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COMPUTING $\sqrt{N}$: A MODERN GENERALIZATION OF ANCIENT TECHNIQUE

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The paper presents a brief account of square rooting by means of algebraic, geometric, indeterminate analysis and iterative methods across the various culture area of the world. The iterative procedure for obtaining square-root of 2 is extended to non-square positive integers, and it is concluded that the evaluation of square-rooting in the pre- and post-historic Indian period was most probably based on this extended procedure. Further, this method of computation of $\sqrt{N}$ compliments Newton-Raphson method and other iterative procedures for root extraction.

Key words: *Āryabhaṭīya*, *Bījgaṇita*, *Dvikaraṇi*, Indeterminate Analysis, *Khaṇḍakhādyaka*, *Līlāvatī*, Newton-Raphson method, *Śulbasūtras*, *Tṛkaraṇī*

1. INTRODUCTION

1.1 Modern algebraic method of root extraction of a non-square positive integer is well known. Apart from this, there are iterative procedures like Newton-Raphson method for finding the root of an equation. Geometric and indeterminate analysis based evaluation of $\sqrt{N}$ are hardly known to mathematical community at large. Incorporating additional aspects of root finding as geometric and indeterminate analysis in the present curricula would certainly benefit students' community.

The concept of irrational numbers seems a land mark in the development of science and technology. Modern historians attempt to ascribe this concept to

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sixth century BC philosopher and mathematician Pythagoras (p. 44)[27]. The notion of irrational numbers may be found in the *Brāhmaṇās* (ca. 2500 BC) and *Saṃhitās* (ca. 3000 BC)[3,12,19,29-31,35,38,40]. Techniques to compute approximate rational values of certain irrational numbers in the *Śulbasūtras* (ca. 1000-200 BC) is enough evidence that irrationality concept is much older than the period of Pythagoras (ca. 540 BC)[31]. It may be mentioned that the *Śulbasūtras* are manuals for constructing altars. This means that the *Śulba* rules are much older than their compositions. See ref (6) for details concerning surd terminology, e.g., *dvikaraṇī* ($\sqrt{2}$), *tṛkaraṇī* ($\sqrt{3}$). However, a proof to the fact that an irrational number is not expressible as a ratio of two integers had to wait for Aristotle (d. 332 BC) (pp. 83-85)[8], (p.55)[24] and (pp. 43-44)[27].

1.2 The aphorism 2.12 of *Baudhāyana Śulbasūtra* (=*BŚS*) gives a value of $\sqrt{2}$ which means **'**The measure to be increased by its third and this (third) again by its own fourth less the thirty-fourth part (of that fourth); this is (the value of) the diagonal of a square (whose side is the measure)' (with an additional term 'we mean this is approximate or more terms than what is prescribed')[31].

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} \text{ (appr)} = \frac{577}{408} \quad \ldots(1.1)$$

(This gives correct value to five places of decimal.)

The same rule is also quoted in the *Āpastamba Śulbasūtra* (=*ĀŚS*) 1.6 and *Katyāyana Śulbasūtra* (=*KŚS*) 2.9. Less accurate (i.e., *sthula*) values of $\sqrt{2}$ are also found in the *Baudhāyana Śulbasūtra* (= *BŚS*) (p.174)[31].

On the *Śulba* pattern, commentator Rāma (15th century AD) gave the following improved approximation (p.85)[24&35].

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} - \frac{1}{3.4.34.33} + \frac{1}{3.4.34.34} \quad \ldots(1.2)$$

$$= \frac{577}{408} - \frac{1}{3.4.34.33.34} = \frac{647393}{457776}$$

(This gives correct value to seven places of decimal.)

In the *BŚS* (2.11), *AŚS* and *KŚS* (3.12), an approximate values of $\sqrt{3}$ is frequently used implicitly (pp. 163-164)[31].

1.3 For the sake of completeness, approximate values of and available in other ancient cultural areas have been cited.

Babylonian value of (ca. 1800-1600 BC) in fractional presentation may be given as below[18] :

…(1.3)

=1.41421296…

Evidently, this Babylonian value is a little more accurate than the *Śulba* value 1.41421568… of $\sqrt{2}$. Many approximations to the value of poorer than (1.23) are known in Greek sources (Part I, p. 155)[21] (see also (p.132)[5] and [ref. 38]).

