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BOOK REVIEW

Kim Plofker: ***Mathematics in India,*** Princeton University Press, Princeton and Oxford, 2009, 357 pages.

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The *Mathematics in India* by Kim Plofker is a well researched publication supported by quotations in English translation of the original sources and extracts from other publications in the field, and a genuine attempt to highlight the mathematical heritage of India in its various dimensions from the early Vedic phase to 1800 AD. The venture for such a volume is not an easy task, since majority of important original sources, archaeological, textual or commentaries, are either not available in translations, and if available, are not always free from faults. The fixation of chronology of these sources is also an added problem which is a burden for historians of mathematics. The author has, however, set two-fold objectives : first, collation of the materials, of both techniques and texts in order to find a trustworthy information on various aspects of Indian mathematical tradition maintaining a chronological order, and then setting an aim to narrow down the gaps in knowledge between the actual achievements attained through various researches in mathematics and the perception of general and interested readers on the subject.

Planned in nine chapters, first three chapters give historical background and perspective details underlining varieties of problems and difficulties in setting the mathematical traditions of India, mathematical thoughts of Vedic Indians and of the early classical period, chapters 4 &5: astronomy and mathematics including a few special works in the siddhântic period (5th to 11th centuries), chapter 6: mathematics of Bhâskara II, Nârâyaṇa Paṇdita and their independent standing as mathematical texts (12-14th centuries), chapter 7 : Kerala mathematics (15-17th centuries), chapter 8 : exchanges with Islamic world, and chapter 9 on the family of astronomers and commentaries in the period during 15-18th centuries.

The Vedic knowledge system, being basically oral, as gleaned from early Vedic texts including *Úulba-sûtras* and *Vedâṅga- jyotiṣa*, has spanned through various types of ceremonies (daily, on the occasion of new and full moon, seasonal, quarterly, annual etc including optional sacrifices) and had more or less a perception of time based on the motions of sun and moon with reference to 28 or 27 *nakṣatras*, subdivision of each *nakṣatra*- space into 124 parts (*bhâmúas*), positions of full and new moon, speed of the moon and sun being 1 7/603 days, and 13 5/9 days per *nakṣatra* respectively. For this purpose, they developed a luni-solar concept of a *yuga* of 5 years, a solar year of 12 months each of 30 day's duration, a *sâvana* (civil) year of 366 days each with intercalation of 1 month in a *yuga*. The concept of *yuga* being luni-solar, the months were in *tithis*, day (sunrise to sunrise), day length, highest and shortest day length ratio, along with idea of equinoxes and solstices, often sidereal and synodic year with luni-solar adjustment. The Vedic numerical system was also unique in the sense that it was based on 'word numerals' consisting of nine units (one to nine), nine tens, and a decimal scale as big as 10^{12} and could express large numbers by arranging higher to lower order of the scale. The nature of oblations/worships, as necessitated in the construction of various shaped fire altars having the same area, led to the discovery of Pythagorian triplets including the general statement, value of $\sqrt{2}$ (correct to 5 decimals) and several improvements in the value of π, besides construction of wide varieties of geometrical figures and their transformation from one figure to another. Some of the problems specially the highest and shortest day-length ratio (3:2), Pythagorian triplets, measurement of time by gnomon shadows, synodic phenomena etc. are found common in both Vedic and ancient Mesopotamian culture on which the author's views appear to be quite realistic which says, "There is nothing in these similarities that necessarily has to be accounted for the transmission, and there are no indisputable traces for transmission such as Akkadian loan-word technical terms in Sanskrit texts. Nor are there any equally clear avenues of transmission for the centuries before the spread of empire of the Persians"(p.42).

