Translation of Verses
with
Notes

Conventions and signs used in the Translation

- This is a prose translation of the verses of the K and M with my notes.
- A pair of angled brackets, ⟨A⟩, indicates that A has been added by me.
- A pair of parentheses, (A), indicates that A is either an explanation or an equivalent of the immediately preceding word(s).
- A pair of square brackets, [A], indicates that A is a number expressed by the word numerals or the so-called *bhūtasaṃkhyā*. For the word numerals used by Devarāja, see Appendix C. Cf. Datta & Singh 2001, part 1, pp. 53–63 and Sarma 2003.
- Before the translations of the verses of the K and M, I have added headings indicating the topics discussed.
- The word *graha* ('a planet') often means the 'distance along the orbit traversed by a planet since the epoch'. When *graha* is used in this sense, I render it as *planet* with italics.
- I use the word 'residue' for *agra* and 'remainder' for *śeṣa*, although Devarāja does not seem to distinguish them. Cf. 'atrāgraśabdaḥ śeṣavacanaḥ' in M 1.3p1.

2.1 Chapter 1: Residual Kuṭṭākāra

M 1.1. Introduction.

> After the salutation to the lover of Śrī (= Viṣṇu) and to the preceptors, this ⟨book called⟩ *Kuṭṭākāraśiromaṇi* ('Crest-Jewel of Pulverizer'), which is a special commentary (*vyākhyā*) told by me for the pleasure of intelligent people on a pair of ⟨verses for⟩ the *kuṭṭa* told by Master Āryabhaṭa, is commented on in details ⟨here in the M by myself⟩ by means of clear words, the conventional ⟨word numerals for⟩ digits, and sentences accompanied by *vṛtta* verses for questions.

K 1.1. Salutation to two goddesses and the aim of the book.

> After having saluted Ramā (= Lakṣmī) and Dharaṇī (= Bhūmidevī), Devarāja, son of Varadārya, lucidly elucidates the *kuṭṭākāra* made (formulated) by Master Āryabhaṭa.

In K 1.1p1, Devarāja derives the word *kuṭṭākāra* as follows. $\sqrt{}$kuṭṭ ('to cut, pulverize') + gha → kuṭṭa ('pulverizer') [AA 3.3.117–18]; kuṭṭa + ṭāp → kuṭṭā [AA 4.1.4]; and kuṭṭā + ākāra ('natural form') → kuṭṭākāra ('natural form of pulverizer') [Tatpuruṣa compound]; where gha = the affix *a* which produces a masculine noun (technical term), in the sense of an instrument (karaṇa) here; and ṭāp = the affix *ā* which produces a feminine noun from a masculine noun ending in -a.

K 1.2. Organization of the book.

> There are two ⟨kinds of⟩ *kuṭṭākāra*s, residual and non-residual. Here (out of the two), the residual ⟨*kuṭṭākāra*⟩ is first shown concisely, and then the non-residual is explained clearly.

K 1.3. Source of the astronomical parameters.

> Two ⟨kinds of⟩ quantities, divisor and dividend, are mentioned here (in this book) according to the way (i.e., the usage or definition) of the *Bhāskarasiddhānta* (= *Sūryasiddhānta*). These should also be applied to the *Āryabhaṭīya* by those who know the destiny.

Devarāja remarks in the commentary that the 'divisor' is the number of the solar months, or of the lunar days, or of the civil days, etc. in a *yuga* and that the 'dividend' is the number of the intercalary ⟨lunar⟩ months, or of the omitted ⟨lunar⟩ days, or of the revolutions of the planets, etc. in the same period. This terminology of 'divisor' and 'dividend' is not of the residual (see K 1.4 below) but of the non-residual *kuṭṭākāra* (see K 2.1). What Devarāja intends to say here is that he adopts the values of astronomical parameters given in the *Sūryasiddhānta* when he solves astronomical examples for *kuṭṭākāra*. See M 1.16 and K 2.5ff.

K 1.4. Type of problems treated in Chapter 1.

> One who can tell instantly the dividend-quantity from the residues of the dividend divided ⟨separately⟩ by two given divisors is an expert on the earth in the residual ⟨*kuṭṭākāra*⟩.

The type of problem of the residual *kuṭṭaka* is:

$$N = a_i x_i + R_i \quad \text{where} \quad 0 \leq R_i < a_i \quad (i = 1, 2, ..., n),$$

where N (dividend) and a_i (divisors) are positive integers and R_i (residues) non-negative integers. The verse only mentions the case $n = 2$ but in the commentary Devarāja says that 'this is an indication' (*etad upalakṣaṇam*) and that there may be many (*bahu*) divisors. The case $n \geq 3$ is in fact alluded to in Āryabhaṭa's rule (M 1.2–3), although Devarāja ignores it and gives a separate rule in K 1.6–7.

K 1.5. Plan for solution.

> Having applied the two Āryā stanzas composed by the teacher Āryabhaṭa which begin with ⟨the words⟩ 'the greater residue ...' (*adhikāgra-*) to ⟨problems for⟩ the residual ⟨*kuṭṭākāra*⟩, one should obtain the dividend by means of a bundle of operations that arise there.

M 1.2–3 (= AB 2.32–33). Āryabhaṭa's rule for *kuṭṭākāra*.

> One should divide the divisor having the greater residue by the divisor having the smaller residue. The remainders are mutually di-

vided.[1] ⟨At any optional stage of the mutual divisions, the remainder is⟩ multiplied by an intelligence ⟨number⟩ (*mati*) and added to the difference of the ⟨given⟩ residues ⟨so that the sum may be divided without remainder by the divisor at that stage⟩. // ⟨Then, one should make a column (called *vallī* or 'a creeper') which consists of the quotients of the mutual divisions and the intelligence number as well as the corresponding quotient. Among the three terms at the bottom of the column⟩, the upper is multiplied by the lower and increased by the last, ⟨and the last is discarded. This procedure is repeated until there remain only two terms in the column.⟩ When ⟨the upper of the two terms is⟩ divided by the divisor having the smaller residue, the remainder multiplied by the divisor having the greater residue and increased by the greater residue is the residue for the two divisors.

Let us first consider the case of $n = 2$. That is,

$$N = a_1x_1 + R_1 = a_2x_2 + R_2, \quad \text{where} \quad 0 \leq R_i < a_i.$$

We assume $R_1 < R_2$, and rewrite the latter relationship as

$$x_1 = \frac{a_2x_2 + (R_2 - R_1)}{a_1},$$

which, for the sake of brevity, we further rewrite as:

$$y = \frac{ax + c}{b}. \tag{I}$$

Āryabhaṭa first performs mutual divisions upon a, b, and the remainders obtained successively:

$$\begin{array}{ll}
(1) & a = bq_1 + r_1 \quad (0 \leq r_1 < b), \\
(2) & b = r_1q_2 + r_2 \quad (0 \leq r_2 < r_1), \\
(3) & r_1 = r_2q_3 + r_3 \quad (0 \leq r_3 < r_2), \\
(4) & r_2 = r_3q_4 + r_4 \quad (0 \leq r_4 < r_3), \\
\vdots & \\
(m) & r_{m-2} = r_{m-1}q_m + r_m \quad (0 \leq r_m < r_{m-1}).
\end{array}$$

[1] *śeṣaparasparabhaktam*. According to Devarāja, the kta affix of bhakta is here used to denote the original concept of the verb root (*bhāve niṣṭhā*) and therefore bhakta = bhajana. Therefore, this passage means 'Mutual divisions of the remainders ⟨are done⟩.'

