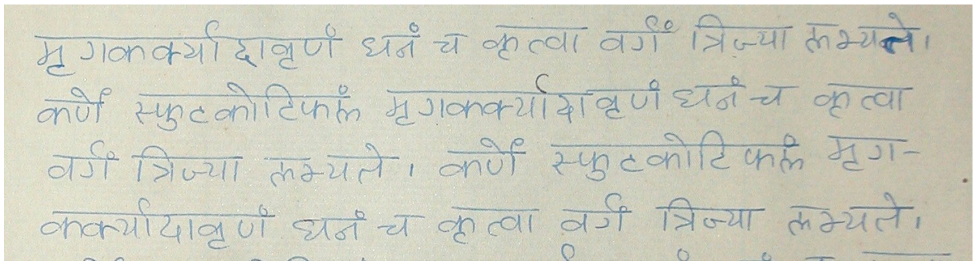


Figure 2: Clip of the manuscript where a string of words get repeated.

As it is pretty obvious that this is due to scribal error, in our version of the edited text this was dropped out. In yet another place (see the fourth line of the clip of the manuscript shown in Figure 3, whose typed version is given below), it was noted that a string of characters were missing from a compound word.

तदर्धं मन्दोच्चोनमा लब्धं अविशिष्टभुजाफलम्।

With reasonable working knowledge in Sanskrit it should not be difficult to make out that there is obviously some error in the second and the third words—though fixing the error requires the knowledge of the subject as well as intelligent guess work. After pondering over the sentence, keeping the context in mind, it became evident to us that the reading should be

तदर्धं मन्दोच्चोनमध्यमाल्लब्धं अविशिष्टभुजाफलम्।

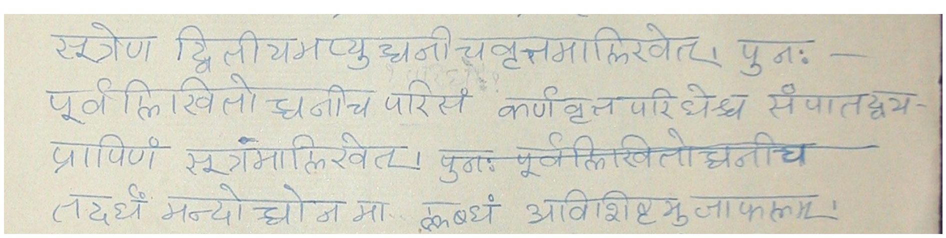


Figure 3: Clip of the manuscript where a few letters are found to be missing.

While presenting an edited version of the text, to the best or our ability we have tried to fix the errors and present a correct version of the original. In doing so, the reading found in the manuscript had to emended at places. Though we have been very careful and judicious in emending the text, for the benefit of scholars to judge for themselves, we have presented the readings found in the manuscript in the form of footnotes with the exception of one type of emendation. On a few places the *daṇḍas*[4] in the manuscript were found to be misplaced and hence had to be removed and inserted in certain other places. Such insertion and deletion of *daṇḍas* have been done silently, where it seemed appropriate to us.

Before we present the edited text and the translation, we offer a brief explanation of the verses that form the *mūla*[5] of the manuscript and also discuss on the authorship of the same.

4 The *mūla* verses

The manuscript essentially presents a detailed commentary on two verses commencing with '*vistṛti*' and '*arkendu*'. In fact, this has been clearly stated in the very opening sentence of the manuscript:

> *'vistṛtī'tyādi | 'arkendvor'ityādi | anayoḥ ślokadvayayoḥ yuktipradarśanāya ...*
>
> [The verse] that commences with *vistṛti*. [The verse] that commences with *arkendvoḥ*. In order to present the rationale behind the [content] of these two verses, ...

Though these two verses happen to be part of the second chapter of the celebrated work *Tantrasaṅgraha* of Nīlakaṇṭha Somayājī, it turns out that Nīlakaṇṭha is not the author of them. Neither is there a mention about the author of the *mūla* verses anywhere in the text of the manuscript itself. However, the quotations made by Nīlakaṇṭha—with due acknowledgement to the authors whom he is quoting—in his works *Tantrasaṅgraha*,[6] *Siddhāntadarpaṇa* and *Āryabhaṭīyabhāṣya* enable us to identify the author of the *mūla* verses to be Mādhava.

[4]This term refers to the vertical bars employed to indicate the end of a sentence particularly while using a *Devanāgarī* script.

[5]The world *mūla* which literally the 'root' or 'basis', is traditionally employed to refer to the text on which commentary is being written.

[6]See for instance, **{TS 1958}**, (II.44).

4.1 About the authorship

Nīlakaṇṭha in his *Tantrasaṅgraha*,[7] soon after quoting the verse *vistṛti* states:

> इति वा कर्णः साध्यः मान्दे सकृदेव माधवप्रोक्तः।
>
> Or, the *karṇa* is to be obtained only once [in this way] in the *manda* process as enunciated by Mādhava.

In the *Nyāyabhāga* of his *Siddhānta-darpaṇa-vyākhyā*[8] (auto-commentary on *Siddhānta-darpaṇa*), in the context of explaining how the problem of mutual dependency involved in the computation of *karṇa* and the radius of the *manda-vṛtta* can be circumvented, Nīlakaṇṭha observes:

> स्ववृत्तकलाप्रमितस्य कर्णस्य सदैव त्रिज्यातुल्यत्वात् तेन मध्यकक्ष्याव्यासार्धा-नयने कर्णानयनविपरीतकर्म कार्यम्। तच्च माधवेनोक्तम् – विस्तृतिदल ...
>
> Since the *karṇa* measured in terms of the minutes of its own circle is [taken to be] equal to *trijyā*, in order to compute the radius of the *kakṣyāmaṇḍala* (the deferent circle), we have to adopt a process that is [exactly] the reverse of the process employed in finding the *karṇa*. And that has been stated by Mādhava [thus] – *vistṛti* ...

In his magnum opus *Āryabhaṭīya-bhāṣya*, while presenting a detailed commentary (that runs to more than 20 pages) on the five verses[9] of Āryabhaṭa that describe the geometrical picture of planetary motion, Nīlakaṇṭha says:

> स्फुटे(न) मध्यमानयने सकृत्कर्म अन्यादृशं माधवोक्तमपि श्रुतम् – 'अर्केन्द्वोः...'
>
> In obtaining the mean from the true, the one-step process enunciated by Mādhava is also heard – '*arkendvoḥ* ...'

These acknowledgements made by Nīlakaṇṭha clearly settle the issue regarding the authorship of the *mūla* verses and leave no room for ambiguity, whatsoever. This, however, is not the case with respect to the prose commentary in the manuscript. Our ascription of it to Mādhava is solely based on the noting made by Sarma—just next to the title (see Figure 1).

[7] See **{TS 2010}**, p. 105.
[8] **{SDA 1976}, p. ...**
[9] *Āryabhaṭīya*, *Kālakriyāpāda*, verses 17–21.

4.2 Explanation of the verse '*vistṛti* ...'

The verse commencing with '*vistṛti*' presents the formula enunciated by Mādhava for finding the *aviśiṣṭa-manda-karṇa*—the iterated hypotenuse associated with the *manda-saṃskāra*. At this stage, it would be useful to include a brief note on *aviśiṣṭa-manda-karṇa* as well as procedure for the computation of the *manda-sphuṭa*[10] in order to make the discussion more edifying. This would also enable the reader to have a better appreciation of the substantial simplification in computation achieved by Mādhava in obtaining *aviśiṣṭa-manda-karṇa* without having to resort to the conventional interative process.