Ptolemy (200 BC) of Greek gives the value of which may be expressed in fractions (pp. 23-24)[22] as:

$$\sqrt{3} = 1 + \frac{43}{60} + \frac{55}{60^2} + \frac{23}{60^2} \qquad \ldots(1.4)$$

$$\sqrt{2} = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^2}$$

=1.7320509…

Proof of the method for obtaining approximation (1.1) is not yet known exactly. Of course, several ways of obtaining this formula appears to have been suggested much later (cf. Datta[11], Gurjar[20] and Thibaut[39]; see also[2-4], [5], [15] and [31]).

1.4 Āryabhaṭa I (b. 476 AD) gave an algebraic method of extracting square- and cube–roots of positive integers[5,7], Mahāvīra (ca. 850 AD), Śridhara (fl. 850-950AD), Āryabhaṭa II (ca. 950 AD), Bhāskara II (b. 1114 AD) and Kamalākara (fl. 1616-1700AD) have also given algebraic methods for extracting square-roots (p.79)[5]. Although the rule of the extraction of square-root is found in the *Āryabhaṭīya*, it does not at all mean that Āryabhaṭa I is the inventor of the rule, which is evident in the reference to Maskarī, Putana etc. who had written books on mathematics in great details[42]. Brahmagupta (b. 598 AD), Mahāvīra, Śridhara, Āryabhaṭa II and Bhāskara II and Kamalākara have attempted to give similar algebraic methods for obtaining cube-roots (p.80)[5]. For details on extraction of roots, See [1,9-10,12-14,25-26,28,32-34]. Square-and cube-root methods available in Hindu and Jaina works are also found in Arabic works from 9th century onwards (Part I, pp.138-139)[36]. In the sixteenth century AD both square-and cube-roots were given by Canteneo which are exactly the same as those of Āryabhaṭa I [op. cit., Part II, p.148]. Moreover, modern methods of square-and cube-root extraction are simply reduced form of Āryabhaṭa I's methods. It seems that essentially Indian methods of root extraction travelled to Europe via Arab and returned back to India from Europe in a slightly cosmetized form. In traditional learning schools of ancient Indian astronomy, these methods of square-and cube-root (especially from the *Līlāvatī* of Bhāskara II) are still taught and used in India

It would not be a miss to mention that square-and cube-root methods as available basically in ancient Indian mathematical compositions are essentially based on the inversion theme of the formulae[10,34]: $(x + y)^2 = x^2 + 2xy + y^2$ and $(x + y)^3 = x^3 + 3xy^2 + 3x^2y + y^3$.

2. Computation of $\sqrt{N}$

2.1 With a view to improving accuracy of approximative calculations obtained by algebraic methods, ancient Indian mathematicians appear to use iterative procedures while solving equations etc. Brahmagupta in his *Khaṇḍakhādyaka* (c. 655 AD) gives an iterative rule for finding the arc or angle, when its *sine* is known[16]. This method is frequently used in Indian system of astronomical calculations (see, for example, *Mahābhāskarīa*, cf. [op. cit.]).

Gurjar proved (1.1) using iterative procedure. This procedure for $\sqrt{2}$ has been modified and extended to square root of *N* in the form of Proposition as below.

2.2 Proposition: Let N>0 be a non-square integer such that $N = A_1^2 + r$ where A_1 is the largest positive integer and | r | the smallest integer. Then

$$A_i = \frac{Nd_{i-1}^2 + n_{i-1}^2}{2d_{i-1}n_{i-1}}, \quad i = 1, 2, 3, \ldots, n \qquad \ldots(2.1)$$

are the successive convergents to the series

And $$\sqrt{N} = A_1 + \frac{r}{m} + \sum_{i=3}^{\infty} \frac{Nd_{i-1}^2 - n_{i-1}^2}{2d_{i-1}n_{i-1}}. \qquad \ldots(2.2)$$

where $d_{i\text{-}1}$ and $n_{i\text{-}1}$ stand for the sum of denominator and numerator up to (i-1) terms respectively and m for a positive integer.

Proof. Let

Step 1: Let , …(2.3)

where R_1 stands for the remainder term.

On squaring (2.3),

This gives

Suppose $\frac{r}{2A_1} = \frac{r}{m} + A$. Then

$$[illegible] = \frac{r}{2A_1} - \frac{R_1^2}{2A_1}$$

and .

