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PROJECT REPORTS

A Study of Devarāja's *Kuṭṭākāraśiromaṇi**

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1. Introduction

The *Kuṭṭākāra Śiromani* (*KŚ*) is an extensive commentary by Devarāja, a medieval Indian mathematician, on verses 32-33 of the second chapter of the *Āryabhaṭīya*. In these two verses Āryabhaṭa gave his *Kuṭṭākāra* (i.e. pulveriser) method for solving linear indeterminate equations with integral solutions. Devarāja also wrote his own commentary *Mahālakṣāmi Muktāvalī* (MLM) on his book *KŚ* to further explain the procedures. The *KŚ* is a rare monograph which is fully devoted to a single mathematical topic among sanskrit mathematical texts. Devarāja, apart from discussing the pulveriser method in detail, has also applied it to a number of astronomical problems.

In the present study we use the text of *KŚ* and *MLM* as edited by Balvantrai Dattatreya Apate and published in 1944 by Ānand Āśrama press at Pune. We have translated all the sanskrit verses of *KŚ* in English. All the verses with mathematical and astronomical problems have also been translated and further interpreted using the text of *MLM*. The translation of all those verses of *MLM* having some historical importance are also included in the translation. This is further supplemented by mathematical notes giving formulae and calculations involved in the modern terms.

The Sanskrit quotations from the earlier mathematical and astronomical treatises and the authors referred to, and a glossary of all the relevant sanskrit technical words with their meaning in English are supplied in the final report as

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separate appendices. A few minor discrepancies noticed at some places in the text and in the workout of the calculations have also been rectified.

2. The author: Devarāja

In the opening verses of the commentary *MLM*, the author bows with reverence to lord Viṣṇu and the teachers. Devarāja then after salutation to goddess Lakṣamī and Bhūmi, in the first verse of *KŚ*, for the smooth completion of the book and describes his lineage. It is mentioned there that *KŚ* has been written by Devarāja son of Vardārya, a distinguished member of *Atri* family. Vardārya was also known as Siddhānta Vallabh and was well versed in the three vedas.

KŚ quotes Bhāskarācārya II, who wrote *Līlāvatī* in 1150 AD and one of the manuscripts of *KŚ* found in grantha script in Tanjore library is dated 1700 AD. These two dates indicate that the author Devarāja might have flourished some time between these two dates.

3. Contents of the book KŚ

The algorithm used in *Kuṭṭākāra* method and developed by Āryabhaṭa consists of repeated divisions of integral numbers. As Devarāja notes "*kuṭṭayte, chindayate*" which means pounding the numbers and making them smaller and smaller. Therefore *Kuṭṭākāra* means pulverising the numbers. The author thus justifies the name *Kuṭṭākāra Śiromaṇi* (i.e. crest jewel of pulveriser method) of the book.

The *Kuṭṭākāra* mathematics in the book is mainly of two kinds: *Sāgra* and *Niragra*. The first chapter of the book is on *Sāgra Kuṭṭākāra*, second chapter deals with *Niragra*, *Saṃśliṣṭa* and *Vāra* and *Vela Kuṭṭākāra*. The third and the last chapter is on *Miśra Śreṇī Miśra Kuṭṭākāra*. Following is the brief discussion of the contents of all the three chapters.

3.1 *Sāgra Kuṭṭākāra*

The first chapter of the book deals with *Sāgra Kuṭṭākāra*. The chapter begins with two *sūtras* written by Āryabhaṭa in his *Āryabhaṭīya*. The *Sāgra* problem is to find a number called *Bhājya rāśi* which when divided by the two given divisors leave two different remainders.