4.2.1 Calculation of *manda-sphuṭa*

The procedure for obtaining the *manda-sphuṭa* from the mean longitude (*madhyama*) of the planet prescribed in the Indian astronomical texts can be understood with the help of Figure 4. Here, $\theta_0 = \Gamma\hat{O}P_0$ represents the longitude of the mean planet (*madhyama-graha*, P_0), $\theta_0 = \Gamma\hat{O}U$ the longitude of *mandocca* (apogee or aphelion) and $\theta_{ms} = \Gamma\hat{O}P$ the longitude of the *manda-sphuṭa* which is to be determined from θ_0. It can be easily seen that

$$\begin{aligned}\theta_{ms} &= \Gamma\hat{O}P \\ &= \Gamma\hat{O}P_0 - P\hat{O}P_0 \\ &= \theta_0 - \Delta\theta. \end{aligned} \tag{1}$$

Since the mean longitude of the planet θ_0 is known, the *manda-sphuṭa* θ_{ms} is obtained by simply subtracting $\Delta\theta$ from the *madhyama*. In the figure, $P_0P = r$ and $OP_0 = R$ represent the radii of the epicycle and deferent circle respectively. Considering the right-angled triangle OPQ, we have

$$\begin{aligned}K = OP &= \sqrt{OQ^2 + OP^2} \\ &= \sqrt{(OP_0 + P_0Q)^2 + OP^2} \\ &= \sqrt{\{R + r\cos(\theta_0 - \theta_m)\}^2 + r^2\sin^2(\theta_0 - \theta_m)}. \end{aligned} \tag{2}$$

Also

$$\begin{aligned}K \sin\Delta\theta &= PQ \\ &= r\sin(\theta_0 - \theta_m). \end{aligned} \tag{3}$$

[10] The longitude of the planet obtained by applying the *manda-saṃskāra* (equation of centre) to the mean longitude of the planet.

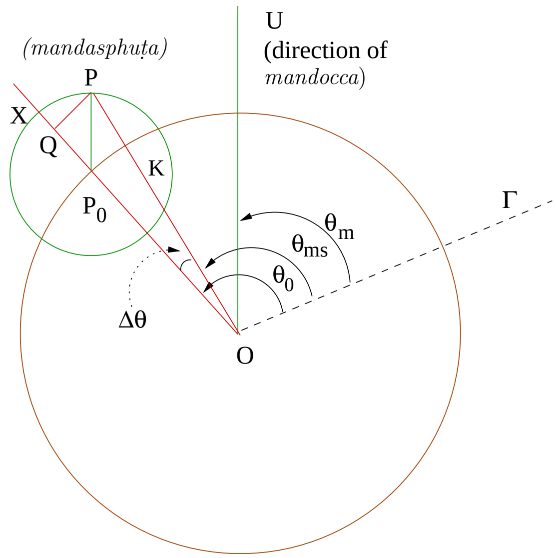


Figure 4: Geometrical construction underlying the rule for obtaining the *manda-sphuṭa* from the *madhyama* using the epicycle approach.

Multiplying the above by R and dividing by K we have

$$R \sin \Delta\theta = \frac{r}{K} R \sin(\theta_0 - \theta_m). \tag{4}$$

According to the geometrical picture of planetary motion given by Bhāskara I (c. 629), the radius of the epicycle *manda-nīcocca-vṛtta* (r) employed in the the *manda* process is not a constant. It varies continuously in consonance with the hypotenuse, the *manda-karṇa* (K), in such a way that their ratio is always maintained constant and is equal to the ratio of the mean epicycle radius (r_0)—whose value is specified in the texts—to the radius of the deferent circle (R). Thus, according to Bhāskara, as far as the *manda* process is concerned, the motion of the planet on the epicycle is such that the following equation is always satisfied:

$$\frac{r}{K} = \frac{r_0}{R}. \tag{5}$$

Thus the relation (4) reduces to

$$R \sin \Delta\theta = \frac{r_0}{R} R \sin(\theta_0 - \theta_m), \tag{6}$$

where r_0 is the mean or tabulated value of the radius of the *manda* epicycle.

4.2.2 Bhāskara's method for obtaining the *aviśiṣṭa-manda-karṇa*

It may be noted that in (5), RHS is the ratio of two fixed values, namely the mean epicycle radius and the *trijyā*. This, however is not the case with respect to quantities appearing in the LHS. Here, both the numerator and the denominator are variables. Hence the question arises, as to how one can obtain the values of r and K at any given instant, though the ratio is always a constant. For this, Bhāskara prescribes an iterative procedure called *asakṛt-karma*, by which both are simultaneously obtained.

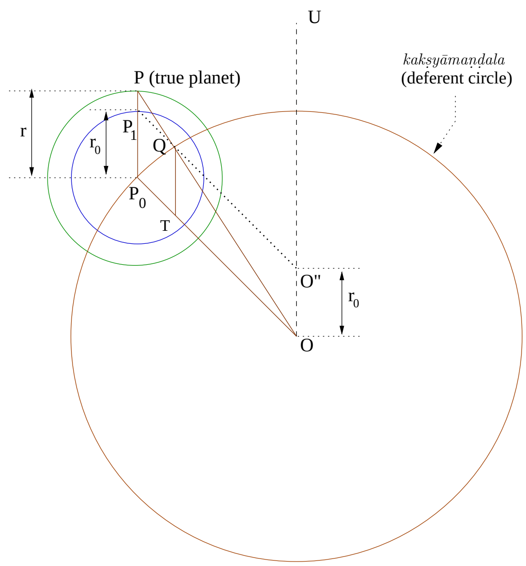


Figure 5: The variation of the radius of the *manda* epicycle with the *manda-karṇa*.

We explain this with the help of Figure 5. Here P_0 represents the mean planet around which an epicycle of radius r_0 is drawn. The point P_1 on the epicycle is chosen such that PP_1 is parallel to the direction of the *mandocca*, OU. Now, the first hypotenuse (*sakṛt-karṇa*) is found from r_0 using the relation

$$OP_1 = K_1 = [(R\sin(\theta_0 - \theta_m))^2 + (R\cos(\theta_0 - \theta_m) + r_0)^2]^{\frac{1}{2}}. \qquad (7)$$

From K_1, using (5), we get the next approximation to the radius $r_1 = \frac{r_0}{R}K_1$. Then, from r_1 we get the next approximation to the *karṇa*,

$$K_2 = [\{R\sin(\theta_0 - \theta_m)\}^2 + \{R\cos(\theta_0 - \theta_m) + r_1\}^2]^{\frac{1}{2}}, \qquad (8)$$

and from that we get $r_2 = \frac{r_0}{R}K_2$. The process is repeated till the radii and the *karṇas* do not change (*aviśeṣa*). The term *aviśeṣa* means 'not distinct'. In the present context it means that the successive *karṇas* are not distinct from each other. That is, $K_{i+1} \approx K_i = K$. If this is satisfied, then $r_{i+1} \approx r_i = r$. Consequently, the equation giving the *manda*-correction (4) becomes

$$R \sin \Delta\theta \;=\; \frac{r}{K}\, R \sin(\theta_0 - \theta_m) = \frac{r_0}{R}\, R \sin(\theta_0 - \theta_m). \qquad (9)$$

Thus the computation of $\Delta\theta$—known as the *mandaphala*, that is to be applied to the *madhyama* to get the *manda-sphuṭa*—involves only the mean epicycle radius and the value of the *trijyā* and not the value of the *manda-karṇa*. It can be shown that the iterated *manda-karṇa* is actually given (in the limit) by OP in Figure 5, where the point P is obtained as follows.[11] Consider a point O'' at a distance of r_0 from O along the direction of *mandocca* OU and draw $O''P_1$ so that it meets the concentric at Q. Then produce OQ to meet the extension of P_0P_1 at P.

Though there is no need to evaluate the *karṇa* to compute $\Delta\theta$, the *manda-karṇa* K is needed in other calculations. For instance, in eclipse computation we need to find the true distance of the Sun or Moon. Similarly, if we have to find the latitude of the planet, we need the heliocentric distance of the planet, which is given by *aviśiṣṭa-manda-karṇa* (iterated hypotenuse). Needless to say that it would have been an arduous task to determine *aviśiṣṭa-manda-karṇa* by iterative procedure in those days, given that fact that it involves repeated computation of squares and square roots.

