Indian Journal of History of Science, 49.2 (2014) 142-156

TRIGONOMETRIC TABLES IN INDIA

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(Received 15 December 2012; revised 25 February 2014)

ABSTRACT

From Sanskrit literature we get various types of numerical tables such as arithmetical tables, astronomical tables, data preservation tables, reference tables, ready reckoners, trigonometric tables and difference tables for interpolation. Accuracy in astronomical determinations depends on the accuracy in tabular values. So for construction of accurate tabular values several methods are formulated. Special iterative techniques were designed for refining the values and also to generate finer tables with shorter arc bits from coarser tables with larger arc bits thereby forming some sort of *second generation* tables from known tables. A brief survey on development of trigonometric tables is carried out here from a few select works.

Key words: *Āryabhaṭa*, *Vaṭeśvara*, Rsines, Versed Rsines, *Golasāra*, *Yuktibhāṣā*, *Trijyā*

1. INTRODUCTION

Numerical tabulation is an ingenious system of representing simple and complex data elements in a compact manner for easy reference and understanding of facts, or for the purpose of storing information or for interpolation of desired functional values or for ready use in further computational procedures so as to simplify computation process to a considerable extent. Numerical tables are in fact functions. As such, study and analysis of tables point towards study and analysis of functions in a broader sense. Study of tables includes study of their history in all aspects, study of computations for preparation of tables, study of computations using tables, study of information provided by tables and study of methods for construction of tables and for reconstruction of old tables so as to improve and update them if need be. Thus an explorative study on numerical tables can be carried out by looking into various aspects such as i) the why, how and when the system of tabulation evolved and the necessity for tabulating their findings, information or data obtained ii) the period and the purpose of the various types of tables iii) the methodology and motivations involved in their constructions iv) how they were used, validity of the tabular data and how they can be updated or upgraded for current use if possible v) how to make modifications and refinements vi) mode of presentation of various tables in compact form such as in versified forms, value based *vākyas*, numerical forms etc and the usage of different systems of expressing numbers such as the *kaṭapayādi* system, *bhūtasankhyā* system, and *Āryabhaṭa*'s alphabetic system vii) various aspects involved in computation *of* tabular values and that of computations *using* the tabular values viii) construction algorithms ix) analysis of table parameters x) techniques for enhancing accuracy xi) mathematical concepts and formulae developed for the purpose of construction of tables and also for using the tabular values xii) type of tabular data and analysis of the contents xiii) classification of tables and xiv) identification and

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study of so far unexplored manuscripts dealing with numerical tables.

Several arithmetical tables can be had from the Vedic literature. Various astronomical texts provide several types of astronomical tables such as planetary tables for computation of longitudes, latitudes of planets; tables for preparation of *pañcāṅga* elements such as *tithi*, *nakṣatra*, *yoga*; eclipse tables; and several other tables giving shadow lengths, ascensions and ascensional differences, mid-heaven longitudes, daylight lengths, time and so on. Moreover they provide various trigonometric tables and their construction techniques.

2. Trigonometric Tables

Construction of trigonometric tables is an important topic for discussion in Indian astronomical works. Accuracy in astronomical determinations depends on accuracy in the functional values and so various methods were developed from time to time (Bag 1969, pp. 79-85). Aiming more accuracy, Indian mathematicians generated finer trigonometric tables with smaller arc bits from coarser tables with larger arc bits and this method deserves special mention. Starting from Āryabhaṭa I (b.476 AD) to Śankara Varman (b.1800 AD) several astronomers have dealt with trigonometric tables with different arc bits. Generally 24 tabular values were constructed at arc bits of 225′ by dividing circumference of a circle into 21600 equal parts called *kala*s (1′). The choice of 225′ is based on observation that 96th part of a circle appears to be straight. Balabhadra's (8th cent AD) observation (from a lost work *Pulisasiddhānta*) quoted in the *Al Birūni's India* (Shukla 1983, p.74; Edward C Sachau 2009, p.266) supports it. "If anybody asks the reason of this, he must know that each of these *Kardajāt* is 1/96 of the circle = 225 minutes (= $3\frac{3}{4}$ degrees). And if we reckon its sine, we find it also to be 225 minutes." Another remark (Shukla 1983, p.75; Edward C Sachau 2009, p.265) supports it further. "Human eye sight reaches to a point distant from the earth and its rotundity the 96th part of 5000 *yojana*, i.e 52 *yojana* (exactly $52\frac{1}{12}$). Therefore a man does not observe its rotundity, and hence the discrepancy of opinions on the subject." Our eyes can span 1/96 of Earth's circumference and so it appears flat. Another justification can be had from the choice of the value 3438′ for *trijyā R* by Āryabhaṭa I and from his geometrical method meant for deriving Rsines of half arcs starting from 30° up to 3°45′=225′. On reaching 225′ the chord nearly becomes the arc itself and so taken as the first tabular value. 24 tabular values are computed at arc bits of 225′ using some prescribed algorithm. The last tabular Rsine is that corresponding to the quadrantal arc which is obviously *R*. Different values for R like 60′, 120′, 3270′, 3438′ and refinements of 3438′ were used by different astronomers. Āryabhaṭa I logically arrived at the value 3438′ for *R*. Since the circumference of a circle of diameter 20000 is nearly 62832 (Shukla and Sarma 1976, verse ii. 10, p.45), the diameter corresponding to circumference 21600′ will be $\frac{21600\times 20000}{62832}'$ so that $R\approx 3438'$. Later astronomers made further refinement to this value some of which are shown in the Table 1.