The knowledge system in the early classical period which passed through considerable Sino- and Greco-Indian interactions were concretized further with the introduction of Brâhmi (left-to-right), Kharoṣṭhi (right-to-left) and Devnâgarî (left-to-right) system of writing in India . The Brâhmî and Nâgarî were comparatively much popular. The word numerals (mostly one to nine, some times word numerals of higher denomination as a unit were used) and zero or zero-point (the word '*úûnya* or *úûnya-bindu*' or its synonyms) following decimal place-value scale *eka-daúa-úata* .. in the left-to-right order of writing were used to express large numbers almost from the beginning of Christian era.The number was, of course, obtained by reversing the scale to read from higher-to-lower order of the scale similar to earlier Vedic tradition. Though the zero(*úûnya*) concept had been the part of philosophical tradition for a long time, the numbers 'in symbols' took a longer time to standardize because of the regional variation of symbols used by scribes, business people and others. There is little doubt that the nine numerical symbols (known as *aṅka* meaning nine symbols) and zero symbol began to appear from 5th century AD onwards in inscriptions and Sanskrit texts, best example of which are found in Bakshâli Mathematical Ms (7th century AD), and created lot of interest among the Arabic and Latin European scholars. Author's favor for a Sumerian or Chinese origin of the discovery of zero symbol does not appear to have much ground for the simple reason that both the early Sumerian sexagesimal (c.8-6th BC) and the Chinese counting board decimal place-value system(from later Han to Tan dynasty, c.200AD to625 AD) had left just a blank space limiting the very spirit of the system. Even a symbol looking like a square frame mostly used to represent the absence of characters in proof-reading of the old texts among the Chinese scholars was found used by Lu Lu Chen Shu in the 12th century to represent gap in a decimal place-value numerical system. It appears that the author was aware of this limitation when she says that 'no documentary evidence survives to conform the conclusion' (p.48). However, during this period, inescapable evidence of the Greek horoscopic astrology and Greco-Mesopotamian zodiacal system are found in three Sanskrit works in India, viz the *Arthaúâstra* of Kauṭilya, *Yavana Jâtaka* (of 269-70 AD) and *Pañcasiddhântikâ* (five *siddhântas*) of Varâhamihira (505 AD). While *Arthaúâstra* had attempted to a large extent the quantification and standardization of measurements including shadow measurement technique by gnomon which was possibly a Greek revelation, the *Yavana Jâtaka* introduced a major framework of dividing the path of the sun (ecliptic) into 12 zodiacal signs each sign being 30^0 as an alternative to the Indian *nakṣatra* system with its system of 27 divisions, along with various technical vocabularies like *drekkâna*, *navâmúa*, etc. The *Pañcasiddhântikâ*, as codified by Lâṭadeva and redacted by Varâhamihira, on the other hand, had used epicyclic and excentric models for interpreting motions of planets, carrying some similarities with those of Greco-Mesopotamian spherical astronomy. Other concepts like the short cycle of 7 days not attached to any astronomical phenomena like the yearly or monthly cycles of the Sun or Moon or to lunar phases, as the names of weekdays after seven planets in order (Ravi, Soma, Maṅgala, Budha, Bṛhaspati, Úani) of astrological origin like that of the Chaldeans who conceived them as god-luminaries controlling the destinies of kings and men, use of ruler's name in power for the purposes of chronology as in Greek empire for the dating purposes, similarities in using of ordinal numbers to indicate fixed number of days for months etc are some of the pointers. Indian novelty in each of these results are extremely interesting. Best example could be found with the use quarter circle which is undoubtedly a far advanced knowledge over the clumsy Greek full-chord geometry of right triangle in a semi-circle. The Indian derivation is very much closer to modern trigonometry and in this context the author's remarks about the hypothesis of common linkages remaining unproven is quite matured and significant (p.52). However, there is no doubt that the new modified concepts of zodiacal signs greatly influenced the luni-solar methodology and helped to lay the foundation of the subsequent modifications and developments in astronomical tradition in India.

The Siddhântic astronomy (Chap 4) gives an idea of the geocentric structure and objectives of the tradition beginning with Âryabhaṭa I (b.496 AD), the pioneer in this tradition and quite famous for his new methodologies following day-reckoning on the basis of sunrise system in his *Âryabhaṭîya* as well as the midnight system of day-counting in his *Âryasiddhânta* and introducing Kali (432 x 10^3 years) as the epochal era for finding mean motions of planets. Among his followers in the sunrise system, Bhâskara I (authors of *Âryabhaṭîya-bhâṣya*, *Mahâbhâskarîya*, *Laghubhâskarîya*), Lalla (author of *Úiṣyadhivṛddhida-tantra*), Vaṭeúvara (*Vaṭeúvara-siddhânta*), Parameúvara (b.1430 AD) and others in the South were quite well known. Brahmagupta (628 AD), a well known scholar of the Siddhântic system and a follower of the sunrise system who started with the epochal year from *Mahâyuga* (432 x 10^4 years) criticized Âryabhaṭa I in his *Brâhmasphuṭasiddhânta* for his midnight system, but later on became an exponent of the system by adopting it in his *Khaṇḍakhâdyaka*. The *Paitâmahasiddhânta* of *Viṣṇudharmottara-purâṇa*, which is a *upa-purâṇa* is supposed to have been followed and compiled as a sequel to *Brâhmasphuṭasiddhânta* of Brahmagupta, and not necessarily before Brahmagupta as presumed, since it is clear that astronomical parameters of sidereal revolution of planets, apogees and ascending nodes are exactly the same as that of Brahmagupta, and the attempt in the *Viṣṇudharmottara-purâṇa* is nothing more than a compilation by a novice unaware of large number of mistakes and meant just for popularization of the system as part of the Upa-Purâṇic tradition in the 8-9th century. The Brahmagupta tradition had been followed by Pṛthudakasvâmî (commentator), Úripati (author of