To circumvent this difficult exercise of performing an iterative process, Mādhava, by carefully analysing the geometry of the problem came up with a brilliant method for finding the *aviśiṣṭa-manda-karṇa*. This method due to Mādhava involves only one step and hence is usually referred to as *sakṛt-karma* as opposed to the *asakṛt-karma* prescribed by Bhāskara.

4.2.3 Mādhava's method for obtaining the *aviśiṣṭa-manda-karṇa* through *sakṛt-karma*

The verses commencing with *vistṛti*, which as mentioned earlier forms one of the *mūla* verses of the manuscript, succinctly presents the *sakṛt-karma* method of finding the *aviśiṣṭa-manda-karṇa* given by Mādhava.

[11] See for instance, the discussion in {MB 1960}, pp. 111–9.

> विस्तृतिदलदोःफलकृतिवियुतिपदं कोटिफलविहीनयुतम्।
> केन्द्रे मृगकर्किगते स खलु विपर्ययकृतो भवेत् कर्णः॥
> तेन हृता त्रिज्याकृतिः अयत्नविहितोऽविशेषकर्णः स्यात् ।
>
> The square of the *doḥphala* is subtracted from the square of the *trijyā* and its square root is taken. The *koṭiphala* is added to or subtracted from this depending upon whether the *kendra* (anomaly) is within six signs beginning from *Karki* (Cancer) or *Mṛga* (Capricorn). This gives the *viparyayakarṇa*. The square of the *trijyā* divided by this *viparyayakarṇa* is the *aviśeṣakarṇa* (iterated hypotenuse) obtained without any effort [of iteration].

As described in the above verse, Mādhava's method involves finding a new quantity called the *viparyayakarṇa* or *viparītakarṇa*. The term *viparītakarṇa* literally means 'inverse hypotenuse', and is nothing but the radius of the *kakṣyāvṛtta* when the measure of *mandakarṇa* is taken to be equal to the *trijyā*, R.

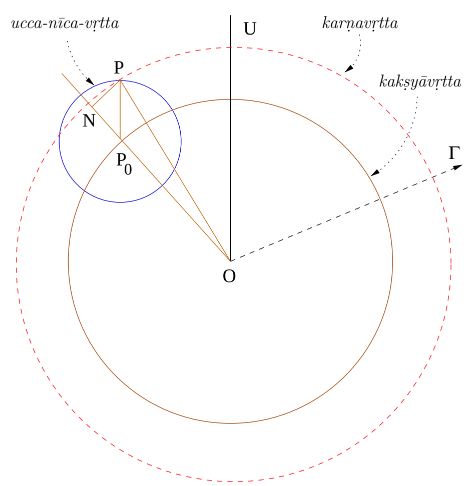


Figure 6: Determination of the *viparītakarṇa* when the *kendra* is in the first quadrant.

The rationale behind the formula for *viparītakarṇa*, presented in this manuscript under study, is also explained in the celebrated Malayalam text *Yuktibhāṣā*,[12] and can be understood with the help of Figures 5 and 6. In these figures P_0 and P represent the mean and the true planet respectively. N denotes the foot of a perpendicular drawn from the true planet P to the line joining the centre of the

[12] See {GYB 2008}, pp. 484–6, 635–40.

circle and the mean planet. NP is equal to the *doḥphala*. Let the radius of the *karṇavṛtta* OP be set equal to the *trijyā* R. Then the radius of the *uccanīcavṛtta* P_0P is r_0, as it is in the measure of the *karṇavṛtta*. In this measure, the radius of the *kakṣyāvṛtta* $OP_0 = R_v$, the *viparītakarṇa*, and is given by

$$\begin{aligned} R_v &= ON \pm P_0N \\ &= \sqrt{R^2 - (r_0 \sin(\theta_0 - \theta_m))^2} \pm |r_0 \cos(\theta_0 - \theta_m)|. \end{aligned} \tag{10}$$

Mādhava, besides besides giving an expression for R_v in terms of the mean anamoly $(\theta_0 - \theta_m))$, as in (10), also seems to have provided an an alternative expression for R_v in terms of the true anamoly $(\theta_{ms} - \theta_m))$, as follows:

$$R_v = \sqrt{R^2 + r_0^2 - 2r_0 R \cos(\theta_{ms} - \theta_m)} \tag{11}$$

This is clear from the triangle OP_0P, where $OP_0 = R_v$, $OP = R$ and $P_0PO = \theta_{ms} - \theta_m$.

In Figure 5, Q is a point where $O''P_1$ meets the concentric. OQ is produced to meet the extension of P_0P_1 at P. Let T be the point on OP_0 such that QT is parallel to P_0P_1. Then it can be shown that $OT = R_v$ is the *viparītakarṇa*. Since triangles OQT and OPP_0 are similar, we have

$$\begin{aligned} \frac{OP}{OP_0} = \frac{OQ}{OT} &= \frac{R}{R_v} \\ \text{or,} \qquad OP = K &= \frac{R^2}{R_v}. \end{aligned} \tag{12}$$

Thus we have obtained an expression for the *aviśiṣṭa-manda-karṇa* in terms of the *trijyā* and the *viparītakarṇa*. Once we find R_v either using (10) or (11), the *aviśiṣṭa-manda-karṇa* K can be calculated using (12) and thereby avoid the iterative process.

4.3 Explanation of the verse '*arkendvoḥ* ...'

The verse commencing with *arkendvoḥ*—the second of the two *mūla* verses commented in the manuscript—outlines the procedure for obtaining the mean positions (*madhyama*) of the Sun and the Moon from their true positions (*manda-sphuṭa*). This verse, originally due to Mādhava and quoted by later Kerala astronomers, runs as follows:

अर्केन्द्वोः स्फुटतो मृदूच्चरहितात् दोःकोटिजाते फले
नीत्वा कर्किमृगादितो विनिमयेनानीय कर्णं सकृत् ।
त्रिज्यादोःफलघाततः श्रुतिहृतं चापीकृतं तत् स्फुटे
केन्द्रे मेषतुलादिगे धनमृणं तन्मध्यसंसिद्धये ॥ ५१ ॥

Subtracting the longitude of their own *mandoccas* from the true positions of the Sun and the Moon, obtain their *doḥphala* and *koṭiphala*. Find the *sakṛt-karṇa* (one-step hypotenuse) once by interchanging the sign [in the cosine term] depending upon whether the *kendra* is within the six signs beginning with *Karki* or *Mṛga*. Multiplying the *doḥphala* and *trijyā*, and dividing this product by the *karṇa* [here referred to as *śruti*], the arc of the result is applied to the true planet to obtain the mean planet. This arc has to be applied positively and negatively depending upon whether the *kendra* lies within the six signs beginning with *Meṣa* or *Tulā* respectively.

The term *mṛdūcca* appearing in the first line of the verse is a synonym for *mandocca*. The *sphuṭa* that is referred to here should be understood as *manda-sphuṭa*, since it has been explicitly mentioned in the verse that the procedure outlined here is applicable only for the Sun and the Moon.[13] If θ_{ms} and θ_m represent the longitudes of the *manda-sphuṭa* and *mandocca* (of the Sun or the Moon), then their *sphuṭa-doḥphala* and *sphuṭa-koṭiphala* are given by

$$\begin{aligned} \textit{doḥphala} &= r_0 \sin(\theta_{ms} - \theta_m) \\ \textit{koṭiphala} &= r_0 \cos(\theta_{ms} - \theta_m), \end{aligned} \tag{13}$$

where r_0 represents the radius of the mean epicycle whose values are provided in the text. With these *doḥ* and *koṭiphalas*, the *sakṛt-karṇa* (expression similar to (2), with opposite sign in *koṭiphala*), may be written as

$$\textit{karṇa} = [(R - r_0 \cos(\theta_{ms} - \theta_m))^2 + (r_0 \sin(\theta_{ms} - \theta_m))^2]^{\frac{1}{2}} \tag{14}$$

It can be easily seen that the above expression is the same as the expression for the *viparīta-karṇa* R_v given by (11). The first half of the above verse essentially prescribes to obtain the expression for R_v. We now proceed to explain the second half of it with the help of Figure 7.