The value thus gradually approaches 1 radian. Evolution of latent concept of *radian* from Āryabhaṭa's basic considerations up to Mādhava's power series concept is worth exploring.

Denote the i^{th} tabular *R*sine, Rcosine and *R*versine by $J_i = R\sin(ih)$, $K_i = R\cos(ih)$ and $V_i = R - R\sin(l-i)h = R - J_{l-i}$ for $i = 0, 1, 2, ..., l$ where $J_0 = 0, l = 3\times 2^m$, $m = 0, 1, 2, \ldots$ and $h = \frac{5400'}{l}$. Also denote tabular *R*sine differences

Table 1: Āryabhaṭa's *R* and some refinements

Āryabhaṭa I	Govindasvāmin	Vaṭeśvara	Mādhava
$R = 3438'$	$R = 3437'\ 44''\ 19'''$		$R = 3437'\ 44''\ 48'''$
$= 57°18'$	$= 57°17'\ 44.316''$	$R = \sqrt{11818047'35''} = (3437'44''19.43''')$	$= 57°17'\ 44.8''$
		$= 3437'\ 44'' = 57°17'\ 44.322''$	

$J_{i+1} - J_i$ by ΔJ_i for $i = 0, 1, 2, ..., l-1$. Generally for $l = 24$, $m = 3$ and $h = 225'$. l values are computed at arc bits of h'. Rsine of a third part of quadrantal arc is $J_{l/3} = R\sin 30^0 = \frac{R}{2}$ and that of whole is $J_l = R\sin 90^0 = R$.

2.1. Āryabhaṭa I (b. 476 AD)

Āryabhaṭa I's geometrical method (Shukla and Sarma 1976, vs ii. 11, p.45) provides two formulae $J_{l-i} = \sqrt{R^2 - J_i^2}$ and $J_{i/2} = \frac{1}{2}\sqrt{J_i^2 + V_i^2}$ with which we can compute l tabular values (Agathe 2006, pp.60-64) at arc bits of h′ or compute the first Rsine J_1 = Rsinh needed for constructing l tabular values using another algorithm. Geometrical method gives the following algorithm for finding $l = 3\times 2^m$ values (m =0, 1,…)

Step 1. $J_k = \frac{R}{2}$, $J_l = R = 3438'$, $k = \frac{l}{3}$, $l = 3\times 2^m$,

$m = 0,1,2,...$ (m = 3 for l=24).

Step 2. Get $J_{l-i} = \sqrt{R^2 - J_i^2}$, $V_i = R - J_{l-i}$,

$J_{i/2} = \frac{1}{2}\sqrt{J_i^2 + V_i^2}$; $i = k, \frac{k}{2}, l-\frac{k}{2}, \frac{k}{4}, l-\frac{k}{4}, \frac{1}{2}\left(l-\frac{k}{2}\right),...$

Step 3. Also go to Step 2 with $i = l, \frac{l}{2}, \frac{l}{4}, l-\frac{l}{4},...$

24 tabular Rsines are thus obtained as per the following scheme (Table 2)

The following algorithm gives Āryabhaṭa's values (Shukla and Sarma 1976, p.51, vs. ii. 12).