Now, if we substitute (1) for a in (I), we have $y = q_1x + \frac{r_1x+c}{b}$, which can be replaced with the two equations,

$$(\text{A, 1}) \quad y = q_1x + y_1, \quad (\text{B, 1}) \quad y_1 = \frac{r_1x+c}{b} \quad \text{or} \quad (\text{B, 1})' \quad x = \frac{by_1-c}{r_1}.$$

If (B, 1) is solved, the value of y is also obtained by (A, 1), and therefore (I) is solved. If not, we substitute (2) for b in (B, 1)′ and obtain $x = q_2y_1 + \frac{r_2y_1-c}{r_1}$, which can be replaced with the two equations,

$$(\text{A, 2}) \quad x = q_2y_1 + x_1, \quad (\text{B, 2}) \quad x_1 = \frac{r_2y_1-c}{r_1} \quad \text{or} \quad (\text{B, 2})' \quad y_1 = \frac{r_1x_1+c}{r_2}.$$

If (B, 2) is solved, x and y are also obtained successively by (A, 2) and (A, 1), and therefore (I) is solved. In this way, we have two series of equations, (A) and (B).

$$\begin{array}{llllll}
(\text{A, 1}) & y = q_1x + y_1, & (\text{B, 1}) & y_1 = \frac{r_1x+c}{b} & \text{or} & (\text{B, 1})' \quad x = \frac{by_1-c}{r_1}, \\
(\text{A, 2}) & x = q_2y_1 + x_1, & (\text{B, 2}) & x_1 = \frac{r_2y_1-c}{r_1} & \text{or} & (\text{B, 2})' \quad y_1 = \frac{r_1x_1+c}{r_2}, \\
(\text{A, 3}) & y_1 = q_3x_1 + y_2, & (\text{B, 3}) & y_2 = \frac{r_3x_1+c}{r_2} & \text{or} & (\text{B, 3})' \quad x_1 = \frac{r_2y_2-c}{r_3}, \\
(\text{A, 4}) & x_1 = q_4y_2 + x_2, & (\text{B, 4}) & x_2 = \frac{r_4y_2-c}{r_3} & \text{or} & (\text{B, 4})' \quad y_2 = \frac{r_3x_2+c}{r_4},
\end{array}$$

etc.

(A, m) and (B, m) have different forms according to whether m is odd or even.

If m is odd,

$$(\text{A, m}) \quad y_{\frac{m-1}{2}} = q_m x_{\frac{m-1}{2}} + y_{\frac{m+1}{2}}, \quad (\text{B, m}) \quad y_{\frac{m+1}{2}} = \frac{r_m x_{\frac{m-1}{2}} + c}{r_{m-1}}.$$

If m is even,

$$(\text{A, m}) \quad x_{\frac{m}{2}-1} = q_m y_{\frac{m}{2}} + x_{\frac{m}{2}}, \quad (\text{B, m}) \quad x_{\frac{m}{2}} = \frac{r_m y_{\frac{m}{2}} - c}{r_{m-1}}.$$

Since we have the relationships, $b > r_1 > r_2 > \cdots > r_m \geq 1$ (hence follows the name, *kuṭṭākāra/kuṭṭaka* or 'pulverizer'), the greater the m, the easier the (B, m) to solve. Once (B, m) is solved somehow, one can easily arrive at x and y by following the series of equations, (A, 1) $\cdots$ (A, m), inversely.

Āryabhaṭa solves (B, m) when m is odd as the expression in his rule, '*added* to the difference of the ⟨given⟩ residues' (*agrāntare kṣiptam*), indicates, and calls $x_{\frac{m-1}{2}}$ 'intelligence ⟨number⟩' (*mati*), which is obtained by trial and error. Let a pair of solutions of (B, m) be (μ, ν), that is :

$$(\text{B, m}) \quad \nu = \frac{r_m\mu + c}{r_{m-1}}.$$

Then μ is an intelligence number.

The actual *kuṭṭaka* procedure carried out by Āryabhaṭa seems to be as follows. First, a is written down above b on a calculating board, $\left|\begin{array}{c} a \\ b \end{array}\right|$, divided by b, and replaced by the remainder of that division, $\left|\begin{array}{c} r_1 \\ b \end{array}\right|$, the quotient q_1 being discarded as it is useless for Āryabhaṭa's *kuṭṭaka* procedure. Then, b is divided by r_1 and replaced by the remainder of that division, $\left|\begin{array}{c} r_1 \\ r_2 \end{array}\right|$, the quotient q_2 being written down elsewhere on the calculating board. Next, r_1 is divided by r_2 and replaced by the remainder of that division, $\left|\begin{array}{c} r_3 \\ r_2 \end{array}\right|$, the quotient q_3 being written down below the previous quotient, $\left|\begin{array}{c} q_2 \\ q_3 \end{array}\right|$. This procedure is repeated. When the two remainders on the calculating board become small enough and when m is odd or, in other words, when the number of the quotients written down, $(q_2, q_3. \dots, q_m)$, is even, an integer called 'intelligence number' is sought for; the intelligence number has to satisfy the condition that, when the upper term of the last pair, $\left|\begin{array}{c} r_m \\ r_{m-1} \end{array}\right|$, is multiplied by that number and added to the difference of the two given residues, the sum is divisible by the lower term. The intelligence number μ obtained thus and the corresponding quotient ν are added to the column of the quotients, which is called *vallī* or 'a creeper' by later mathematicians including Devarāja.

To this column, from the bottom upward, the iterative procedure, 'the upper is multiplied by the lower and increased by the last, ⟨and the last is discarded⟩' is applied, and a pair of numbers, say (α, β_1), is obtained finally.[2] This procedure which Devarāja calls 'creeper-contraction' (*vallī-upasaṃhāra*) corresponds to the series of computations from (A, m) to (A, 2). The following diagram shows the successive changes of the initial column of quotients on the calculating board.

[2] In the non-residual *kuṭṭākāra*, q_1 is taken into consideration, and (β, α) is obtained finally. See K 2.2 below.

$$\begin{array}{llllll} q_2 & \to & q_2 & \to & \cdots & \to \alpha \\ q_3 & & q_3 & & & \beta_1 \\ q_4 & & q_4 & & & \\ \vdots & & \vdots & & & \\ q_m & & q_m\mu+\nu & & & \\ \mu & & \mu & & & \\ \nu & & & & & \end{array}$$

Then, $x = \alpha$ is a solution of (I). By dividing it by b $(= a_1)$,

$$\alpha = bq + \alpha_0 \quad (0 \leq \alpha_0 < b),$$

Āryabhaṭa obtains the least positive solution of (I), $x = x_2 = \alpha_0$, and finally calculates:

$$N_0 = a_2\alpha_0 + R_2.$$

This is the least positive solution for N.

As Devarāja points out in M 1.3p2, since $x = \alpha_0 + bk$ is also a solution of (I), other solutions for N are obtained by:

$$N = a_2(\alpha_0 + bk) + R_2.$$

Devarāja calls α_0 *dhruva* or 'fixed ⟨value⟩'.

As mentioned briefly under K 1.4 above, the case $n \geq 3$ is alluded to in Āryabhaṭa's rule by the compound *dvicchedāgram* or 'the residue for the two divisors'. That is, if $n \geq 3$, the N_0 is regarded as 'the residue for the two divisors', i.e., the remainder when N is divided by the least common multiple (a_0) of a_1 and a_2, and the above method is again applied to:

$$N = a_0x_0 + N_0 = a_3x_3 + R_3.$$

Repeating this procedure, one can finally obtain such an N that satisfies all the conditions.