Substituting this value of R_1 in (2.3) yields

$$\sqrt{N} = A_1 + \frac{r}{m} + R_2 \qquad \text{…(2.4)}$$

That is

, where …(2.5)

Now squaring (2.5), we obtain

, where

And, therefore from (2.5),

$$\sqrt{N} = A_1 + \frac{r}{m} + \frac{N - A_2^2}{2A_2} + R_3 \qquad \ldots(2.6)$$

That is

, where …(2.7)

Repeating the above iterative process gives

$$\sqrt{N} = A_1 + \frac{r}{m} + \sum_{i=3}^{\infty} \frac{N - A_{i-1}^2}{2A_{i-1}} \qquad \ldots(2.8)$$

and the ith convergent is given by

$$A_i = \frac{N + A_{i-1}^2}{2A_{i-1}} \qquad \ldots(2.9)$$

$$\sqrt{N} = A_3 + R_3, \quad A_3 = A_2 + \frac{N - A_2^2}{2A_2}$$

$$A_2 = A_1 + \frac{r}{m} = 1 + \frac{1}{3} = \frac{4}{3}$$

$$A_3 = A_2 + \frac{Nd_2^2 - n_2^2}{2d_2n_2} = \frac{4}{3} + \frac{2}{2.3.4} = \frac{17}{12};$$

Taking $A_{i-1} = n_{i-1} / d_{i-1}$ in (2.8) and (2.9) establishes the Proposition.

Remark: The above Proposition can easily be extended to $\sqrt[n]{N}$.

Example

For $N = 2$; $A_1 = 1$; $r = 1$. Choose $m = 3$. Now we shall apply our Proposition here.

(Notice that $d_2 = 3$, $n_2 = 4$)

($d_3 = 12$, $n_3 = 17$)

$$A_4 = A_3 + \frac{Nd_3^2 - n_3^2}{2d_3n_3} = \frac{17}{12} - \frac{1}{2.12.17} = \frac{577}{408} \quad (d_4 = 408,\ n_4 = 577)$$

Therefore we obtain a remarkable infinite series expansion representation

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} - \frac{1}{3.4.34.1154} + \ldots \qquad \ldots(2.10)$$

The successive convergents are:

, , $A_3 = \frac{17}{12} = 1.4166667$, $A_4 = \frac{577}{408} = 1.4142157$,

$A_5 = \frac{665857}{470832} = 1.4142316$, etc.

Datta[11] (1932) in his book, provides the *Śulbākāras* plausible geometrical proof of (2.10) based on the combination of two unit squares.

If we take m = 2 in the above example, the corresponding series becomes $= \frac{577}{408} - \frac{1}{2.408.577} = \frac{665857}{470832}$

$$\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{2.6} - \frac{1}{2.6.34} - \frac{1}{2.6.34.1154} + \ldots \qquad \ldots(2.11)$$

Thus we observe that a large number of infinite series representations for are available for different values of *m*.

An analogous treatment will yield a large variety of infinite series representations for . For example, for m = 3,

…(2.12)

3. Conclusion

A. The evaluation of square-root of non-square integers in the period seems to be very much based on the series (2.2) with $m = 2A_1 + 1$ (odd). The reasons being:

(a) The series (1.2) with m = 2.1 + 1 = 3 takes exactly the same form as (2.10).

(b) The *ĀŚS* (3.3), *BŚS* (2.11) and *KŚS* (3.12) give a rule for quadrature of a circle that mathematically takes the form:

…(3.1)

wherein 2a is the side of the square and d its diameter.

According to (pp.146-147][11] (see also (pp.162-164)[31], the rule (3.1) corresponds to

if we take $\sqrt{3} = 1 + \frac{2}{3} + \frac{1}{15} = \frac{26}{15}$.

This pattern of $\sqrt{3}$ is akin to (2.12). (Notice that here m = 2x1+1=3)

$$\frac{N + A_1^2}{2A_1} = \frac{Nd_i^2 + n_i^2}{2d_i n_i}$$

The geometers of the post-*Śulba* period (including Jainas and the *Bakhsali Manuscript*) use the same series (2.2) with m = 2A_1 (even). For example to evaluate "Jainas" prefer m = 2x3 = 6.