In Figure 7, O is the observer and P_0 is the mean planet (the Sun or the Moon). The point P represents their true position. The distance $P_0P = OO'$

[13]The reason as to why the domain of applicability is restricted only to the Sun and the Moon, is explained by Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya*. We discuss this in the next section.

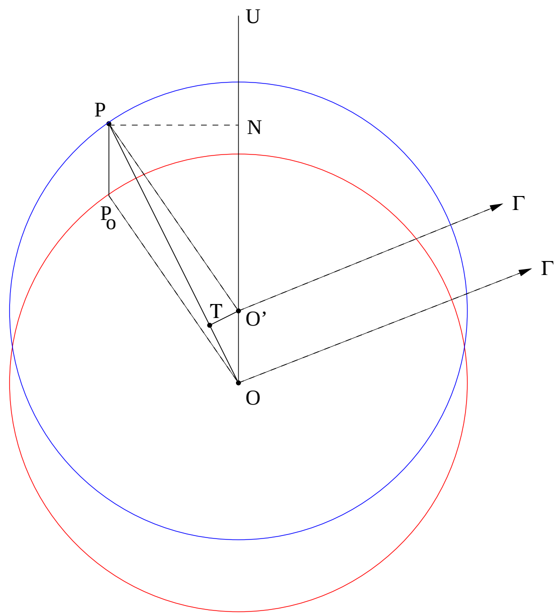


Figure 7: Obtaining the *madhyama* (the mean position) from the *sphuṭa* (the true position).

represents the radius of the actual variable epicyle that we denote as r. The angles $P_0\hat{O}P = OPO' = (\theta_{ms} - \theta_m)$. Considering the triangle $OO'P$, we draw a perpendicular from O' that intersects OP at T. Now, in the triangle $O'PT$,

$$\begin{aligned} O'T &= O'P\sin(O'\hat{P}T) \\ &= O'P\sin(P\hat{O}P_0) \\ &= R\sin(\theta_0 - \theta_{ms}). \end{aligned} \tag{15}$$

$$\text{Also} \qquad O'T = r\sin(\theta_{ms} - \theta_m). \tag{16}$$

Equating the above two expressions for $O'T$,

$$\begin{aligned} R\sin(\theta_0 - \theta_{ms}) &= r\sin(\theta_{ms} - \theta_m) \\ \text{or} \qquad R\sin(\theta_0 - \theta_{ms}) &= r_0\sin(\theta_{ms} - \theta_m)\frac{R}{R_v}, \end{aligned} \tag{17}$$

where we have used (2.135) and (2.153). Hence,

$$\theta_0 - \theta_{ms} = (R\sin)^{-1}\left[r_0\sin(\theta_{ms} - \theta_m)\frac{R}{R_v}\right]. \tag{18}$$

Thus the mean planet θ_0 can be obtained by adding the above difference to the true planet θ. $\theta_0 - \theta_{ms}$ is positive when the *kendra* (anomaly) $\theta_{ms} - \theta_m$

is within the six signs beginning with *Meṣa*, viz., $0° \leq \theta_{ms} - \theta_m \leq 180°$, and negative when the *kendra* is within the six signs beginning with *Tulā*, viz., $180° \leq \theta_{ms} - \theta_m \leq 360°$.

Having briefly explained the content of the verse, we now proceed to discuss it in greater detail in the light of the edifying commentary authored by Nīlakaṇṭha on it in his *Āryabhaṭīya-bhāṣya*. As the commentary is quite elaborate, runs to more than two pages, we only present excerpts from it. The excerpts have been chosen primarily to highlight some of special features pointed out be Nīlakaṇṭha, particularly with respect to the distinction that must be maintained in applying the procedure, outlined by the above verse, for obtaining the *manda-sphuṭa* from the *śīghra-sphuṭa* of the planets.

4.3.1 Obtaining the *manda-sphuṭa* from the *śīghra-sphuṭa*

Since the geometrical construction involved in the *manda* as well as the *śīghra* process is essentially the same—both just involve a deferent circle and an epicyle, though the significance of them widely varies in the two processes—Nīlakaṇṭha, in his *Āryabhaṭīya-bhāṣya*, outlines the procedure for obtaining the *manda-sphuṭa* from the *śīghra-sphuṭa*.

> अत एव शीघ्रस्फुटेन स्फुटमध्यमानयने शीघ्रस्फुटतदुच्चविवरभुजां स्वपरिधिहतां चक्रांशैः अशीत्या वा हृत्वा लब्धं दोःफलं केवलमेव चापीकृत्य शीघ्रस्फुटे व्यत्ययेन संस्कुर्यात्। तदा स्फुटमध्यमं स्यात्।...
>
> अत उक्तं – 'मृदूच्चरहितादि'ति।
>
> It is for this reason, it has been prescribed that the arc corresponding to the *doḥphala*—obtained from the Rsine of the difference between the *śīghra-sphuṭa* and *śīghrocca* multiplied by its own circumference (*sva-paridhihatām*[14]) and divided by either 360 or 80—should be applied to the *śīghra-sphuṭa* inversely, in order to the *manda-sphuṭa* from the *śīghra-sphuṭa*.

We may explain the content of the above passage with the help of Figure 5. In this figure, O is the observer, S is the Sun and P any of the three exterior planets Mars, Jupiter or Saturn. $OS = r_s$ represents the radius of the *śīghra* epicycle

[14]Since the discussion is on the *śīghra* process, the word 'sva' refers to the *śīghra*.

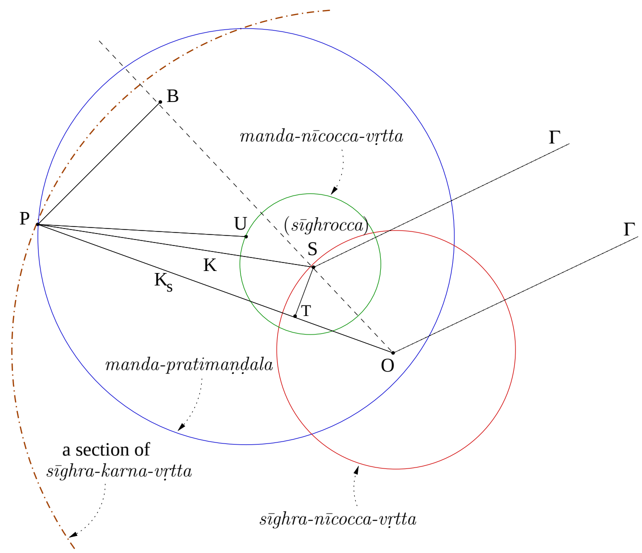


Figure 8: Obtaining the *manda-sphuṭa* from the *śīghra-sphuṭa*.

and $SP = K$ the *aviśiṣṭa-manda-karṇa*. The angles $\Gamma\hat{S}P = \theta_{ms}$ and $\Gamma\hat{O}P = \theta$ represent the *manda-sphuṭa* and *śīghra-sphuṭa* respectively. θ_s represents the longitude of *śīghrocca* which is the same as the longitude of the mean Sun that is known. The objective is to find the θ_{ms} from θ.

Considering the triangle OPS, and applying the sine formula we have

$$\frac{r_s}{\sin(\theta_{ms} - \theta)} = \frac{K_s}{\sin(\theta_{ms} - \theta_s)} = \frac{K}{\sin(\theta - \theta_s)}. \tag{19}$$

Hence,

$$\begin{aligned} \sin(\theta_{ms} - \theta) &= \frac{r_s}{K} \sin(\theta - \theta_s) \\ \text{or,} \qquad (\theta_{ms} - \theta) &= \sin^{-1}\left(\frac{r_s}{K}\sin(\theta - \theta_s)\right) \\ \text{or,} \qquad \theta_{ms} &= \theta + \sin^{-1}\left(\frac{r_s}{K}\sin(\theta - \theta_s)\right). \end{aligned} \tag{20}$$

Thus the *manda-sphuṭa* can be obtained from the *śīghra-sphuṭa* using the above relation.