Devarāja's interpretation of *dvicchedāgram*, which is given at the end of M 1.3p1, is different from this. According to him, it simply means that the resulting value (N_0) is 'another' dividend-quantity (*bhājya-rāśy-antaram*) 'that possesses the two divisors and the ⟨corresponding⟩ residues'. For this interpretation, Devarāja applies the *matvarthīya ac*-suffix interpretation (vowel suffix in the sense of possession) to the compound *dvicchedāgram* and supplies the irrelevant neuter word *antaram* (another) to the faked modifié *bhājya-rāśi* in order to change its gender (masculine) to the neuter, the gender of that compound which he regarded as its modifier. Later on in the commentary, he changes the gender of the compound to the masculine so that it can modify *bhājya-rāśi* itself. See 'ayaṃ dvicchedāgraḥ prathamo bhājyarāśiḥ' in M 1.4p1 and 'ayaṃ dvicchedāgro bhājyarāśiḥ' in M 1.16p. Devarāja gives his own rule for the case $n \geq 3$ in K 1.6–7.

In M 1.3p3, Devarāja points out the following.

1. It is sometimes necessary to find and use more than one intelligence number. See Ex. 2 (M 1.5).
2. If $r_1 = 0$ (and $R_2 - R_1 = a_1 q$), then $N = ka_2 + R_2$. See Ex. 3 (M 1.6).
3. If $r_2 = 0$ or $r_3 = 0$, then solve (B, 1) and obtain $x = x_2 = \mu$. Then, $N = a_2(\mu + a_1 k) + R_2$ $(k = 0, 1, 2, ...)$. See Exs. 4–6 (M 1.7–9).

Hereafter, in my notes on the examples, I express the procedure of mutual divisions, (1) to (m), as follows.

$$\left|\begin{matrix} a \\ b \end{matrix}\right| \to [q_1]\left|\begin{matrix} r_1 \\ b \end{matrix}\right| \to [q_2]\left|\begin{matrix} r_1 \\ r_2 \end{matrix}\right| \to [q_3]\left|\begin{matrix} r_3 \\ r_2 \end{matrix}\right| \to [q_4]\left|\begin{matrix} r_3 \\ r_4 \end{matrix}\right| \to \cdots$$

$$\to \begin{cases} [q_m]\left|\begin{matrix} r_m \\ r_{m-1} \end{matrix}\right| & (m\text{: odd}) \\ [q_m]\left|\begin{matrix} r_{m-1} \\ r_m \end{matrix}\right| & (m\text{: even}) \end{cases}$$

In Āryabhaṭa's *kuṭṭaka* procedure, q_1 is ignored and m is taken to be odd. In order to save space, I will rotate the 'creeper' anticlockwise by 90 degrees:

$$[q_2, q_3, q_4, \ldots, q_m, \mu, \nu] \to [q_2, q_3, q_4, \ldots, q_m\mu + \nu, \mu] \to \cdots$$
$$\to [\alpha, \beta_1].$$

M 1.4. Example 1. Standard problem.

> O learned one, tell such a quantity that would have the residues,
> one and seven, when divided by eighteen and twenty-nine ⟨respectively⟩.

That is to say,

$$N = 18x_1 + 1 = 29x_2 + 7.$$

Mutual divisions:

$$\left|\begin{array}{c}29\\18\end{array}\right| \rightarrow \left|\begin{array}{c}11\\18\end{array}\right| \rightarrow [1]\left|\begin{array}{c}11\\7\end{array}\right| \rightarrow [1]\left|\begin{array}{c}4\\7\end{array}\right| \rightarrow [1]\left|\begin{array}{c}4\\3\end{array}\right| \rightarrow [1]\left|\begin{array}{c}1\\3\end{array}\right|.$$

Computation of intelligence number:

$$\frac{1 \times 3 + (7-1)}{3} = 3 \quad \rightarrow \quad \mu = \nu = 3.$$

Therefore the 'creeper' is [1,1,1,1,3,3], to which the iterative procedure called 'creeper-contraction' is applied.

$$[1,1,1,1,3,3] \rightarrow [1,1,1,6,3] \rightarrow [1,1,9,6] \rightarrow [1,15,9] \rightarrow [24,15]$$

Then one divides the upper term 24 by the divisor having the smaller residue 18:

$$24 = 18 \times 1 + 6.$$

Hence follows the least positive solution, $\alpha_0 = 6$, which is the 'fixed' value. Therefore,

$$N = 29 \times 6 + 7 = 181.$$

Since $x_2 = 6 + 18k$ $(k = 1, 2, ...)$ are also solutions for x_2, other values of N are obtained by

$$N = 29 \times (6 + 18k) + 7.$$

Devarāja in M 1.4p2 actually obtains $N = 703$ $(k = 1)$ and 1225 $(k = 2)$.

Hereafter, Devarāja often gives the answers to his examples without a working process. In such cases, I will try to indicate the possible working process.

M 1.5. Example 2. Solution by more than one intelligence number.

> O best of the intelligent, tell eight successive quantities which have the residues eleven and seven when divided by [Ākṛti-meter] (22) and [Manus] (14) respectively.

That is to say,

$$N = 14x_1 + 7 = 22x_2 + 11.$$

Mutual divisions: $\left|\begin{array}{c}22\\14\end{array}\right| \rightarrow \left|\begin{array}{c}8\\14\end{array}\right| \rightarrow [1]\left|\begin{array}{c}8\\6\end{array}\right| \rightarrow [1]\left|\begin{array}{c}2\\6\end{array}\right|.$

Case I. Computation of intelligence number:

$$\frac{2 \times 1 + (11 - 7)}{6} = 1 \quad \to \quad \mu = \nu = 1.$$

Creeper and its contraction: $[1, 1, 1, 1] \to [1, 2, 1] \to [3, 2]$.

Hence follow $\alpha_0 = 3$ and $x_2 = 3+14k$, and therefore $N = 22\times(3+14k)+11$. Substituting 0, 1, 2, and 3 for k, Devarāja obtains the 'dividends': $N = 77$, 385, 693, and 1001.

Case II. Computation of intelligence number:

$$\frac{2 \times 4 + (11 - 7)}{6} = 2 \quad \to \quad \mu = 4,\ \nu = 2.$$

Creeper and its contraction: $[1, 1, 4, 2] \to [1, 6, 4] \to [10, 6]$.

Hence follow $\alpha_0 = 10$ and $x_2 = 10 + 14k$, and therefore $N = 22 \times (10 + 14k) + 11$. Substituting 0, 1, 2, and 3 for k, Devarāja obtains: $N = 231$, 539, 847, and 1155.

By combining these two cases Devarāja obtains the first 'eight successive quantities' that satisfy the conditions. If one applies the *kuṭṭaka* procedure after having canceled the two divisors and the difference of the residues by two, one gets $N = 22 \times (3 + 7k) + 11$, which would yield the same solutions when $k = 0, 1, 2, \ldots, 7$.

M 1.6. Example 3. $r_1 = 0$, $R_2 - R_1 = a_1q$.

> A quantity when divided ⟨separately⟩ by seven and seventy has the residues three and ten respectively. Tell that quantity, O the foremost of the fortune-tellers, if you know.

That is to say,

$$N = 7x_1 + 3 = 70x_2 + 10.$$

Devarāja in the commentary substitutes $k = 1$ and 2 in $N = 70k + 10$, and obtains $N = 80$ and 150. Generally speaking, if $r_1 = 0$ (or $a_2 = a_1q_1$) and $R_2 - R_1 = a_1q$ (though the latter condition is not mentioned in the commentary) for the equations, $N = a_1x_1+R_1 = a_2x_2+R_2$, then $N = ka_2+R_2$ $(k = 1, 2, ...)$ are the solutions.

M 1.7. Example 4. $r_2 = 0$.

> A quantity when divided ⟨separately⟩ by ten and three has the residues three and one respectively. O best of calculators, tell quickly that quantity if you know.