Siddhântaúekhara) and Bhâskara II (author of *Siddhântaúiromaṇi*) and others. The modern *Sûrya-siddhânta* (c 800 AD), also a follower of Âryabhaṭa I's mid-night system, gave sidereal revolutions of planets with reference to a *Mahâyuga* which explains the system in a much simplified version. The *Kalpa* (432 x 10^7 years), *Mahâyuga* (432 x 10^4 years) and *Kali* (432 x 10^3) are the measures of overflowing time, *Kalpa* being 1000 times larger than the *Mahâyuga*, and the *Mahâyuga* 10 times than the *Kaliyuga*. The *Mahâyuga*, *Caturyuga* or simply *yuga* are synonymous, and divided into Satya (or *Kṛta*), Tretâ, Dvâpara, and Kali are connected by a ratio 4:3:2:1, though. Âryabhaṭa I 's scheme of four *yugas* were taken to be of equal periods, and it was based on the assumption that three-fourths of the Caturyugas since the beginning of Kalpa had elapsed and the last quarter started with the beginning of the current Kaliyuga. The planets make an integral number of revolutions in a Kalpa or Mahâyuga and are based on the assumption that they were at a great conjunction at the commencement of the present Kali (i.e. Friday, 18 February, 3102 BC, Ujjain midnight, according to Bentley) with the initial point at the zodiac.

The Siddhântic texts, as categorized under *Âryapakṣa*, *Brahmapakṣa* and *Saurapakṣa* by the author do not apparently appear to have much significance other than the use of bigger cycles, of course, yielding more improved parameters For finding mean positions of planets, the *kuṭṭakâra* technique and *bîja* corrections, have been widely used. The purpose of corrections arises only when there were discrepancy in the computed value or findings and the observations, not the other way round as suggested. For true positions, the unevenness of the motions were mapped with the help of eccentric and epicyclic models, and its solution by applying trigonometrical tables. The *manda anomaly* is found to occur only for Sun and Moon for obliquity of their corresponding orbits, where as both *manda and úîghra* are considered for other five planets, the latter anomaly of course is due to the relative orbital motion of the planet and the earth about the Sun causing retrograde motion. A large number of iterative corrections were adopted by Siddhântic astronomers for better and better values of true positions by adjusting the sine and cosine values and that of R, the distance of the earth from the planet. For finding lunar and solar eclipses the geometrical methods of shadow triangles for calculating ecliptic limits, half-duration of the eclipse, and other related parameters were referred and not discussed in detail. The Indian sine table, used for the purpose, is very similar to our modern methods of trigonometry, with the difference, the former expressed R in minutes, while R=1 radian in modern system.

The methods found in the *Brâhmasphuṭasiddhânta* of Brahmagupta and *Úṣyavṛddhidatantra* of Lalla and a few others, cited as examples, are short and instructive. The reference of *karaṇa* work or handbook used for simplifying siddhantic procedures in matters like selecting a recent epoch, usually a date within the lifetime of the author and of calendars or *pañcâṅgas* with elements, *vâra* (weekday), *tithi* (one-thirtieth of synodic month, the difference of eastward motion of sun and moon being 12^0), *karaṇa* (half- *tithi*), *nakṣatra* (27^{th} part of a sidereal month,13^0 20´), and *yoga* (sum of the eastward motion of sun and moon equaling *nakṣatra* space, 13^0 20´) to find times of new and full moon, beginning of solar and intercalary months considered very important for rituals and human wellbeing were also mentioned.

Next is discussed the pattern of Siddhântic period mathematics (Chap 5) as it appears in proper astronomical works and in exclusive mathematical texts like *Bakhshâlî Manuscript, Gaṇitasârasamgraha* of Mahâvîra with their contents, structure and illustrations with extracts. It is somewhat puzling to note that there is no discussion on Úridhara's *Pâṭîgaṇita* and Úrîpati's *Gaṇitatilaka*, two very important works, the former of course was composed before Mahâvîra, and are ignored in the process. Úridhara seems to have written a separate book on algebra which was known to Bhâskara II, but now lost. The (*Pâṭî*)*gaṇita* appeared originally as a part of the Siddhântic astronomical tradition but slowly separated out when eight fundamental operations of arithmetic (addition, subtraction, multiplication, division, square, square-root, cube, cube-root) became increasingly popular after the invention of decimal place-value notation. These operations were carried out on a dust computing board (*pâṭî*, and hence the name *pâṭîgaṇita*).The Arabic word for *pâṭî* is *takht*, and according to Al-Khwarizmi (825 AD) and Al-Uqlidisi(952 AD), the Hindu *pâṭî* arithmetic entered Islam with dust abacus as an intrinsic tool of it. An idea of the fundamental operations including fractions, *trairâúika*, area of a triangle, circle, volume of a sphere, diagonals of triangle and quadrilateral in arithmetic & geometric, solutions of first and second degree equations, trigonometric processes and values as appeared in the works of Âryabhaṭa I, Bhâskara I, Braghmagupta, Bakhshâli manuscript, and Mahâvîra have been discussed in a nutshell.