4.3.2 Applicability of the procedure outlined in the verse *arkendvoḥ* to other planets

Though by conceding the analogy between the geoemtrical constructions involved in the *manda* and the *śīghra* processes, though Nīlakaṇṭha prescribed a procedure for getting the *manda-sphuṭa* from the *śīghra-sphuṭa*, he clearly maintains the distinction between the two. In fact, to avoid any confusion he explicitly points out that the *sakṛt-karma* procedure given for the Sun and the Moon cannot be simply generalized to find the *madhyama* from the *sphuṭa* in the case of the planets Mars, Jupiter and so on. He says:[15]

> अर्केन्दुग्रहणं भौमादीनां निवृत्त्यर्थम्। तेषां दोर्ज्यावशात् परिधिभेदात् ...
> Specific mention of the Sun and the Moon is to desist the entry of Mars, etc. For them, since the circumference changes as per *dorjyā*, ...

The justification presented by Nīlakaṇṭha as to why the procedure outlined by Mādhava for obtaining the the mean position from the true position using *sakṛt-karma*, is not applicable to the other five planets is quite involved and requires an elaborate explanation. As this does not fall under the scope of the present paper, we skip further discussion on this topic and move on to the next section.

5 The Edited text and translation

The edited version of the text presented below, is identical with the manuscript but for the insertion of numbers before the beginning of the paragraphs. These numbers have been inserted for the purpose of easily identifying the original text with the explanatory notes that has been prepared in four sections.[16] The sectioning too has been done in manner that seemed most appropriate and convenient for our discussion.

> (1) विस्तृतीत्यादि। अर्केन्द्वोरित्यादि। अनयोः श्लोकद्वययोर्युक्तिप्रदर्शनाय त्रिज्या-तुल्यव्यासार्धं कक्ष्यामण्डलमालिख्य मातृपितृरेखां कृत्वा तत्परिधौ रविमध्यमा-

[15] **{ABB 1931}**, p. 50; This passage is quoted in a different context in **{LB 1979}** as well.

[16] Para (1) forms the first section, Para (2) the second, Para (3)–(5) the third and Para (6) the last.

धिष्ठितप्रदेशं केन्द्रं कृत्वा अविशिष्ठान्त्यफलत्रिज्यातुल्येन सूत्रेण उच्चनीचवृत्त-मालिख्य तत्परिधौ मन्दोच्चदिशि स्फुटग्रहमालिख्य ग्रहाधिष्ठितप्रदेशात् कक्ष्या-मण्डलकेन्द्रप्रापिणं सूत्रं कुर्यात्। कक्ष्यामण्डलकेन्द्रात् स्ववृत्तगतोच्चप्रापिणं सूत्रं च कुर्यात्। पुनः कक्ष्यामण्डलकेन्द्रमेव केन्द्रं कृत्वा अविशिष्टान्त्यफलसूत्रेण द्वितीयमप्युच्चनीचवृत्तमालिखेत्। पुनः पूर्वलिखितोच्चनीचपरिधेः[17] कर्णवृत्तपरिधेश्च सम्पातद्वयप्रापिणं सूत्रमालिखेत्। तदर्धं मन्दोच्चोनमध्यमाल्लब्धं[18] अविशिष्टभुजा-फलम्। एतत्सूत्रमध्यात् मध्यमग्रहप्रापिसूत्रं अविशिष्टमध्यमकोटिफलम्। पुनर्द्वि-तीयोच्चन्नीचवृत्तस्य कक्ष्यामण्डलस्य च परिधिसम्पातद्वयप्रापिणं सूत्रं कुर्यात्। तदर्धं मन्दोच्चोनात् अर्कस्फुटाल्लब्धं अविशिष्टभुजाफलम्। तन्मध्यात् स्फुटग्रहप्रापि सूत्रमविशिष्टस्फुटकोटिफलम्। स्फुटभुजाकोटिफलाभ्यां कर्णानयनं त्रिज्यावर्गात् भुजाफलवर्गं विशोध्य मूले[19] कोटिफलं मृगकर्क्यादितः स्वर्णं कृत्वा साध्यम्। तस्य कर्णस्य पुनरविशेषणं पूर्ववदेव।

(2) एवं साधनचतुष्टये लिखिते तैर्विषमचतुरश्रं क्षेत्रं भवति। तत्र मध्यमोद्भवयोः भुजाकोटिफलयोः स्फुटोद्भवयोश्च साधारणं कर्णो मध्यमग्रहात् स्फुटग्रहप्रापि-सूत्रम्। एवं क्षेत्रे लिखिते कक्ष्यामण्डल स्फुटग्रहप्रापिकर्णसूत्रात्[20] त्रिज्याया आनयनं[21] निरूप्यम्। तद्यथा कर्णवर्गात् मध्यमदोःफलवर्गं विशोध्य यन्मूलं तस्मिन् मध्यमकोटिफलं मृगकर्क्यादावृणं धनं च कृत्वा त्रिज्या[22] लभ्यते।[23] कर्णे स्फुटकोटिफलं मृगकर्क्यादावृणं धनं च कृत्वा तद्वर्गे[24] स्फुटदोःफलवर्गं क्षिप्त्वा मूलं च त्रिज्या भवति। एवं द्विविधं त्रिज्यानयनमवगम्य पुनर्विपरीतकर्णा-नयनं निरूप्यते।

(3) विपरीतकर्णस्तु क इति चेत्, कर्णे त्रिज्यातुल्ये सति कियती त्रिज्या इति लब्धा त्रिज्या विपरीतकर्णः। उक्तप्रकारेण तदानयनाय अवशिष्टस्य वृत्तगते मध्यमकोटिभुजाफले स्फुटकोटिभुजाफले अविशिष्टकर्णं त्रिज्यां च त्रिज्यया हत्वा कर्णेन हरेत्। तदानीं कर्णस्त्रिज्यातुल्यो भवति। मध्यमकोटिभुजाफले प्रथमं मन्दोच्चोनमध्यमाल्लब्धयोर्बाहुकोटिज्ययोः परिधिगुणनाशीतिहरणे कृत्वा लब्धाभ्यां भुजाफलकोटिफलाभ्यां तुल्ये च भवतः। त्रिज्या, त्रिज्यावर्गं[25] अविशिष्टकर्णेन

[17]नीच परिसं

[18]मन्दोच्चोनमा ल्लब्धं

[19]मूलं

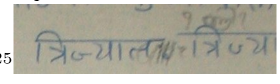
[20]प्रापिणं कर्णसूत्रात्

[21]आनयनयनं

[22]कृत्वा वर्गं त्रिज्या

[23]Here the sentence "कर्णे स्फुटकोटिफलं मृगकर्क्यादावृणं धनं च कृत्वा वर्गं त्रिज्या लभ्यते।" appears twice in the manuscript.