That is to say,

$$N = 3x_1 + 1 = 10x_2 + 3.$$

The first division only is made: $\left|\begin{matrix}10\\3\end{matrix}\right| \to \left|\begin{matrix}1\\3\end{matrix}\right|$. Computation of intelligence number: $\frac{1\times1+(3-1)}{3} = 1 \to \mu = \nu = 1$. Creeper: $[1,1]$. Hence follows $x_2 = 1 + 3k$, and therefore, $N = 10 \times (1 + 3k) + 3$, which yields $N = 13$ $(k = 0)$, $N = 43$ $(k = 1)$, etc.

M 1.8. Example 5. $r_2 = 0$. Solution by more than one intelligence number.

> Tell the first four of such quantities which when divided ⟨separately⟩ by eight and twelve have the residues four and eight respectively.

That is to say,

$$N = 8x_1 + 4 = 12x_2 + 8.$$

Mutual divisions: $\left|\begin{matrix}12\\8\end{matrix}\right| \to \left|\begin{matrix}4\\8\end{matrix}\right| \to [2]\left|\begin{matrix}4\\0\end{matrix}\right|$. Therefore, the intelligence number is calculated at the second (or 'initial' in Devarāja's expression) step.

Case I. Computation of intelligence number: $\frac{4\times1+(8-4)}{8} = 1 \to \mu = \nu = 1$. Creeper: $[1,1]$. Hence follow $\alpha_0 = 1$ and $x_2 = 1 + 8k$, and therefore $N = 12 \times (1 + 8k) + 8 \quad (N = 20,\ 116,\ ...)$.

Case II. Computation of intelligence number: $\frac{4\times3+(8-4)}{8} = 2 \to \mu = 3,\ \nu = 2$. Creeper: $[3,2]$. Hence follow $\alpha_0 = 3$ and $x_2 = 3 + 8k$, and therefore $N = 12 \times (3 + 8k) + 8 \quad (N = 44,\ 140,\ ...)$.

Case III. Computation of intelligence number: $\frac{4\times5+(8-4)}{8} = 3 \to \mu = 5,\ \nu = 3$. Creeper: $[5,3]$. Hence follow $\alpha_0 = 5$ and $x_2 = 5 + 8k$, and therefore $N = 12 \times (5 + 8k) + 8 \quad (N = 68,\ 164,\ ...)$.

Case IV. Computation of intelligence number: $\frac{4\times7+(8-4)}{8} = 4 \to \mu = 7,\ \nu = 4$. Creeper: $[7,4]$. Hence follow $\alpha_0 = 7$ and $x_2 = 7 + 8k$, and therefore $N = 12 \times (7 + 8k) + 8 \quad (N = 92,\ 188,\ ...)$.

From these four cases 'the first four' solutions are known to be: $N = 20$, 44, 68, 92. This is the method adopted by Devarāja but, as in M 1.6, if one applies the *kuṭṭaka* procedure after having canceled the two divisors and the difference of the residues by four, one gets $N = 12 \times (1 + 2k) + 8$, which would cover all the four cases when $k = 0, 1, 2, 3$.

M 1.9. Example 6. $r_3 = 0$, $a_1 > a_2$.

> Tell quickly such a quantity which when divided ⟨separately⟩ by eight and eighteen has the residues four and two respectively, if there exists ⟨in you any⟩ familiarity with the residual ⟨*kuṭṭākāra*⟩.

That is to say,

$$N = 18x_1 + 2 = 8x_2 + 4.$$

Mutual divisions: $\begin{vmatrix} 8 \\ 18 \end{vmatrix} \rightarrow \begin{vmatrix} 8 \\ 18 \end{vmatrix} \rightarrow [2] \begin{vmatrix} 8 \\ 2 \end{vmatrix} \rightarrow [4] \begin{vmatrix} 0 \\ 2 \end{vmatrix}$. Therefore, the intelligence number is calculated at the second step, which is identical with the first step. Computation of intelligence number: $\frac{8\times 2+(4-2)}{18} = 1 \rightarrow \mu = 2,\ \nu = 1$. Creeper: $[2, 1]$. Hence follow $\alpha_0 = 2$ and $\langle x_2 = 2 + 18k$, and therefore $N = 8 \times (2 + 18k) + 4$. When $k = 0\rangle$, $N = 20$. For the general solutions one has to use the reduced divisor 9 instead of the given 18: $N = 8 \times (2 + 9k) + 4$.

M 1.10. Example 7. Two kinds of remainders for more than two divisors.

> Tell me quickly such a quantity that has the residue two when divided by the three ⟨numbers⟩ beginning with [Prakṛti-meter] (21), but fourteen by the three ⟨numbers⟩ beginning with [Saṃkṛti-meter] (24).

That is to say,

$$N = 21x_{11} + 2 = 22x_{12} + 2 = 23x_{13} + 2$$

$$= 24x_{21} + 14 = 25x_{22} + 14 = 26x_{23} + 14.$$

Devarāja reduces these equations to:

$$N = 10626x_1 + 2 = 7800x_2 + 14,$$

where 10626 and 7800 are the least common multiples of the two triplets, $(21, 22, 23)$ and $(24, 25, 26)$, respectively. For the calculation of the least common multiples, Devarāja cites a half Āryā stanza from an unknown source:

> 'Having reduced the divisors two by two, one should multiply the product of those ⟨reduced numbers⟩ by the reducer.'

Devarāja then gives the answer $N = 12347414$ without a working process, which must have been as follows.

Mutual divisions: $\begin{vmatrix} 7800 \\ 10626 \end{vmatrix} \rightarrow \begin{vmatrix} 7800 \\ 10626 \end{vmatrix} \rightarrow [1]\begin{vmatrix} 7800 \\ 2826 \end{vmatrix} \rightarrow [2]\begin{vmatrix} 2148 \\ 2826 \end{vmatrix} \rightarrow [1]\begin{vmatrix} 2148 \\ 678 \end{vmatrix}$ $\rightarrow [3]\begin{vmatrix} 114 \\ 678 \end{vmatrix} \rightarrow [5]\begin{vmatrix} 114 \\ 108 \end{vmatrix} \rightarrow [1]\begin{vmatrix} 6 \\ 108 \end{vmatrix} \rightarrow [18]\begin{vmatrix} 6 \\ 0 \end{vmatrix}$. Computation of intelligence number: $\frac{6\times16+(14-2)}{108} = 1 \rightarrow \mu = 16,\ \nu = 1$. Creeper and its contraction: $[1,2,1,3,5,1,16,1] \rightarrow [1,2,1,3,5,17,16] \rightarrow [1,2,1,3,101,17] \rightarrow [1,2,1,320,101] \rightarrow [1,2,421,320] \rightarrow [1,1162,421] \rightarrow [1583,1162]$. Hence follow $\alpha_0 = 1583$ and $x_2 = 1583 + 10626k$, and therefore $N = 7800 \times (1583 + 10626k) + 14$. When $k = 0$, the value of N given by Devarāja is obtained. For the general solutions one has to use the reduced divisor 1771 instead of 10626: $N = 7800 \times (1583 + 1771k) + 14$.

M 1.11. Example 8. $R_1 = 0$.

> A quantity has the residue one ⟨when divided⟩ by ninety-one and no residue ⟨when divided⟩ by twenty. Tell quickly, O fortune-teller, such a quantity to me who is asking.

That is to say,

$$N = 20x_1 + 0 = 91x_2 + 1.$$

Devarāja gives the answer $N = 820$ without a working process, which must have been like the following.

Mutual divisions: $\begin{vmatrix} 91 \\ 20 \end{vmatrix} \rightarrow \begin{vmatrix} 11 \\ 20 \end{vmatrix} \rightarrow [1]\begin{vmatrix} 11 \\ 9 \end{vmatrix} \rightarrow [1]\begin{vmatrix} 2 \\ 9 \end{vmatrix}$. Computation of intelligence number: $\frac{2\times4+(1-0)}{9} = 1 \rightarrow \mu = 4,\ \nu = 1$. Creeper and its contraction: $[1,1,4,1] \rightarrow [1,5,4] \rightarrow [9,5]$. Hence follow $\alpha_0 = 9$ and $x_2 = 9 + 20k$, and therefore $N = 91 \times (9 + 20k) + 1$. When $k = 0, 1, 2, ...$, $N = 820$, etc.