Then follows the section on development and standardization of mathematics including trigonometry in the texts of Bhâskara II (c.1114 AD) and Nârâyaṇa Paṇdita (Chap 6). The works of Bhâskara II, viz. *Lîlâvatî* (on

Arithmetic), *Bîjagaṇita* (Algebra), *Jyotpatti* (Trigonometry), *Siddhânta-úiromaṇi* (Mathematical Astronomy) with a *Vâsanâ* (commentary) on it, became extremely popular and attained almost canonical status. Such a popularity, as reasoned by the author, was possibly for his scholarly lineage, royal and noble patronage. The justification does not appear to be convincing since Bhâskara II and his family are not known to have any close associations or linkages with any wellknown royal court or family . It will not be out of place to mention that a further investigation is perhaps needed to explore whether it is the clear exposition, great poetic skill, careful subject wise treatment and methodology of mathematical topics with large number of analogy & examples and modern planning which made the works distinct, simple, useful and more imaginative meeting the demands of the time, and could be the reason for the popularity of his works or otherwise. The *Gaṇita Kaumudî* of Nârâyaṇa Paṇdita (c.1356 AD) had also some unique features on the pulveriser, factorization of integers, permutations and combinations, combinations of diagonals in cyclic quadrilateral, magic squares etc, though originally planned after Bhâskara II's works, has its own merit. However, the *Vâsanâ* of Bhâskara II, written in prose, laid a special emphasis on the role of logical demonstration or rational in astronomical cum mathematical discourse.The traditional system, as envisaged in the works of Bhâskara II and Nârâyaṇa Paṇḍita according to author, still remained not open taking the shape of an independent discipline, rather kept a close liaison among their own group of scholars, some fall out of course could be found in other fields of *úâstras* like metrics, grammar, logic, and other technical texts. The author describes the Indian system as an individual mathematician's ingenuity and makes it a point how the Indian system of logic differed from axioms and proposition based universal proof system of Euclid, a formal methodology of mathematical sequence of steps leading to a logically unassailable conclusion.The reason behind these two approaches is of course not easy to appreciate when the philosophical trend of both Indian and European literary traditions are so different.

The Kerala mathematical tradition (Chap 7) had a strong lineage of very competent scholars following Mâdhava (c.1400) near Kochi/ Cochin with their writings in Sanskrit and in local Malayalam prose. The lineage had Namputhiri Brahmins and other lower castes scholars, and their astronomy mostly followed the Âryabhaṭa school. Mâdhava's major works are lost and his contribution on infinite series, sine and co-sine power infinite series is mostly known fom the later citations by his students and others. Parameúvara, the direct disciple of Mâdhava wrote as many as twenty-five works, and his *Dṛggaṇita* was based on the consistent studies and observations of eclipses, planetary conjunctions by means of instruments. Nîlakaṇṭha, another important scholar of this tradition composed about dozen works in astronomy of which his *Âryabhaṭîya-bhâṣya* and *Tantrasaṅgraha* are quite popular and well known. He also laid a great emphasis on observations, and made serious attempt to resolve the computational inconsistencies of the earlier models by suggesting that the heliocentric models for star planets is more suitable than geocentric one. The *Yukti-bhâṣâ* of Jyeṣṭhadeva, another important work in Malayalam prose, justifies the mathematical operations and spherical geometry of Nîlakaṇṭha's *Tantrasaṅgraha* with rationale in great detail. Úankara, another scholar of this tradition and the disciple of Nîlakaṇṭha and Jyeṣṭhadeva, is also well known for his elaborate *Yukti-dîpikâ* commentary in verse and short commentary *Laghuvivṛti* on the *Tantrasaṅgraha* following *Yukti-bhâṣâ*, *Kriyâ-kramakarî* a commentary on *Lîlâvatî* in prose, and others. The infinite series and its (n-1)th term correction for attaining the better values of π, sine and co-sine power series had developed a notion as well as application of the knowledge of calculus. The Kerala mathematics with many such features and the tradition which treated explanations and rational as valid subjects for mathematical creativity is indeed unique in its own way.