Step 1: $R = 3438'$, $J_k = \frac{R}{2}$, $J_l = R$, $k = \frac{l}{3}$, $l = 3 \times 2^m$, $m = 0, 1, 2, ...$ ($m = 3$ for $l = 24$).

Step 2: For $i = k, \frac{k}{2}, \frac{k}{4}, ..., \frac{k}{2^{m-1}}$, compute

$J_{l-i} = \sqrt{R^2 - J_i^2}$, $V_i = R - J_{l-i}$, $J_{i/2} = \frac{1}{2}\sqrt{J_i^2 + V_i^2}$

Step 3: With $J_1 = J_{k/2^m}$ find

$\Delta J_i = J_1 - \frac{J_1 + J_2 + ... + J_i}{J_1}$, $J_{i+1} = J_i + \Delta J_i$,

$i = 1, 2, 3, ..., l-1$ [One version says:

$\Delta J_i = \Delta J_{i-1} - \frac{\Delta J_0 + \Delta J_1 + ... + \Delta J_{i-1}}{J_1}$, $\Delta J_0 = J_1$ (i.e.;

$\Delta J_i = \Delta J_{i-1} - \frac{J_i}{J_1}$, $J_{i+1} = J_i + \Delta J_i$) and another:

$\Delta J_i = \Delta J_{i-1} - \frac{J_i}{J_1}(\Delta J_0 - \Delta J_1)$, $i = 2, ..., l-1$,

$\Delta J_0 = J_1$, $\Delta J_1 = J_1 - \frac{J_1}{J_1}$, $J_2 = J_1 + \Delta J_1$]

Table 3 shows Āryabhaṭa's Rsine differences (Shukla and Sarma 1976, p. 29, verse i.12), Rsines obtained from it or by geometrical method along with actual values $\sin\theta \times (10800/\pi)$.

Table 2: Āryabhaṭa's computation scheme (l = 24)

i	J_i	i	J_i
0	$J_0 = 0$	$24 = l$	$J_l = R$, *known*
$1 = \frac{k}{2^3}$	Step 3 on J_2	$23 = l - \frac{k}{2^3}$	Step 2 on J_1
$2 = \frac{k}{2^2}$	Step 3 on J_4	$22 = l - \frac{k}{2^2}$	Step 2 on J_2
$3 = \frac{l}{2^3}$	*Step 3 on J_6 (with k=l)*	$21 = l - \frac{l}{2^3}$	Step 2 on J_3 (*with k=l*)
$4 = \frac{k}{2}$	Step 3 on J_8	$20 = l - \frac{k}{2}$	Step 2 on J_4
$5 = \frac{1}{2^2}\left(l - \frac{k}{2}\right)$	Step 3 on J_{10}	$19 = l - \frac{1}{2^2}\left(l - \frac{k}{2}\right)$	Step 2 on J_5
$6 = \frac{l}{2^2}$	*Step 3 on J_{12} (with k=l)*	$18 = l - \frac{l}{2^2}$	Step 2 on J_6 (*with k=l*)
$7 = \frac{1}{2}\left(l - \frac{1}{2}\left(l - \frac{k}{2}\right)\right)$	Step 3 on J_{14}	$17 = l - \frac{1}{2}\left(l - \frac{1}{2}\left(l - \frac{k}{2}\right)\right)$	Step 2 on J_7
$8 = k = l/3$	$J_k = R/2$, known	$16 = l - k$	Step 2 on J_8
$9 = \frac{1}{2}\left(l - \frac{l}{2^2}\right)$	*Step 3 on J_{18} (with k=l)*	$15 = l - \frac{1}{2}\left(l - \frac{l}{2^2}\right)$	*Step 2 on J_9 (with k=l)*
$10 = \frac{1}{2}\left(l - \frac{k}{2}\right)$	Step 3 on J_{20}	$14 = l - \frac{1}{2}\left(l - \frac{k}{2}\right)$	Step 2 on J_{10}
$11 = \frac{1}{2}\left(l - \frac{k}{2^2}\right)$	Step 3 on J_{22}	$13 = l - \frac{1}{2}\left(l - \frac{k}{2^2}\right)$	Step 2 on J_{11}
$12 = l/2$	*Step 3 on J_{24} (with k=l)*	****************************↑	*************************↑

2.2. Sūryasiddhānta

The *Sūryasiddhānta* has given a method for construction of tabular Rsines and Rversines along with their values (Phanindralal 2000, pp.58-59, vs.ii.15-16; 22c-d;17-22a-b;23-27):

Step 1: With $J_1 = 225'$ compute

$$J_{i+1} = J_i + J_1 - \frac{J_1}{J_1} - \frac{J_2}{J_1} - \ldots - \frac{J_i}{J_1} \quad \text{for } i = 1, 2, ..., 23$$

Step 2: With $R = 3438'$ compute $V_i = R - J_{24-i}$ for $i = 1, 2, 3, ..., 24$.