[24]तद्वर्गगे

[25] 

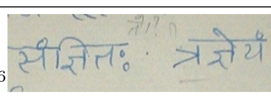
हृत्वा लब्धेन तुल्या भवति। तद्विपरीतकर्णसंज्ञितोऽत्र ज्ञेयम्।[26] तदर्थं त्रिज्याभूतस्य कर्णस्य वर्गात्[27] मध्यमभुजाफलवर्गमपहाय यन्मूलं तस्मिन् मध्यमकोटिफलं मृगकर्क्यादावृणं[28] धनं कुर्यात्। ततो विपरीतकर्णभूता त्रिज्या[29] लभ्यते। अत उक्तं 'विस्तृतिदले'त्यादिना। त्रिज्याभूते कर्णे सकृत्कर्मणा स्फुटतो लब्धं कोटिफलं मृगकर्कितः ऋणं धनं च विधाय तद्वर्गे सकृत्कर्मलब्धस्य स्फुटभुजाफलस्य वर्गं क्षिप्त्वा मूलं च विपरीतकर्णभूता त्रिज्या भवति। अत उक्तं – 'अर्केन्द्वोः स्फुटत' इत्यादिना।

(4) एवं विपरीतकर्णे ज्ञाते अविशेषकर्णानयनाय विपरीतकर्णभूतायां त्रिज्यायां त्रिज्यातुल्यायां जातायां त्रिज्याभूतः कर्णः कियानिति त्रैराशिकम्। तत्र फलस्य इच्छायाश्च त्रिज्या तुल्यत्वात् त्रिज्यावर्गो विपरीतकर्णेन भाज्यम्। लब्धं अविशेषकर्णश्च भवति। अत उक्तं – तेन हृता त्रिज्याकृतिरयत्नविहितोऽविशेषकर्णः स्यात्। – इति।

(5) एवं मध्यमतः स्फुटतश्च सकृत्कर्मणा अविशेषकर्णलब्धिश्च भवति। त्रिज्याविशेषकर्णाभ्यां त्रैराशिके कर्तव्ये सर्वत्र त्रिज्यास्थाने विपरीतकर्णः, अविशेषकर्णस्थाने त्रिज्यां च कृत्वा कर्म करणीयं भवति। तत्र स्फुटतो मध्यमानयनं[30] करिष्यते।

(6) एवं कर्णानयने ज्ञाते स्फुटतो मध्यमानयनं निरूप्यते – पूर्वलिखितमविशिष्टवृत्तगतं[31] स्फुटभुजाफलं कक्ष्यामण्डलकेन्द्रात् मध्यमस्फुटग्रहप्रापिसूत्रयोरन्तरालं[32]कक्ष्यामण्डलपरिधिगतम्[33]। अतः अविशिष्टस्फुटभुजाफलमेव मध्यमस्फुटान्तरम्। तज्ज्ञानाय सकृत्कर्मलब्धं स्फुटभुजाफलमविशेषकर्णेन हत्वा त्रिज्यया हर्तव्यम्। अथवा त्रिज्यया हत्वा विपरीतकर्णेन विभज्यापि लभ्यते। लब्धचापं मेषादौ मध्यमसूत्रयोः मध्यमसूत्रस्याग्रगतत्वात् स्फुटे धनं कार्यम्। तुलादौ पृष्ठगतत्वात् ऋणं च कार्यम्। अतः – 'त्रिज्यादोःफलघाततः श्रुतिहृतं चापीकृतं तत्स्फुटे केन्द्रे मेषतुलादिगे धनमृणं तन्मध्यसंसिद्धये' इत्युक्तम्।

(1) [The verse] that commences with *vistṛti*. [The verse] that commences with *arkendvoḥ*. In order to present the rationale behind

26 

27 वार्गात्

28 मृगकक्यादावृणं

29 कर्णभूतानि त्रिज्या

30 आनयने

31 अविशिष्टस्य वृत्तगतं

32 मध्यमस्फुटग्रहप्रापिणं च सूत्रयोरन्तरालं

33 परिधिगतः

the [content] of these two verses, having drawn the deferent circle (*kakṣyā-maṇḍala*) with radius equal to the *trijyā*, mark the east–west line (*mātṛ-pitṛrekhā*) in it. Then, having marked the position of the mean Sun on its circumference, draw an epicycle with that as centre and with the radius equal to *aviśiṣṭāntyaphala*. Marking the position of the true planet on its circumference along the direction of the apogee (*mandocca*), draw a line from the location of the planet to the center of the deferent circle. Also draw a line, that reaches the apogee corresponding to that circle from the center of the *kakṣyā-maṇḍala*. Again, with the center of the deferent circle as the center, draw a second epicycle with radius equal to the *aviśiṣṭāntya-phala*. Then, draw a line connecting the points of intersection of the circumference of the epicycle that was drawn earlier and the circumference of the *karṇavṛtta*. Half of that [line] is *aviśiṣṭa-bhujāphala* obtained from the mean Sun diminished by the apogee. The line reaching the mean planet from the center of this line is the *aviśiṣṭakoṭiphala* of the mean planet. Then, draw a line joining the points of intersection of the circumference of the second epicycle and the circumference of the *kakṣyāmaṇḍala*. Half of that [line is] *aviṣiṣṭa-bhujāphala* obtained from the true Sun diminished by the apogee. The line reaching the true planet from the middle of this line [is the] *aviśiṣṭakoṭiphala* of the true [planet]. The *karṇa* is obtained from the true *bhujā* and *koṭiphalas* by first finding the square root of the difference of the the squares of the *bhujāphala* and the *trijyā* and applying the *koṭiphala* to that positively or negatively depending upon [whether the *kendra* is] *Mṛgādi* or *Karkyādi* respectively. The iteration of that *karṇa* is to be done as before.

(2) Thus, the four means (*sādhanacatuṣṭaya*) [namely, the *bhujā* and the *koṭiphalas* corresponding to the mean and the true planets] when represented, form a quadrilateral of unequal sides (*viṣamacaturaśra*). In that, the line joining the mean and the true planet, forms the common hypotenuse of the *bhujā* and the *koṭiphalas* corresponding to the mean and the true [*kendras*]. Thus, having drawn the figure, now the procedure for obtaining the *trijyā* from the *karṇa*, which joins the centre of the *kakṣyāmaṇḍala* and the true planet, is to be explained. This is as follows: By applying the *madhyama-koṭiphala* negatively or positively—depending upon whether the *kendra* is *Mṛgādi* or *Karkyādi* respectively—to the the square root of the difference of the the squares of the *karṇa* and the *madhyama-doḥphala*, *trijyā* is obtained. [Similarly] having applied the true *koṭiphala* to the *karṇa* negatively or positively depending upon whether the

kendra is *Mṛgādi* or *Karkyādi* respectively, the square of that has to be taken. By adding the square of the *sphuṭa-doḥphala* to the square of that, and taking the square root, *trijyā* is obtained. Thus, having understood the two ways of obtaining the *trijyā*, we now proceed to explain the procedure for obtaining the inverse hypotenuse.

(3) [If you ask] what *viparītakarṇa* is, [we say], when the *karṇa* is taken to be equal to the measure of the *trijyā*, then the value of *trijyā* obtained in that measure is the *viparītakarṇa*. To obtain that along the lines described above multiply the *trijyā* with itself [and] divide by the *karṇa*. Then, *karṇa* will be equal to the *trijyā*. The *bhujā* and the *koṭiphalas* [thus obtained] will be equal to the *bāhu* and the *koṭijyās* obtained earlier by subtracting the apogee from the mean planet, when multiplied by the circumference [of the epicycle] and divided by eighty. It is to be known here, that the quantity obtained by dividing the square of the *trijyā* by the *aviśiṣṭakarṇa*, is the *viparītakarṇa*. In order to obtain that, we first find the square root of the difference of the squares of *karṇa* taken in the measure of the *trijyā* and the square of *madhyama-bhujāphala*. To the square root of that, the *madhyama-koṭiphala* is applied. negatively or positively depending upon whether the *kendra* is *Mṛgādi* or *Karkyādi*. Now, the *viparīta-karṇa* in the measure of *trijyā* is obtained. It is therefore said "*vistṛtidala*" etc. To the *karṇa* which is in the measure of *trijyā*, apply the *sphuṭa-koṭiphala* obtained through *sakṛt-karma* negatively or positively depending upon whnether the *kendra* is *Mṛgādi* or *Karkyādi* respectively. To the square of that, add the square of the *sphuṭa-bhujāphala* [again] obtained through *sakṛt-karma*, and find its square root. The result will be the *viparītakarṇa*, in the measure of the *trijyā*. It is therefore said – "*arkendvoḥ sphuṭataḥ*".