In mathematical representation,

$$N = ax + r_1 = by + r_2,$$

where N is *Bhājya rāśi*, a and b are two given divisors and r_1, r_2 are corresponding remainders. All the quantities here including x and y are taken to be positive integers. The method is explained in detail clearly with the help of illustrations. The method is as follows:

Divide the divisor with greater remainder by divisor with smaller remainder and the quotient obtained is not used. With the remainder further divide the divisor with smaller remainder. Mutually divide the remainders until the last remainder is small. Multiply the last remainder by an appropriate number called *Mati* and add the difference of remainders to it so that it is divisible by the remainder before the last remainder. The quotients of mutual divisions are placed below each other in a column and *Mati* is placed below the last quotient and below this is placed the quotient just obtained. Now the last but one number in the column is multiplied by the number above it and the last term is added to it and then discard the last number. This procedure is continued until two numbers remains. The lower number is not used. Now divide the upper number by the divisor with smaller remainder and multiply the remainder by the divisor with greater remainder and add the greater remainder to it and *Dvichedāgra* is obtained. This is the first *Bhājya rāśi* which has two divisors and two remainders. From this an unlimited number of *Bhājya rāśis* can be obtained.

There are in all fourteen examples given which cover almost all types of *Sāgra* problems. The first example is as follows:

Example (*KŚ* 1.1/8)[1]: O! learned one, tell that number which gives the remainder 1, when divided by 18 and from 29 the remainder 7 is obtained.

In mathematical representation,

$$N = 18x + 1 = 29y + 7$$

N is the *Bhājya rāśi* to be obtained and x and y are the integers. By using *Kuṭṭā kāra* method given in the text, the first *Bhājya rāśi* comes out to be 181, the other numbers obtained are 703, 1225, ... and so on. All these give the remainder 1 and 7 when divided by 18 and 29 respectively.

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[1] The numbering of the verses from *KŚ* and *MLM* is as given in the project report preceded by the chapter number.

After fully explaining the method with the help of various examples, the author applies the method to an astronomical problem.

Example (*KŚ* 1.12/19): The Sun's remainder of the orbit (Kakṣāgra) is 1260644522217217200 and the moon's remainder of the orbit is 76811829552000000. With these two remainders given, tell the *Bhājya rāśi*, number of desired days and the number of elapsed revolutions of both sun and moon.

Using *Sāgra Kuṭṭākāra*, the *Bhājya rāśi* obtained is the number 467802021600000000, number of days is 1000 and it gives the number of elapsed revolutions of sun to be 2 and for moon to be 36.

The last two examples deal with the problems which are not solvable. The last example which is not solvable is as follows:

Example (*KŚ* 1.14/21): If you know *Sāgra*, tell quickly the number which when divided by 42 gives 2 as remainder and when divided by 26 gives 1 as remainder.

Here in this problem, the divisors have a common factor 2 but it does not divide the difference of the remainders equal to 1, therefore the *Bhājya rāśi* should be seen as not solvable. In case these three have the same common factor the problem should be taken as solvable.

Further to motivate the *Niragra* version of *Kuṭṭākāra* to be discussed in the next chapter, the author gives here a problem in which remainder is zero. The word *Niragra* refers to linear indeterminate equation with zero remainder. The example is as follows:

Example (*MLM* 1.17/24): O! foremost astronomer, tell that number which when divided by numbers two to ten gives zero as remainder.

Here from the nine divisors 2, 3 to 10 the first two are 2 and 3 which have no common factor and there multiple is 6. This is the *Niragra Bhājya rāśi* for the first two divisions. Now this and the third divisor 4 have a common factor 2. After removing the common factor from these (3, 2) respectively are obtained and their multiple is 6. This is when multiplied by common factor 2 gives the number 12. This is the *Niragra Bhājya* for the first three numbers.

Following the same procedure for the nine divisions from 2 to 10 we can get the *Niragra Bhājya rāśi* as 2520.

3.2 *Niragra Kuṭṭākāra*

The second chapter deals with the *Niragra Kuṭṭākāra*. The first verse gives the problem on the subject of *Niragra* which is as follows:

Problem (*KŚ* 2.1): Whoever tells the multiplier (*Guṇa*) and the quotient (*Phala*) fo the given divisor, *Bhājya* and interpolator (*Kṣepa*), he is the best astronomer on the earth who knows *Niragra*.