Why ancients have not used m = 2A_1 + k (k 0, 1; k being an integer) is a matter of great concern. This untold computational tradition needs further investigation.

B. Hence as per the remark on the above Proposition, the formula is exactly derived from Newton-Raphson iterative method. Let . Then and . So the first and second approximations of $\sqrt{N}$ are respectively A_1 and

Continuing of above technique yield third and subsequent approximations.

C. Yet another concept from Narāyaṇa's method (ca. 1356) was also explored. The following Lemma due to Brahmagupta is less known:

Lemma: Let *N* be a positive integer. If two integral solutions of the equation

…(3.2)

are known then any number of other solutions can be found.

That is, if (p, q) and (p',q') then ($pq' + p'q, qq' + Npp'$) is also a solution of (3.2). In particular, if we take (p, q) = (a, b) in this Lemma then (2ab, $b^2 + Na^2$) is also a solution of (3.2). Notice that (0,1) is the trivial solution of (3.2). For finding an approximate value of square root, Narāyaṇa in his *Bijagaṇita*, gave the rule which is put as:

If (a, b) is a solution of (3.1), then , the first approximation. From

(3.2) . …(3.3)

Narāyaṇa anticipated that if y (so also x) is large, $\frac{y}{x}$ is a close approximation of the second approximation would be . Continuing in this way gives subsequent approximations. Interestingly, Binomial expansion may be applied to the square-root term on the right hand side of (3.3). Of course, there exist many parallel abstractions of $\sqrt{N}$ in the other culture areas of the world.

D. In order to find , Greek Pythagoreans studied the so called Pell's equation (pp. 55-56)[22]

…(3.4)

In Greek terminology, the pair (x, y) was called side- and diameter (diagonal) –numbers respectively. As the values increase the ratio of y to x approximates more and more closely to . Pythagoreans found a way of generating larger and larger solutions by means of recurrence relations

$$y_{n+1} = x_n + y_n$$

Further if (x_n, y_n) satisfies (3.4) then (x_{n+1}, y_{n+1}) satisfies . Let (1,0) be the trivial solution of and we generate the larger solutions. For plausible geometrical proof refer to Stillwell (pp. 33-34)[38].

E. The values of in various culture areas of world are shown graphically.

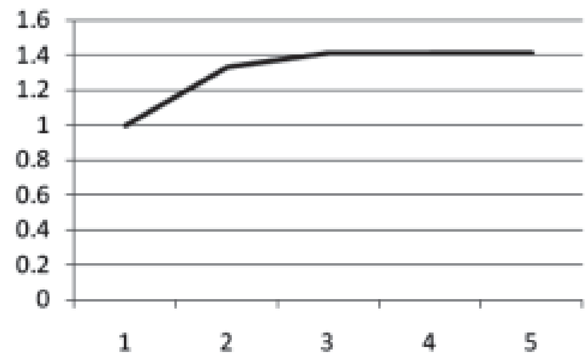

Fig. 1. The values of (A_i vs i) as in Proposition (with m = 3)

$\sqrt{2}\,2^2 - y^2 - 1{+}1$

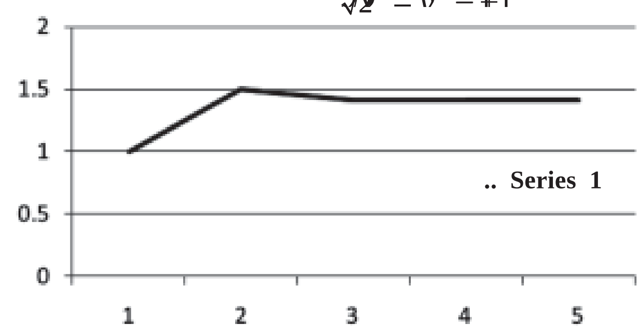

Fig. 2. The values of (A_i vs i)

This curve corresponds to Narāyaṇa's, Pythagoreans, the *Bakhsali Manuscript*, Jainas and Newton-Raphson approach for with $m = 2$ is inferred directly or indirectly. The integer 1 is chosen as initial approximation to Newton-Raphson method.

4. Program

Most often undergraduate students in engineering and science with major in Mathematics undergo Course in Numerical Methods (Theory and Computational

Lab). For making program for Newton-Raphson Method to find the root of a positive integer or an equation, in particular the square-root in C-Language, the author may be consulted.

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