(4) Thus having known the *viparīta-karṇa*, the rule of three that is to be employed for obtaining the iterated hypotenuse is : If the *viparīta-karṇa* were to be taken to be equal to the measure of the *trijyā*, then what would be the value of the *karṇa* which was previously taken to be equal to the *trijyā*. Since here, the *pramāṇaphala* as well as *icchā* are equal to the *trijyā*, the square of the *trijyā* has to be divided by the *viparīta-karṇa*. The resultant is the iterated hypotenuse. It is therefore said – 'The square of the *trijyā* divided by that [*viparyayakarṇa*] is the *aviśeṣakarṇa* (iterated hypotenuse) obtained without any effort [of doing iteration]'.

(5) Thus either from the mean or the true the iterated hypotenuse is

obtained through the *sakṛt-karma*. Since the rule of three should be applied using the *trijyā* and the iterated hypotenuse, all the operations have to be executed by replacing *trijyā* with the *viparīta-karṇa* and the *aviśeṣakarṇa* with the *trijyā*. Now, the mean [planet] from the true will be obtained.

(6) Thus having known how to find the *karṇa*, the procedure for obtaining the mean from the true is now explained. The *sphuṭa-bhujāphala* which lies inside the iterated epicycle drawn earlier, which is essentially equal to the distance of seperation between the lines drawn from the centre of the *kakṣyā-maṇḍala*, is [also] inside the circumference of the *kakṣyā-maṇḍala*. Hence, the *aviśiṣṭa-sphuṭa-bhujāphala* is the same as difference between the mean and the true [positions of the planet]. In order to obtain that, the *sphuṭa-bhujāphala* obtained through the *sakṛt-karma* should be multiplied by the iterated hypotenus and divided by the *trijyā*. The same result may also be obtained by multipling by the *trijyā* and dividing by the *viparītakarṇa*. If the *kendra* is *Meṣādi*, then the resulting arc should be applied positively to the true planet, since of the two lines joining the mean and the true [from the centre of the *kakṣyāmaṇḍala*], the one joining the mean leads the other. If the *kendra* is *Tulādi*, then the arc should be applied negatively since the line joining the mean planet lags behind the other. It is therefore said that – 'Multiplying the *doḥphala* and *trijyā*, and dividing this product by the *karṇa* [here referred to as *śruti*], the arc of the result is applied to the true planet to obtain the mean planet. This arc has to be applied positively and negatively depending upon whether the *kendra* lies within the six signs beginning with *Meṣa* or *Tulā* respectively.'

6 Explanatory notes

Transcending the confines of immediate utility of merely explaining the content of the verses, the commentary in the manuscript gradually develops the background material that would enable the reader to have a fuller appreciation of the content of the *mūla*. It also attempts to present the rationale behind the procedures outlined in the verses. In what follows, we present the content of the manuscript using modern mathematical notations. For the purpose of convenience we have divided it into four sections.

6.1 Definition of *sādhanacatuṣṭaya*

The manuscript commences with detailed description of how the geometrical figure needs to be constructed with which the two verses to be commented upon can be understood. The geometrical figure described therein may be depicted as indicated in Figure 9.

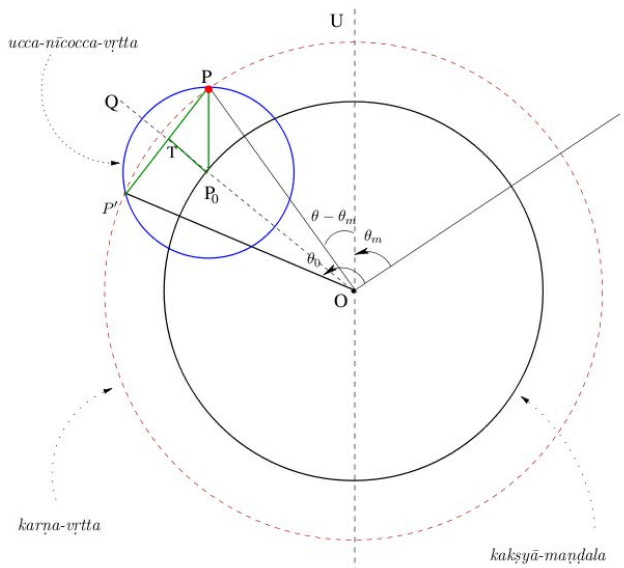


Figure 9: The geometrical construction described in the manuscript as a tool to understand the *mūla* verses.

Here O represents the centre of the deferent circle known as *kakṣyāmaṇḍala*, whose radius is R. The mean planet P_0 is located on this circle, whose longitude is denoted as θ_0. The circle centered around P_0 and with radius r_0 is called *ucca-nīca-vṛtta* (epicycle). A line parallel to the direction of apogee (OU) drawn from the P_0 meets the epicycle at P. This gives the position of the *manda-sphuṭa* (the *manda* corrected planet). Now, the circle centered at O and having a radius $OP = K$, known as *karṇa-vṛtta* is drawn. This intersects the epicycle at two points, namely P and P'. The line OQ, which is an extension of the line joining the centre of the deferent circle and the mean planet bisects the line PP' at T.

Now considering the $\triangle PP_0T$,

$$P\hat{P}_0T = \theta_0 - \theta_m,$$

where θ_0 and θ_m are longitudes of mean planet and apogee respectively. In this triangle which is a right-angled at the vertex T the hypotenuse P_0 represents the *aviśiṣṭāntyaphala* (r). Thus, we have

$$PT = r\sin(\theta_0 - \theta_m) \quad (21)$$

$$P_0T = r\cos(\theta_0 - \theta_m) \quad (22)$$

The quantities PT and P_0T, defined in the above equations are referred to as *aviśiṣṭa-madhyama-bhujāphala* and *aviśiṣṭa-madhyama-koṭiphala* respectively.

A section of Figure 9 is blown up and depicted in Figure 10. Here $OP_0 = R$ represents the *trijyā* and $OP = K$ denotes the *manda-karṇa*. The foot of perpendicular drawn from P_0 to OP, intersects the latter at B. In the $\triangle PP_0B$, $P_0\hat{P}B = \theta_{ms} - \theta_m$. Hence,

$$P_0B = r\sin(\theta_{ms} - \theta_m) \quad (23)$$

$$PB = r\cos(\theta_{ms} - \theta_m). \quad (24)$$

These two quantities P_0B and PB are known as *sphuṭabhuja* and *sphuṭakoṭi* respectively. It may be noted that the two triangles constitute a quadrilateral of unequal sides. It is the four sides of this quadrilateral—representing the *bhujā* and *koṭi* of the *madhyama* and *sphuṭa-kendras*—that are referred as *sādhanacatuśṭaya*. Literally the term *sadhanacatuṣṭaya* means "a group consisting of four-means".

Having defined *sadhanacatuṣṭaya*, the text presents a formula for the *karṇa* OP in terms of the *sphuṭabhujā* and *koṭiphalas*. In the $\triangle OP_0B$,

$$\begin{aligned} OB &= \sqrt{OP_0^2 - P_0B^2} \\ &= \sqrt{R^2 - (r\sin(\theta_{ms} - \theta_m))^2} \end{aligned}$$

Also, $OB = OP - BP$. Hence,

$$OP = \sqrt{R^2 - (r\sin(\theta_{ms} - \theta_m))^2} + BP \quad (25)$$

$$K = \sqrt{R^2 - (r\sin(\theta_{ms} - \theta_m))^2} + r\cos(\theta_{ms} - \theta_m). \quad (26)$$

Thus *manda-karṇa* can be obtained in terms of *trijyā* and *sphuṭa-kendra*.

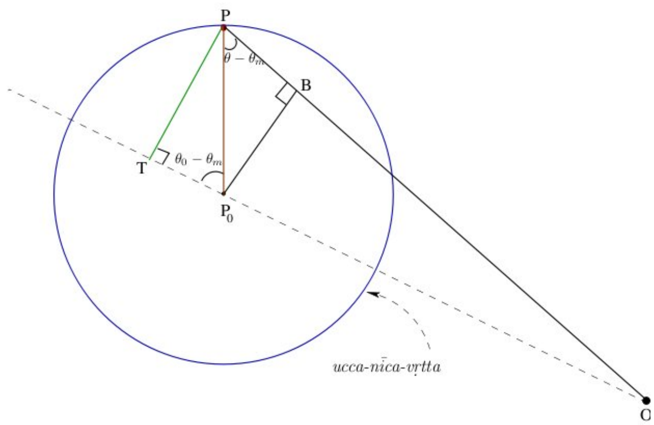


Figure 10: The *sādhana-catuṣṭaya* employed in finding the *manda-karṇa*.