K 1.6–7. Rule 1. Different remainders for more than two divisors ($n \geq 3$).

> When a *kuṭṭākāra* has many different divisors together with many different residues, a dividend ⟨obtained⟩ for two divisors shall be the residue, and the divisor for it shall be the product of the two divisors. This ⟨procedure⟩ should be performed repeatedly until there remain two divisors. Then, a dividend should be obtained as before.

This case is already alluded to by the compound *dvicchedāgram* ('the residue for the two divisors') in Āryabhaṭa's rule (M 1.2–3) but Devarāja's interpretation of this compound is different. See my notes on M 1.3p1 under M 1.2–3 above. Note that not the least common multiple but 'the product of the two divisors' is here taken to be the new 'divisor' corresponding to the new residue. This means that Devarāja is here concerned neither with the least positive solution nor with the general solutions but with a single solution or at most a type of solutions. See under M 1.15 below.

M 1.12. Example 9. Standard problem.

> O expert in ⟨astronomical⟩ doctrine, tell such a quantity which when divided by the three ⟨numbers⟩ beginning with seven has successively the remainders unity, [arrows] (5), and [oceans] (4).

That is to say,

$$N = 7x_1 + 1 = 8x_2 + 5 = 9x_3 + 4.$$

First consider the first pair:

$$N = 7x_1 + 1 = 8x_2 + 5.$$

Mutual divisions: $\begin{vmatrix} 8 \\ 7 \end{vmatrix} \rightarrow \begin{vmatrix} 1 \\ 7 \end{vmatrix}$. Computation of intelligence number: $\frac{1\times 3+(5-1)}{7} = 1 \rightarrow \mu = 3$, $\nu = 1$. Creeper: $[3, 1]$. Hence follow $\alpha_0 = 3$ and $x_2 = 3 + 7k$. When $k = 0$, we have $x_2 = 3$, $N_0 = 8 \times 3 + 5 = 29$, and $a_0 = 7 \times 8 = 56$. Regarding these a_0 and N_0 as a divisor and the corresponding residue, consider the next pair:

$$N = 9x_3 + 4 = 56x_0 + 29.$$

Mutual divisions: $\begin{vmatrix} 56 \\ 9 \end{vmatrix} \rightarrow \begin{vmatrix} 2 \\ 9 \end{vmatrix}$. Computation of intelligence number: $\frac{2\times 1+(29-4)}{9} = 3 \rightarrow \mu = 1$, $\nu = 3$. Creeper: $[1, 3]$. Hence follow $\alpha_0 = 1$ and $x_0 = 1 + 9k$, and therefore $N = 56 \times (1 + 9k) + 29$. When $k = 0, 1, 2, ...$, $N = 85$, etc.

This problem is exactly the same as ex. 3 for BAB 2.32–33 (Shukla's edition, p. 134).

M 1.13. Example 10. With additional equations.

> What quantity, when divided by what different divisors, has different remainders? If the same ⟨quantity⟩ is increased by the sum of the residues, it becomes clear (i.e., can be divided without remainder) by ⟨each of⟩ those divisors.

That is to say,

$$N = a_i x_i + R_i \quad \text{where} \quad 0 \leq R_i < a_i \quad (i = 1, 2, ..., n),$$

and

$$N + (R_1 + R_2 + \cdots + R_n) = a_i q_i,$$

where none of the N, a_i, x_i, R_i, and q_i is known. This is solved by means of the following rule.

M 1.14. Auxiliary rule. Meant for Exs. 10 and 11.

> The residues are optional. Their sum increased by each residue becomes the divisors when the dividend is increased by the sum of the residues. When decreased, however, the sum of the residues is decreased by ⟨each⟩ residue.

That is to say, R_i $(i = 1, 2, ..., n)$ are optionally chosen and a_i are taken to be

$$a_i = (R_1 + R_2 + \cdots + R_n) \pm R_i.$$

The minus sign is for the next example (M 1.15). This rule will be verified as follows. When an N is obtained for the R_i and a_i determined as above,

$$N \pm (R_1 + R_2 + \cdots + R_n) = (a_i x_i + R_i) \pm (a_i \mp R_i) = a_i(x_i \pm 1).$$

In M 1.14p2, Devarāja applies this rule to the case $n = 3$ of Example 10. He assumes that $R_1 = 3$, $R_2 = 4$, and $R_3 = 5$. Then, according to M 1.14, $R_1 + R_2 + R_3 = 12$, and $a_1 = 12 + 3 = 15$, $a_2 = 12 + 4 = 16$, and $a_3 = 12 + 5 = 17$. That is, the problem to be solved is:

$$N = 15x_1 + 3 = 16x_2 + 4 = 17x_3 + 5.$$

Mutual divisions for the first equation: $\left|\begin{matrix} 16 \\ 15 \end{matrix}\right| \rightarrow \left|\begin{matrix} 1 \\ 15 \end{matrix}\right|$. Computation of intelligence number: $\frac{1\times 14+(4-3)}{15} = 1 \rightarrow \mu = 14,\ \nu = 1$. Creeper: $[14, 1]$.

Hence follow $\alpha_0 = 14$ and $x_2 = 14+15k$, and therefore $N = 16\times(14+15k)+4$. When $k = 0$, $N_0 = 228$, which is the residue when N is divided by the divisor, $a_0 = a_1a_2 = 240$. That is, the problem to be solved next is:

$$N = 17x_3 + 5 = 240x_0 + 228.$$

Mutual divisions: $\left|\begin{matrix}240\\17\end{matrix}\right| \to \left|\begin{matrix}2\\17\end{matrix}\right|$. Computation of intelligence number: $\frac{2\times16+(228-5)}{17} = 15 \;\to\; \mu = 16,\ \nu = 15$. Creeper: [16, 15]. Hence follow $\alpha_0 = 16$ and $x_0 = 16 + 17k$, and therefore $N = 240 \times (16 + 17k) + 228$. When $k = 0$, we have $N = 4068$, which has the residues, 3, 4, and 5 when divided by 15, 16, and 17, respectively, and $N + (R_1 + R_2 + R_3) = 4080 = 15 \times 272 = 16 \times 255 = 17 \times 240$.

M 1.15. Example 11. With additional equations.

> What quantity, when divided by what different divisors, has different remainders? If the same ⟨quantity⟩ is decreased by the sum of the residues, it becomes clear (i.e., can be divided without remainder) by ⟨each of⟩ those divisors.

That is to say,

$$N = a_ix_i + R_i \quad \text{where} \quad 0 \le R_i < a_i \quad (i = 1, 2, ..., n),$$

and

$$N - (R_1 + R_2 + \cdots + R_n) = a_iq_i,$$

where none of the N, a_i, x_i, R_i, and q_i is known. Cf. M 1.13.

Devarāja in M 1.15p1 assumes that n is equal to 3 and that $R_1 = 5$, $R_2 = 7$, and $R_3 = 9$. Then, according to M 1.14, $R_1 + R_2 + R_3 = 21$, and $a_1 = 21 - 5 = 16$, $a_2 = 21 - 7 = 14$, and $a_3 = 21 - 9 = 12$. That is, the problem to be solved is:

$$N = 16x_1 + 5 = 14x_2 + 7 = 12x_3 + 9.$$

Devarāja gives the answer $N = 1365$, which seems to have been obtained as follows.