Then comes the section on Exchanges with the Islamic World (Chap 8). Perhaps it is originally through the Sasanian empire of the pre-Islamic Iran the West Asia received the Indian canonical and astronomical works and idea of decimal place-value concepts and numerals including zero, arithmetical & algebraic operations and application of trigonometric functions etc. A number of works, *Âryabhaṭîya* of of Âryabhaṭa, *Brâhmasphuṭasiddhânta* & *Khaṇḍakhâdyaka* of Brahmagupta, *Lîlâvatî*, *Bîjagaṇita* & *Karaṇa-kutûhala* of Bhâskara II and others were translated into Persian. Indian school was quite popular in the dynasties of Abbasid Khalifs, in Cordova in Spain and other centers. By 11-12th century, Islamic West started receiving the Greek school of thoughts in philosophy and astronomical ideas, nurtured them side by side for some time with those of the Indian school and was taken over completely by the Hellenistic tradition. In India, likewise, the interest in Islamic works got generated at the Muslim and Mughal courts and elsewhere through translation of some Islamic works in Sanskrit, specially the works on astrolabe by Mahendra Sûri and others, *tâjikas*- a type of Islamic astrology, *koṣṭhakas*- Sanskrit tables following *zij* astronomical tables. The Sanskrit astronomical tradition took a considerable interest in astronomical tables and astrolabes, it is intriguing to understand why this tradition showed no interest either in axiomatic deductive geometry, or in the other types of mathematical systems achieved through the Greek thoughts which was being pursued by the Islamic mathematicians so dearly.

The final section (Chap 9) covers the status of astronomical and mathematical activities during 1500-1800 in India and describes the lives of the individual scholars, their family, academic lineage, centers of study, patterns of migration of scholars to well known schools in Benares, Maharashtra, Mughal courts in Delhi & other places, contacts with Danish and Jesuit missionaries, the interest of European scholars in the field etc. Major interest was found on the writing of elaborate commentaries with detail demonstration and examples on the earlier works of *Lîlâvatî, Bîjagaṇita, Karaṇakutûhala* of Bhâskara II, *Sûryasiddhânta*, and the works like *Siddhânta-tattvaviveka* of Kamalâkara, *Siddhânta-sârvabhauma* of Munîúvara incorporating Greco-Islamic cosmological notions, celestial co-ordinates based on spherical geometry including composition of Sanskrit tables based on *zijes* in order to use them to generate yearly calendars and prediction of planetary positions. Jayasimha built up stone observatories to recheck the available parameters and took considerable interest in practical astronomy, preparation of accurate *zij* tables, astrolabes, in rendering translations to Sanskrit of Euclid and Ptolemy's works, took help of missionaries, even sent a delegation to Portugal to have more advanced knowledge in astronomy, but his effort failed to create any tangible ideas and debate for an all round effort for achievements or break through in mathematics cum astronomical traditions in India. The volume ends with two appendices on some basic features of Sanskrit language & literature and biographical data on Indian mathematicians and the bibliography.

The publication, indeed, is an excellent survey of both Indian Mathematics and Astronomy from its early phases to 1800 AD. The inclusion of mathematical astronomy in this volume of mathematic is quite obvious since they were closely connected to each other in their development. Author's twin objectives of focusing details on and about the subject including its various features, contributions, interactions and minimizing the gaps in perception of knowledge from its various stand points are considerably met. Her unbiased and independent standing as far as transmission of knowledge from Greco-Babylonian and Persian interactions to India in this period is praiseworthy. This is undoubtedly a happy trend. Perhaps it is the result of her extensive research, long association, involvement with colleagues and friends despite many serious observations made by her mentors including late Professor David Pingree of the Brown University. In this context it is still not clear what is her standing on some of the major Indian contributions to world mathematics during this period as to the decimal place-value numeration, the geometry of half chord (*jyâ* or r sine θ, *co-jyâ* or r cosine θ) and development of trigonometry, infinite series expressions and their penultimate term correction for quick convergent values for π, sine and co-sine series, the knowledge of calculus and some such results which have attracted the world-wide attention of many a scholars. A separate chapter on these items with detail discussion from international perspectives would have been useful and could enhance greatly the value of the book, just casual opinion will not lead us anywhere. However, we need similar such books by foreign and Indian experts. The book, though mainly meant for the foreign audience, will cater all types of scholars interested in Indian tradition of mathematics and astronomy, and is a must for all library.

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