More explicitly the problem can be formulated as follows:

What should be the multiplier for the given number (*Bhājya*) which is added or substracted by the given interpolator (*Kṣepa*) so that it is divisible by the given divisor. Tell what is the multiplier (*Guṇa*) and quotient (*Phala*). In the modern notation the problem can be stated as follows:

$$\frac{ax \pm c}{b} = y$$

where *a* and *b* are the *Bhājya* and the divisor respectively, c is the interpolator. The multiplier (*Guṇa*) *x* and quotient (*Phala*) *y* are to be determined. All quantities are positive integers.

Here *Niragra Kuṭṭākāra* method is extracted from the same two *sūtras* on *Kuṭṭākāra* written by Āryabhaṭa and the method is explained in detail. The multiplier and the quotient are obtained. If the common factor of *Bhājya* and divisor is not a factor of the interpolator then it is not a solvable problem. In this way first of all determine whether the problem is solvable and if solvable then take away the common factor from *Bhājya*, divisor and the interpolator. A number of examples are given which cover various possible cases depending on (i) whether *Bhājya* is greater or lesser than the divsor (ii) whether the column of quotients in *Kuṭṭākāra* method is even or odd. The first example given here is as follows:

Example (*KŚ* 2.4): What should be multiplied to 340 so that after adding 5 it is divisible by 550. Tell quickly the multiplier and the quotient. In notations, it can be written as

$$\frac{340\ x + 5}{550} = \text{y}$$

Here, *Bhājya* = 340, Divisor = 550, interpolator = 5 where x and y are respectively the multiplier and the quotient to be determined.

Since the common factor between 340 and 550 is 10 and interpolator 5 is not divisible by 10, this problem does not have a solution.

The first solvable problem in which *Bhājya* is less than the divisor and there is no common factor is as follows:

Example (*KŚ* 2.5): What should be multiplied to the number (111) to which when 1 is added is divisible by 302. O! learned in *Niragra*, tell quickly – the multiplier and quotient.

By using the method of *Niragra Kuṭṭākāra* the multiplier obtained in 185 and the quotient is 68. The multiplier and the quotient so obtained when added to 1,2, 3 – times the divisor and *Bhājya* respectively gives infinite number of multipliers and quotients. Therefore the earlier multiplier 185 when added to one time the divisor 302 gives another multiplier 487. Similarly when earlier obtained quotient 68 is added to one time the divisor (111) and her quotient179 is obtained. There are seven more examples giving various kinds of *Niragra* problems.

3.2.1. Three *rāśi* method

The following example illustrates the Three *rāśi* method.

Example: What should be multiplied to 550 so that when 10 is subtracted it is divisible by 340. Tell quickly the multiplier and the quotient.

Mathematically,

$$\frac{550x\text{-}10}{340} = y \text{ i.e. } \frac{55x - 1}{34} = y$$

Here, the common factor is 10. The *Bhājya rāśi* reduced by this is 55. The divisor is 34. Let the negative interpolator be 5 instead of 1 as here. With these using *Kuṭṭākāra* method the multiplier obtained is 31. The quotient is the number 50. This multiplier and the quotient can also be obtained by three *rāśi* method. Now in the first example after removing the common factor interpolator is 1 and in the later example interpolator is 5 where as the *Bhājya* 55, divisor 34 are same in both the cases. If 1 is taken as the interpolator the multiplier and the quotient obtained are 13 and 21 respectively. Then by three *rāśi* method multiplier and the quotient is obtained to be 13×5 = 65 and 21×5=105 when interpolator is 5. These when divided by reduced numbers 34 and 55, the remainders, 31 and 50 are the multiplier and the quotient respectively. The multiplier and the quotient obtained by three *rāśi* method are same as obtained directly by *Kuṭṭākāra* method. Similarly in case of positive interpolator and the case when there is no common factor three *rāśi* method can be used.

3.2.2 *Samśliśṭa Kuṭṭākāra*

The later part of this chapter deals with *Samśliśṭa Kuṭṭākāra* which consists of solving many simultaneous linear indeterminate equations. There are three examples on *Samśliśṭa Kuṭṭākāra*.