Mutual divisions for the first equation: $\left|\begin{matrix}14\\16\end{matrix}\right| \to \left|\begin{matrix}14\\16\end{matrix}\right| \to [1]\left|\begin{matrix}14\\2\end{matrix}\right| \to [7]\left|\begin{matrix}0\\2\end{matrix}\right|$. Computation of intelligence number: $\frac{14\times1+(7-5)}{16} = 1 \;\to\; \mu = \nu = 1$. Creeper:

$[1, 1]$. Hence follow $\alpha_0 = 1$ and $x_2 = 1 + 16k$, and therefore $N = 14 \times (1 + 16k) + 7$. When $k = 0$, $N_0 = 21$, which is the residue when N is divided by the divisor, $a_0 = a_1 a_2 = 224$. That is, the problem to be solved next is:

$$N = 12x_3 + 9 = 224x_0 + 21.$$

Mutual divisions: $\left|\begin{array}{c} 224 \\ 12 \end{array}\right| \to \left|\begin{array}{c} 8 \\ 12 \end{array}\right| \to [1]\left|\begin{array}{c} 8 \\ 4 \end{array}\right| \to [2]\left|\begin{array}{c} 0 \\ 4 \end{array}\right|$. Computation of intelligence number: $\frac{8\times6+(21-9)}{12} = 5 \ \to\ \mu = 6,\ \nu = 5$. Creeper: $[6, 5]$. Hence follow $\alpha_0 = 6$ and $x_0 = 6+12k$, and therefore $N = 224 \times (6+12k) + 21$. When $k = 0$, $N = 1365$, which has the residues, 5, 7, and 9 when divided by 16, 14, and 12, respectively, and $N - (R_1 + R_2 + R_3) = 1344 = 16 \times 84 = 14 \times 96 = 12 \times 112$.

Note that Devarāja's answer, $N = 1365$, is not the least positive solution to the problem posed by him. It is 357, which can be obtained as follows. The least common multiple of a_1 and a_2 is taken for the divisor corresponding to the new residue 21:

$$N = 12x_3 + 9 = 112x_0 + 21.$$

Mutual divisions: $\left|\begin{array}{c} 112 \\ 12 \end{array}\right| \to \left|\begin{array}{c} 4 \\ 12 \end{array}\right| \to [3]\left|\begin{array}{c} 4 \\ 0 \end{array}\right|$. Computation of intelligence number: $\frac{4\times3+(21-9)}{12} = 2 \ \to\ \mu = 3,\ \nu = 2$. Creeper: $[3, 2]$. Hence follow $\alpha_0 = 3$ and $x_0 = 3 + 3k$ (with the reduced divisor 3), and therefore $N = 112 \times (3 + 3k) + 21$. When $k = 0$, $N = 357$

In M 1.15p2, Devarāja, assuming the objection that Examples 10 and 11 contain the condition that the quantity, $N \pm (R_1 + R_2 + \cdots + R_n)$, is 'divisible' by each divisor and that this contradicts the fact that the present chapter is devoted to the 'residual' *kuṭṭākāra*, points out that the main condition of these examples is that the quantity N 'has the residue' R_i when divided by each divisor.

M 1.16. Example 12. Orbital *kuṭṭaka*: $R_p \to d$ and $\varpi_{p,0}$.

> The orbital residue of the sun measures [sky, sky, eyes, mountains, earth, eyes, mountains, moon, Ākṛti-meter, two, objects, oceans, oceans, tastes, sky, doctrines, hands, earth] (126064452221721 7200), while ⟨that⟩ of the moon [sky, hands, objects, arrows, nine, two, eight, Īśa, serpents, seasons, mountains] (768118295520) multiplied by *lakṣa* (100000). Tell the dividend corresponding to the two

residues, the day-collection, and the elapsed revolutions of the sun and the moon.

Let us use the following notation.

D: the number of civil days (*bhūdinas*) in a *yuga*.
D_K: the number of civil days in a *kalpa*.
O_p: the orbit of a planet in *yojanas* (*grahakakṣyā*).
O_u: the orbit (great circle) of the universe or the sky in *yojanas* (*khakakṣyā*).
B_p: the number of revolutions (*bhagaṇa*) of a planet in a *yuga*.
m (used as a subscript): the moon.
s (used as a subscript): the sun.
d: the day-collection or the number of elapsed civil days (*ahargaṇa*).
R_p: the orbital residue (*kakṣyāgra*) of a planet. See below for its definition.

According to the *Sūryasiddhānta* (abbr. SS),

$D = 1577917828$ days (SS 1.37),
$D_K = 1000D = 1577917828000$ days (SS 1.40),
$O_m = 324000$ *yojanas* (SS 12.85),
$O_s = 4331500$ *yojanas* (SS 12.86),
$O_u = 18712080864000000$ *yojanas* (SS 12.90),
$B_m = 57753336$ (SS 1.30),
$B_s = 4320000$ (SS 1.29).

For O_s, however, Devarāja uses not 4331500 given in SS 12.86 but $4331500\frac{12}{60}$, which he obtains by means of the equation, $O_u = O_s \cdot 1000B_s$.

One of the basic assumptions of Indian astronomy is that in either *yuga* or *kalpa* all the planets make an integral number of revolutions, which exactly amount to the distance along the orbit traversed by the moon in the same period, which in turn is regarded as equal to the orbit (great circle) of the universe, O_u. In the *Sūryasiddhānta*, the *kalpa*, which comprises one thousand *yugas*, is taken to be the standard period. Hence follow the relationships:

$$O_u = O_m \cdot 1000B_m = O_p \cdot 1000B_p.$$

Now, if a planet has made b_p revolutions when d civil days have passed since the beginning of the current Kalpa (to be precise, since 17064000 years after the beginning),[3] then the distance traversed by the planet may be calculated (or expressed) in two different ways, that is, by $O_u \cdot (d/D_K)$ and by $O_p \cdot b_p$. Hence we have the relation,

$$O_u \cdot \frac{d}{D_K} = O_p \cdot b_p, \quad \text{or} \quad \frac{O_u d}{O_p D_K} = b_p = \varpi_{p,0} + \varepsilon_{p,0},$$

where $\varpi_{p,0}$ is the integer part of the revolutions of the planet and $\varepsilon_{p,0}$ the fractional part.

In M 1.16p0, Devarāja speaks of 'three methods' (*trayaḥ prakārāḥ*) for applying the *kuṭṭaka* to this kind of problems.

(1) Let $N = O_u d$, $a_p = O_p D_K = O_p \cdot 1000D$, and $R_p = a_p \varepsilon_{p,0}$. Then,

$$N = a_p \varpi_{p,0} + R_p \quad (0 \leq R_p < a_p),$$

where R_p is called the 'orbital residue' (*kakṣyāgra*).

(2) Cancel O_u in N and a_p by the 'reducer' (*apavartaka*) or the common factor:

$$N' = O'_u d \quad \text{where} \quad O'_u = O_u/k, \quad \text{and} \quad a'_p = a_p/k.$$

(3) Cancel O_u in N and D in a_p by the reducer:

$$N' = O'_u d, \quad \text{where} \quad O'_u = O_u/k, \quad \text{and}$$

$$a'_p = O_p \cdot 1000D', \quad \text{where} \quad D' = D/k.$$

In M 1.16p, Devarāja employs the third 'method'. In any case, when a_p and R_p are given for two or more planets, one can calculate N (or N') and $\varpi_{p,0}$ by means of the *kuṭṭaka*, and d from N (or N').

In M 1.16 (Ex. 12), the 'orbital residues' of the sun and the moon,

$$R_s = 1260644522217217200 \quad \text{and} \quad R_m = 768118295520000000,$$

at the moment when d civil days have elapsed are given and the four quantities, N (or N'), d, $\varpi_{s,0}$, and $\varpi_{m,0}$, are required.