6.2 Obtaining *trijyā* from *karṇa*

Having defined a relation to obtain the *karṇa* K, in terms of the *trijyā* R, the commentary presents two methods by which one can obtain *trijyā* in terms of the *karṇa*.

Method 1: Here *trijyā* is obtained in terms of the *karṇa* and the *madhymabhujā-koṭiphalas*. The expression given can be understood with the help of Figure 10. Considering the $\triangle OTP$, we have

$$\begin{aligned} OT^2 &= OP^2 - PT^2 \\ \text{or} \qquad (OP_0 + P_0T)^2 &= K^2 - (r\sin(\theta_0 - \theta_m))^2 \end{aligned} \tag{27}$$

Hence,

$$\begin{aligned} OP_0 &= \sqrt{K^2 - (r\sin(\theta_0 - \theta_m))^2} - P_0T \\ \text{or} \qquad R &= \sqrt{K^2 - (r\sin(\theta_0 - \theta_m))^2} - r\cos(\theta_0 - \theta_m). \end{aligned} \tag{28}$$

Method 2: In this method R is found in terms of K and the *sphuṭabhujā-koṭiphalas*. In the $\triangle OP_0B$ (see Figure 10),

$$\begin{aligned} OP_0^2 &= OB^2 + BP_0^2 \\ \text{or} \quad R^2 &= (OP - BP)^2 + (r\sin(\theta - \theta_m))^2 \\ \text{or} \quad R &= \sqrt{(K - r\cos(\theta-\theta_m))^2 + (r\sin(\theta-\theta_m))^2}. \end{aligned} \tag{29}$$

6.3 Expression for *viparītakarṇa* and its application

The third paragraph of the *mūla* starts with the definition of *viparītakarṇa* and then proceeds to give two different expressions for the same—one in terms of the *trijyā* and the *manda-kendra* and the other in terms of the *trijyā* and the *sphuṭa-kendra*. The definition given here may be stated as follows: If the measure of the *karṇa* were to be taken to be equal to the *trijyā*in a certian scale, then whatever that turns out to be the magnitude of *trijyā* in the same scale is defined as *viparīta-karṇa*. Symbolically this may be represented as a problem of rule of three:

$$\begin{aligned} K &: R \\ R &: R_v(?) \\ \text{or} \quad R_v &= \left(\frac{R}{K}\right) R \end{aligned} \tag{30}$$

The two expressions for R_v given here are the same as the equations (10) and (11) discussed in section 4.2.3. As a detailed derivation of the two equations are presented there itself, we do not repeat the same here. It would suffice to recount that one of the main purposes for introducing this mathematical device R_v is to find K without having to resort to *aviśeṣa-karma*. Para (4) essentially states that *aviśiṣṭa-manda-karṇa* or *aviśeṣa-karṇa* K can be expressed in terms of *trijyā* and *viparīta-karṇa* as:

$$\begin{aligned} \textit{aviśeṣakarṇa} &= \frac{\textit{trijyā}^2}{\textit{viparītakarṇa}} \\ \text{or} \quad K &= \frac{R^2}{R_v}. \end{aligned} \tag{31}$$

In Para (5) it is prescribed that the *trijyā* and *aviśeṣakarṇa* can be replaced by the *viparīta-karṇa* and *trijyā* respectively in all the operations that are to be carried out based on rule of three. That is,

$$\begin{aligned} K &\longrightarrow R \\ R &\longrightarrow R_v. \end{aligned} \tag{32}$$

The above prescription can be better appreciated with the help of relations that can be derived from the triangles OP_0P and OTQ in the Figure 5. Considering these two triangles and applying the sine formula we have

$$\frac{R}{\sin(\theta_{ms} - \theta_m)} = \frac{K}{\sin(\theta_0 - \theta_m)} = \frac{r}{\sin(\theta_0 - \theta_{ms})} \tag{33}$$

$$\frac{R_v}{\sin(\theta_{ms} - \theta_m)} = \frac{R}{\sin(\theta_0 - \theta_m)} = \frac{r_0}{\sin(\theta_0 - \theta_{ms})}. \tag{34}$$

From (33) the following equations may be obtained.

$$\sin(\theta_0 - \theta_m) = \frac{K}{R}\sin(\theta_{ms} - \theta_m) \tag{35a}$$

$$\sin(\theta_0 - \theta_{ms}) = \frac{r}{K}\sin(\theta_0 - \theta_m) \tag{35b}$$

$$\sin(\theta_0 - \theta_{ms}) = \frac{r}{R}\sin(\theta_{ms} - \theta_m). \tag{35c}$$

Similary from (34), we have

$$\sin(\theta_0 - \theta_m) = \frac{R}{R_v}\sin(\theta_{ms} - \theta_m) \tag{36a}$$

$$\sin(\theta_0 - \theta_{ms}) = \frac{r_0}{R}\sin(\theta_0 - \theta_m) \tag{36b}$$

$$\sin(\theta_0 - \theta_{ms}) = \frac{r_0}{R_v}\sin(\theta_{ms} - \theta_m) \tag{36c}$$

It may be observed that LHS, in the pair of three sets of equations presented above, is one and the same. So too is the argument of the sine function in RHS. This only forces us to conclude that multipying factors—in the form of ratios of two quantities—that appear in the RHS must also be the same though they look apparently different. Thus we have

$$\frac{K}{R} = \frac{R}{R_v}; \qquad \frac{r}{K} = \frac{r_0}{R}; \qquad \text{and} \qquad \frac{r}{R} = \frac{r_0}{R_v}, \tag{37}$$

which only proves the validity of the prescription given in (32).

6.4 Finding *madhyama* from *manda-sphuṭa*

The last section (Para (6)) of the manuscript deliniates the procedure for obtaining the *madhyama* from the *manda-sphuṭa*. In fact the author commences with the declaration:

एवं कर्णानयने ज्ञाते स्फुटतो मध्यमानयनं निरूप्यते –

Thus having known how to find the *karṇa*, the procedure for obtaining the mean from the true is now explained.

The *karṇa* that is being referred to above is the *aviśeṣakarṇa* given by (31). It is said that having known this *karṇa* K, it should be multiplied by *sphuṭabhujāphala* and divided by *trijyā*. The term *sphuṭabhujāphala* refers to $r_0 \sin(\theta_{ms} - \theta_m)$ and hence the prescrition given amounts to finding

$$\frac{K}{R} r_0 \sin(\theta_{ms} - \theta_m). \tag{38}$$

From (37), this is the same as

$$\frac{R}{R_v} r_0 \sin(\theta_{ms} - \theta_m). \tag{39}$$

Now it can be easily seen that the above expression is the same as the RHS of (36c) but for the multiplying factor *trijyā* R. Hence we have,

$$R \sin(\theta_0 - \theta_{ms}) \quad = \quad r_0 \sin(\theta_{ms} - \theta_m) \frac{R}{R_v}, \tag{40}$$

Or, equivalently

$$\theta_0 - \theta_{ms} = (R \sin)^{-1} \left[r_0 \sin(\theta_{ms} - \theta_m) \frac{R}{R_v} \right], \tag{41}$$

from which the *madhyama* θ_0 can be obtained, since θ_{ms} and θ_m are already known. It may also be recalled that R_v can be obtained in terms of θ_{ms} and θ_m using (11). Thus an elegant procedure[34] for obtaining the *madhyama* from *manda-sphuṭa* has been described, which explains that title of the manuscript *madhyamānayanaprakāra*.

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[34] which does away with the arduous iterative procedure

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