Among these the first problem has the same divisor and its remainders and the *Bhājya* are different. The example is as follows:

Example (*KŚ* 2.14): Tell that number which when multiplied by numbers 11, 13, 5 and 47 and from the product numbers 1, 11, 7 and 1 are subtracted respectively, such that these are divisible by 18.

Mathematically these can be written as

$$\begin{aligned} 11x - 1 &= 18y_1 \\ 13x - 11 &= 18y_2 \\ 5x - 7 &= 18y_3 \\ 47x - 1 &= 18y_4 \end{aligned}$$

where x, y_1, y_2, y_3, y_4 are integers.

Adding these equations we get

$$76x - 20 = 18y$$

i.e.

$$38x - 10 = 9y$$

Solving this equation by the *Kuṭṭākāra* method, multiplier x is found to be 5. In the second case *Bhājya* is same and divisors and its remainders are different. In the third example all *Bhājya*, divisors and their remainders are different and common multiplier is to be obtained.

3.3 Applications of *Niragra Kuṭṭākāra* to Astronomy

In these problems the intercalary months, omitted days and planetary revolutions etc. are taken as *Bhājya rāśi*. The solar months, lunar days and civil days are taken as divisors and residue of intercalary months (*Adhimāsa śeṣa*), residue of omitted lunar days (*Avamadina śeṣa*) and residue to revolutions (*Bhagana śeṣa*) are taken as negative interpolators and then *Niragra Kuṭṭākāra* is applied to obtain multiplier and quotient for the planets.

First of all, the three problems involving *Adhimāsa śeṣa* are discussed as follows:

Problem (*KŚ* 2.19): Whoever can tell intercalary months (*Adhimāsa*), solar months and the desired number of days (*iṣṭadina*) from residue of intercalary months (*Adhimāsa śeṣa*) is the foremost astronomer on the ocean - girdled earth.

Next three verses *KŚ* 2.20, 2.21, 2.22 give the method of solving these problems. An example usign this method is given below:

Example (*KŚ* 2.23): O! learned one, the number 7319712 is seen here by me as the residue of intercalary months (*Adhimāsa śeṣa*). From this tell the number of elapsed intercalary months and elapsed solar months and by adding these get the number of desired days (*iṣṭadina*).

Using *Niragra Kuṭṭākāra* method given in the above verses we get

elapsed intercalary months	=	1687
elapsed solar months	=	54892
Total number of elapsed lunar months	=	1687 + 54892
	=	56579
Thus the total number of desired days (*iṣṭadina*)	=	1670812

Here the number of *iṣṭadina* are counted from the beginning of Kaliyuga era i.e. Friday 18, February 3102 BC.

The next set of four problems is given as follows:

Problem (*KŚ* 2.24): Whoever tells number of desired days (*iṣṭadina*) from the residue of omitted days and then without using desired number of days can tell the *graha* (mean planets) and their residues and the elapsed intercalary months, he is an expert astronomer.

For solving these four problems the five verses *KŚ* 2.25, 2.26, 2.27, 2.28, 2.29 are given. The method is illustrated by the example (*KŚ* 2.30). In this example also the number of *iṣṭadina* obtained are same as the number obtained in the previous example i.e. 1670812.

Now in the following verse seven problems based on residue of seconds (*Vikalā śeṣa*) of the planet are given

Problem (*KŚ* 3.35): Whoever tells *iṣṭadina* from *Vikalā śeṣa* and *madhya* (mean longitude) for the given planet and *madhya* and its residue for the other planets, omitted days and its residue and intercalary months from *Vikalā śeṣa* itself without using *iṣṭadina* is the learned one.

Now for solving the first two problems the multiplier and quotient calculated by *Kuṭṭākāra* method for the planets are needed.