In M 1.16p, Devarāja first cancels out O_u and D by 4:

$$O'_u = 4678020216000000, \quad D' = 394479457.$$

[3] See under K 2.48cd–55ab

Then, the new dividend and divisors which correspond to these reduced values of O_u and D are:

$$N' = O'_u d,$$

$$a'_m = O_m \cdot 1000D' = 324000 \cdot 394479457000 = 127811344068000000,$$

$$a'_s = O_s \cdot 1000D' = 1708687846891391400.$$

Therefore, the problem to be solved is:

$$N' = a'_m \varpi_{m,0} + R_m = a'_s \varpi_{s,0} + R_s.$$

Now, the two divisors and the difference of the orbital residues,

$$R_s - R_m = 1183832692665217200,$$

can be reduced by the reducer, 710063022600 $(= f)$:

$$\frac{a'_m}{f} = 180000, \quad \frac{a'_s}{f} = 2406389, \quad \frac{R_s - R_m}{f} = 1667222.$$

Applying Āryabhaṭa's rule (M 1.2–3 = AB 2.32–33) to these values, which are regarded respectively as the divisor having the smaller residue, the divisor having the greater residue, and the difference of the residues, Devarāja obtains:

$$\frac{a'_s}{f} \cdot \varpi_{s,0} = 4812778 \quad (\text{as } \varpi_{s,0} = \alpha_0 = 2),$$

and

$$\begin{aligned} N' &= \left(\frac{a'_s}{f} \cdot \varpi_{s,0}\right) \cdot f + R_s \\ &= 4812778 \times 710063022600 + 1260644522217217200 \\ &= 4678020216000000000. \end{aligned}$$

The last result is expressed as '[Aṣṭi-meter, two, nails, sky, eight, mountains, tastes, oceans] (4678020216) multiplied by *arbuda*' (*arbudāhatāṣṭidvinakhakhāṣṭādrirasasindhu*), which means '467802021600000000' as has been supplied in the two published editions. Certainly the word *arbuda* means 10^8 in the commonest list of Sanskrit decimal numerals, but the calculation here requires not 10^8 but 10^9. Interestingly, Ṭhakkura Pherū (14th century) lists the Apabhraṃśa word *avva* (Skt. arbuda) for 10^9 in his table of the first 25 decimal places (GSK 1.12–14). Presumably, Devarāja too used *arbuda* in the sense of 10^9 but the person who added the numerical figures took it to mean 10^8 according to the common practice.

Then, Devarāja calculates:

$$d = \frac{N'}{O'_u} = 1000, \quad \varpi_{m,0} = 36, \quad \text{and} \quad \varpi_{s,0} = 2.$$

He obtains these values of $\varpi_{m,0}$ and $\varpi_{s,0}$ by actually dividing N' by the divisors, a'_m and a'_s, respectively. In reality, the latter value, $\varpi_{s,0} = 2$, has already been obtained in the course of the calculation of $(a'_s/f)\varpi_{s,0}$ but he does not mention it.

In conclusion, Devarāja remarks that the 'multiplication by 1000' is not necessary for Āryabhaṭa's system, where the standard period is taken to be not the *kalpa* but the *yuga*.

M 1.17. Example 13. Improper problem (*anupapanna-praśna*).

> Tell quickly such a quantity that has four as the residue ⟨when divided⟩ by eight, nine, six, and ten, but one ⟨when divided⟩ by the four numbers begining with two.

That is to say,

$$\begin{aligned} N &= 2x_1 + 1 = 3x_2 + 1 = 4x_3 + 1 = 5x_4 + 1 \\ &= 6x_5 + 4 = 8x_6 + 4 = 9x_7 + 4 = 10x_8 + 4. \end{aligned}$$

Devarāja, in the commentary, remarks: 'Here, how can such a quantity that has four as the residue ⟨when divided⟩ by six or by eight have one as the residue ⟨when divided⟩ by four or by two? Having pondered thus, one should declare that this problem is improper (*praśno 'yam anupapannaḥ*).' What Devarāja intended with these words seems to be that N cannot be both odd and even at the same time, though he does not explicitly say so.

M 1.18. Example 14. Impossible dividend (*khila-bhājya-rāśi*).

> Tell quickly such a quantity that has two as the residue ⟨when divided⟩ by forty-two and one ⟨when divided⟩ by twenty-six, if you know the residual ⟨*kuṭṭākāra*⟩.

That is to say,

$$N = 26x_1 + 1 = 42x_2 + 2.$$

Mutual divisions: $\left|\begin{matrix}42\\26\end{matrix}\right| \to \left|\begin{matrix}16\\26\end{matrix}\right| \to [1]\left|\begin{matrix}16\\10\end{matrix}\right| \to [1]\left|\begin{matrix}6\\10\end{matrix}\right| \to [1]\left|\begin{matrix}6\\4\end{matrix}\right| \to [1]\left|\begin{matrix}2\\4\end{matrix}\right| \to [1]\left|\begin{matrix}2\\0\end{matrix}\right|$. Computation of intelligence number: $\frac{2\times\mu+(2-1)}{4} = \nu$. $\to$ This is insoluble.

Devarāja remarks: 'Here, it is impossible to assume an intelligence number. It should, therefore, be said that this dividend is barren (*khila*) or impossible. When there is a reducer (i.e., a common factor) of the two divisors and it is not ⟨a reducer⟩ of the difference of the residues, then it should be known that the dividend has the state of being impossible (*khilatva*). Either when there is a common reducer of these three ⟨terms⟩ or when there is none, then it should be known that ⟨the dividend has⟩ the state of being possible (*akhilatva*).'

K 1.8p0.

As we have seen above (under M 1.2–3), Āryabhaṭa solves the reduced equation (B, m) by trial and error when m is odd, and makes 'a creeper' or a sequence of the quotients of the m mutual divisions excepting the first. Every 'creeper', then, contains an even number of quotients according to Āryabhṭa's rule. Devarāja, in K 1.8p0, points out that one may make a sequence of an odd number of quotients, provided that in that case one should subtract the difference of the residues from the product of the remainder and the intelligence number. That is, one may solve

$$\text{(B, m)} \quad \nu = \frac{r_m\mu - c}{r_{m-1}},$$

when m is even.

For example, Ex. 1 ($N = 18x_1 + 1 = 29x_2 + 7$) in M 1.4 may be solved as follows.

Mutual divisions: $\left|\begin{matrix}29\\18\end{matrix}\right| \to \left|\begin{matrix}11\\18\end{matrix}\right| \to [1]\left|\begin{matrix}11\\7\end{matrix}\right| \to [1]\left|\begin{matrix}4\\7\end{matrix}\right| \to [1]\left|\begin{matrix}4\\3\end{matrix}\right|$. Computation of intelligence number: $\frac{3\times6-(7-1)}{4} = 3 \to \mu = 6,\ \nu = 3$. Creeper and its contraction: $[1,1,1,6,3] \to [1,1,9,6] \to [1,15,9] \to [24,15]$. The rest is the same as before.

According to Devarāja, Āryabhaṭa's rule includes this case also because one can read the instrumental sense, 'by', in the locative case of the word *agrāntare* ('to the difference of the residues') by means of the interpretation technique called *vibhakti-vipariṇāma* (case change), and also because one can regard the word *kṣiptam* (lit. 'thrown') as implying *avahīna* ('abondoned' or 'decreased'). This interpretation of the phrase *agrāntare kṣiptam* plays an important role in the next rule also.

K 1.8–9. Rule 2. Āryabhaṭa's rule modified for both even and odd number of quotients.

> One should divide the divisor having the smaller residue by the divisor having the greater residue. The remainders are mutually divided. ⟨At any optional stage of the mutual divisions, the remainder is⟩ multiplied by an intelligence ⟨number⟩ (*mati*) and decreased by (or added to)[4] the difference of the ⟨given⟩ residues ⟨so that the difference (or sum) may be divided without remainder by the divisor at that stage⟩. // ⟨Then, one should make a column (called *vallī* or 'a creeper') which consists of the quotients of the mutual divisions and the intelligence number as well as the corresponding quotient. Among the three terms at the bottom of the column⟩, the upper is multiplied by the lower and increased by the last, ⟨and the last is discarded. This procedure is repeated until there remain only two terms in the column.⟩ When ⟨the upper of the two terms is⟩ divided by the divisor having the greater residue, the remainder multiplied by the divisor having the smaller residue and increased by the smaller residue is the residue for the two divisors.

Since the equations,

$$N = a_1x_1 + R_1 = a_2x_2 + R_2 \quad (\text{where } R_1 < R_2 \text{ and } 0 \leq R_i < a_i),$$

can be rewritten as:

$$x_2 = \frac{a_1x_1 - (R_2 - R_1)}{a_2},$$

my comment on Āryabhaṭa's rule (see under M 1.2–3) holds true also under the two conditions: (1) a_1 and a_2 are regarded as a and b respectively and (2) c is replaced by $-c$. Condition (1) can be realized by the exchange of the 'smaller' residue and the 'greater' residue in Āryabhaṭa's rule, and condition (2) by the new interpretation of the phrase *agrāntare kṣiptam* proposed by Devarāja in K 1.8p0 (see above). Rule 2 is the same as Āryabhaṭa's rule except for the underlined words which produce the two conditions.

[4] For this double meaning see under M 1.18 above.

According to Rule 2, as Devarāja remarks in K 1.9p, the reduced equation

$$\text{(B, m)} \quad \nu = \frac{r_m \mu - c}{r_{m-1}}$$

is solved when the sequence called 'creeper' contains an even number of quotients (i.e., when m is odd), but

$$\text{(B, m)} \quad \nu = \frac{r_m \mu + c}{r_{m-1}}$$

is solved when the creeper contains an odd number of quotients (i.e., when m is even).

K 1.10. Auxiliary rule. $R_1 \neq R_2$: a condition for Rule 2 (K 1.8–9).

> When the remainders are different ⟨from each other⟩ or when one remainder ⟨only⟩ has the state of being zero, then these two versified rules come down (i.e., become applicable). For other cases, a⟨n other⟩ computation is told ⟨in the next stanza⟩.

That is to say, Rule 2 (K 1.8–9) is applicable either when $0 < R_1 < R_2$ or when $0 = R_1 < R_2$.

K 1.11. Rule 3. For $R_1 = R_2$.

> When there is no remainder, then, having canceled out the two divisors ⟨by the largest possible reducer⟩ and multiplied the two ⟨results⟩ mutually, one should multiply ⟨the product⟩ by the reducer. When there is equality of the remainders, one should also add the remainder.

That is to say, when $R_1 = R_2 = 0$, the least positive solution of

$$N = a_1 x_1 + R_1 = a_2 x_2 + R_2,$$

is the least common multiple of the two divisors, that is,

$$N = a'_1 a'_2 f,$$

where f is the largest common factor of a_1 and a_2 and $a_1 = a'_1 f$ and $a_2 = a'_2 f$. When $R_1 = R_2 = R \neq 0$, the solution is:

$$N = a'_1 a'_2 f + R.$$

This stanza is followed by the conclusion of Chapter 1 of the K: 'Thus the first chapter on the residual ⟨*kuṭṭākāra*⟩ in the *Kuṭṭākāraśiromaṇi* composed by Devarāja' (*iti śrīdevarājaviracite kuṭṭākāraśiromaṇau sāgraparicchedaḥ prathamaḥ*). And then come two stanzas of the M which provide examples for Rule 3.

M 1.19. Example 15. $R_1 = R_2 = \cdots = R_9 = 0$.

> Tell me quickly, O the foremost of the experts in the ⟨astronomical⟩ doctrine, such a quantity that has zero as the remainder ⟨when divided⟩ by ⟨nine divisors⟩ beginning with two and ending with ten.

That is to say,

$$N = (i+1)x_i \quad (i = 1, 2, ..., 9).$$

The solution obtained is $N = 2520$.

M 1.20. Example 16. $R_1 = R_2 = \cdots = R_9 = 1$.

> Tell quickly such a quantity that has one as the residue when divided by nine divisors beginning with two and ending with ten.

That is to say,

$$N = (i+1)x_i + 1 \quad (i = 1, 2, ..., 9).$$

The solution obtained is $N = 2521$.

In M 1.20p1–p2, Devarāja refers to the problems of conjunctions of nine planetary entities, that is, the sun, moon, Jupiter, Mars, Saturn, the apogees of fast motion (*śīghra-ucca*) of Mercury and Venus, and the apogee (*ucca*) and the ascending node (*pāta*) of the moon. 'For fear of voluminousness,' he does not illustrate each conjunction but points out two facts. (1) The number of the 'problems' (*praśnas*) of conjunctions of these nine planetary entities amounts to 511. That is, the number of conjunctions of one $= {}_9C_1 = 9$, of two $= {}_9C_2 = 36$, ..., and ${}_9C_1 + {}_9C_2 + \cdots + {}_9C_9 = 511$. In order to compute ${}_nC_r$, he cites a versified rule from Varāhamihira's *Bṛhatsaṃhitā* (76.22). See M 1.21 below. (2) The 'zero-residue quantity' (*śūnya-agra-rāśi*) N for all the nine planetary entities is 1577917828 according to the *Sūryasiddhānta* (1.37) and 1577917500 according to Āryabhaṭa.[5] This is the number of civil days in the *yuga* (D, cf. M 1.16).

[5] For the latter value see MB 7.8.

9								
8	36							
7	28	84						
6	21	56	126					
5	15	35	70	126				
4	10	20	35	56	84			
3	6	10	15	21	28	36		
2	3	4	5	6	7	8	9	
1	1	1	1	1	1	1	1	1

Figure 2.1: Computation of ${}_9C_r$ $(r = 1, 2, \ldots, 9)$.

M 1.21 (= BS 76.22). Varāhamihira's rule for combinatorics.

> Each place, except the last, is increased one by one by the preceding ⟨place⟩. They point out the ⟨last⟩ numbers ⟨of those columns as the number of combinations⟩. When one has moved ⟨the tokens⟩ one by one ⟨except the first⟩ according to the desired options, and when moved ⟨completely⟩, there are termination ⟨of the first option⟩ and further moves ⟨of the tokens⟩.[6]

The first half of the stanza tells how to calculate the number of combinations, ${}_nC_r$. The resulting table of numbers is equivalent to the so-called Pascal's triangle. Figure 2.1 shows the computation when $n = 9$.

The second half of the stanza tells how to enumerate all the possible combinations when r out of n options are taken at a time. This method is called 'the spread by lump of clay or token' (*loṣṭaka-prastāra*) by the commentator Bhaṭṭotpala in his commentary on BJ 12.19.

Figure 2.2 shows the moves of tokens when three out of five options (expressed by A_i) are taken at a time. The sign $\bullet$ indicates a 'fixed token' (*sthira-loṣṭaka*) and $\circ$ a 'movable token' (*cara-loṣṭaka*).[7]

[6] The latter half of the stanza occurs also in BJ 13.4cd.
[7] Cf. Hayashi 1979.

	A_1	A_2	A_3	A_4	A_5
1	•	•	○		
2	•	•		○	
3	•	•			○
4	•		•	○	
5	•		•		○
6	•			•	○
7		•	•	○	
8		•	•		○
9		•		•	○
10			•	•	○

Figure 2.2: 'Spread by token' for ${}_5C_3$.