The multiplier and quotient relevant for Moon, Mars, Jupiter, Saturn, Sighrocce of Mercury and Venus, Moon's apogee and ascending node respectively are each given in a separate verse (*KŚ* 2.38-2.44) whereas for Sun, Mercury and Venus, the multiplier and quotient are given in one verse (*KŚ* 2.37). Then a method of calculating *iṣṭadina* and mean longitude in *Vikalā* is given using *Vikalā śeṣa*, multiplier and the quotient of the planet. Now the method for calculating from the mean longitude in *Vikalā* and its *Vikalā śeṣa*, the mean longitude in *Vikalā* and its *śeṣa* for other planets, omitted days and its *śeṣa* and intercalary months is given. It is illustrated with the help of an example (*KŚ* 2.52). The number of *iṣṭadina* obtained is again 1670812 as in the previous two examples.

3.3.1 *Vāra Kuṭṭākāra*

Vāra Kuṭṭākāra deals with the astronomical problems in which platentary data such as mean longitude, residue of revolution etc. of the planets is specified at the end of the day.

Here is an example of *Vāra Kuṭṭākāra* based on the residue of the elapsed revolutions (*Gata Bhagaṇa śeṣa*)

Example (*KŚ* 2.64): The Sun's residue of the elapsed revolutions on Thursday at the end of the day is the number 172800000. When would this happen again on Friday, Monday, Tuesday and Wednesday at the end of the day. Whoever can tell this quickly is the one who knows method of *Vāra Kuṭṭākāra*.

Here in the given example residue of the solar revolutions (*Ravi Bhagaṇa śeṣa*) is 172800000 and this is the negative interpolator. Solar revolutions in a *yuga* is 4320000 and this is *Bhājya rāśi*. Civil days in a *yuga* are 1577917828 which is the divisor. From these *Bhājya*, divisor and interpolator as told earlier, the multiplier is obtained to be 40. These are the *iṣṭadina* which are counted from the beginning of the era of creation to the end of Thursday. This is called the first *kāla*. This added to one times abraded civil days 394479457 gives the number 394479497. These are the *iṣṭadina* from the beginning of the era of creation to the end of Friday. This is the second *kāla*. Now the earlier obtained number 40 added to 4 times abraded civil days gives the number 1577917868. These are the *iṣṭadina* counted from the era of creation to the end of Monday. This is the third *kāla*. Now 40 added to 5 times abraded civil days gives the number 1972397325. These are the number of days from the era of creation to the end of Tuesday. This is the fourth *kāla*. Now 40 added to 6 times abraded civil days gives the number 2366876782. This is the number of days counted from the epoch of creation upto the end of Wednesday. This is fifth *kāla*.

The value of residue of the solar revolutions *Ravi Bhagaṇa śeṣa* which is the number 172800000 is taken from *Sūrya Siddhānta*. In the same way sun's *Vāra Kuṭṭākāra* from *rāśi śeṣa* etc. should be done.

3.3.2 *Velā Kuṭṭakāra*

Velā Kuṭṭākāra is done when the data like mean longitude or the residue of the planet's revolutions are specified for any time other than the end of the day. Following is an example of *Velā Kuṭṭākāra* in which residue of the revolutions (*Bhagaṇa śeṣa*) is given.

Example (*KŚ* 2.72): Mercury's *Bhagaṇa śeṣa* was 2218093032 at the noon time in Lanka. From this if you have understood *Velā Kuṭṭākāra* properly, tell quickly the elapsed revolutions and the number of *iṣṭadina* along with its *Aṃśa*.

In *Velā Kuṭṭākāra*, civil days in a *yuga* multiplied by *cheda* is the divisor and negative interpolator and *Bhājya* are given again like *Vāra Kuṭṭākāra*. From these multiplier and quotient are obtained and among them quotient is the elapsed *Bhagaṇa* for the planet for which *Bhagaṇa śeṣa* is given and the multiplier divided by *cheda* is the *iṣṭadina* and the remainder is the *Aṃśa* related to this *cheda*.

Now for doing *Vāra* and *Velā Kuṭṭākāra* based on *Graha*, first of all *Bhagaṇa śeṣa* etc. of the planet should be obtained and then one can proceed as before. The chapter two ends with many more examples on *Vāra* and *Velā Kuṭṭākāra*.

The author also quotes at the end of the second chapter *Niragra Kuṭṭākā Sūtra* written by Bhāskarācārya II in his book *Līlāvatī* which gives a somewhat different procedure for solving *Niragra* problems. The *Kuṭṭākāra* method given in *Līlāvatī* does not calculate *Mati* as is done by Āryabhaṭa. Instead below the column of quotient, place the interpolator and at the end zero is placed. The column of quotients is concluded until two numbers remain as before. The upper value is divided by abraded *Bhājya* and quotient (*Phala*) is obtained. The lower value is divided by divisor and the remainder is multiplier (*Guṇa*). This is so if the column of quotients is even and if it is odd then *Phala* and *Guṇa* so obtained are substracted from their respective divisors and the remainder becomes the *Phala* and *Guṇa* respectively. This shows *Kuṭṭaka Gaṇita* can be done without assuming *Mati*. The chapter ends with the *Kuṭṭāka* example from *Līlāvatī*.

3.4 *Miśra Śreṇī Miśra Kuṭṭākāra*

In all, there are 19 verses in chapter III. The first 16 verses give problems and their solutions and the last three are the concluding verses for the book. For solving the problems in this chapter the tools needed are sum of the natural numbers (*Saṅkalita*), sum of the squares of natural numbers (*Varga citighana*), sum of the cubes of natural numbers (*Ghana citighana*) together with the operations involving square (*Varga*), squareroot (*Varga Mūla*), cube (*Ghana*), cuberroot (*Ghana Mūla*) etc. These sums are taken from the *Gaṇita* chapter of *Āryabhaṭīya*. To give an idea of the kind of problems discussed in this chapter the following problem given in the first verse is given here.

Problem (*KŚ* 3.1): The intelligent person should find the days from *Vikalā śeṣa* produced by the *Saṅkalita Ghanaikya*, *Ghanapada Yoga* and *Varga Ghana Yoga*.

These terms respectively refer to the expressions M given by

(i) $M = (1 + 2 + 3 + ... + N) + N^3$

(ii) $M = N^3 + N$

(iii) $M = N^2 + N^3$

The problem is to find N which is *Vikalā śeṣa* in these problems, given the quantity M. In all these problems the *Vikalā śeṣa* N is given by the integral part of the cube root of M. The calculation of the number of days from the given *Vikalā śeṣa* has already been discussed earlier and is therefore not discussed here again. The next 14 verses give the similar problems and their solutions.

In the concluding verse according to the author whoever studies *Kuṭṭākāra Śiromaṇi* with the firm mind by the grace of *Sūrya* right away becomes best among the astronomers.

Epilogue: Date of Devarāja

The above description of the project is based upon the final project report submitted to INSA on September 2, 2008. We had noted there that Devarāja must have flourished after 1150 AD (the date of *Līlāvatī* of Bhāskarācārya II) and before 1700 AD (the date of one of the manuscripts of *KŚ*). Soon after the submission of the report we realized that the astronomical calculations in the three problems *KŚ* 2.23, 2.30 and 2.52 are valuable in fixing the date of the author more precisely. In all three of them the calculation of *iṣṭadina* (i.e. the number of desired civil days) is given to be 1670812. These days are reckoned from the beginning of the *Kali*-era i.e. Friday, 18 February 3102 BC. To convert these *iṣṭadina* to Christian era, we use the procedure given by A.K. Bag in his paper "Luni-Solar Calendar, Kali Ahargaṇa and Julian days" [*IJHS* 38.1 (2003) 17-37]. Decomposing 1670812 as 1643942 + 26663 + 207 and using tables 1-3 in that paper, we obtain the date 26 July 1473 AD as the date for Devarāja. It may be noted that while the *iṣṭadina* are calculated in a number of other astronomical problems in the *KŚ* and *MLM* as well, they are obviously irrelevant in this context as they fall completely outside the known period bracket for Devarāja (1150-1700 AD).