Rsines are 225, 449, 671; 890, 1105, 1315; 1520, 1719; 1910, 2093; 2267, 2431; 2585, 2728; 2859, 2978; 3084, 3177; 3256, 3321; 3372, 3409; 3431, 3438 and *R*versines are 7, 29, 66, 117, 182, 261, 354, 460, 579, 710, 853, 1007, 1171, 1345, 1528, 1719, 1918, 2123, 2333, 2548, 2767, 2989, 3213, 3438.

Table 3: Āryabhaṭa's tabular values

i	Arc ih'	$\Delta J'_i$	J_i'	Modern value $\sin\theta \times (10800/\pi)$	i	Arc ih'	$\Delta J'_i$	J_i'	Modern value $\sin\theta \times (10800/\pi)$
0	0		0	0′	12	2700		2431	2430′51″14‴35.6⁗
		225					154		
1	225		225	224′50″21‴49.6⁗	13	2925		2585	2584′38″05‴32.0⁗
		224					143		
2	450		449	448′42″57‴35.1⁗	14	3150		2728	2727′20″52‴22.9⁗
		222					131		
3	675		671	670′40″16‴02.9⁗	15	3375		2859	2858′22″55‴06.5⁗
		219					119		
4	900		890	889′45″15‴36.7⁗	16	3600		2978	2977′10″33‴43.6⁗
		215					106		
5	1125		1105	1105′01″38‴56.5⁗	17	3825		3084	3083′13″16‴56.1⁗
		210					93		
6	1350		1315	1315′34″07‴26.5′′′′′	18	4050		3177	3176′03″49‴58.0⁗
		205					79		
7	1575		1520	1520′28″35‴27.6⁗	19	4275		3256	3255′18″21‴34.9⁗
		199					65		
8	1800		1719	1718′52″24‴11.2⁗	20	4500		3321	3320′36″30‴12.3⁗
		191					51		
9	2025		1910	1909′54″35‴11.3⁗	21	4725		3372	3371′41″29‴09.0⁗
		183					37		
10	2250		2093	2092′46″03‴29.5⁗	22	4950		3409	3408′20″10‴56.0⁗
		174					22		
11	2475		2267	2266′39″50‴12.5⁗	23	5175		3431	3430′23″10‴38.9⁗
		164					7		
**↑					24	5400		3438	3437′44″48‴22.5⁗

2.3. Varāha Mihira (6th cent AD)

Varāha Mihira's method (Thibaut and Dvivedi 1997, p.22, vs. iv. 2-5) is based on repeated application of $J_{\frac{1}{2}} = \frac{1}{2}\sqrt{J_i^2 + V_i^2}$ or $J_{\frac{1}{2}} = \sqrt{\left(\frac{R}{2}\right)V_i}$ starting from *R*sines of 30°, 45° and 60°. Rsines for every increase of 225′ (Thibaut and Dvivedi 1997, vs. iv. 6-15) with $R = 120'$ are 7′51″, 15′40″, 23′25″, 31′4″, 38′34″, 45′56″, 53′5″, 60′, 66′40″, 73′3″, 79′7″, 84′51″, 90′13″, 95′12″, 99′46″, 103′55″, 107′37″, 110′52″, 113′37″, 115′55″, 117′42″, 118′59″, 119′44″, 120′ and Rsine differences are 7′51″, 7′49″, 7′45″, 7′39″, 7′30″, 7′22″, 7′9″, 6′55″, 6′40″, 6′23″, 6′4″, 5′44″, 5′22″, 4′59″, 4′34″, 4′9″, 3′42″, 3′15″, 2′45″, 2′18″, 1′47″, 1′17″, 0′45″, 0′16″

2.4. Brahmagupta (7th cent AD)

Two methods are given in the *Brahmasphuṭa siddhānta* (Sharma, 1996, IV.pp.1349–1358, vs xxi.17-23) of which one is geometrical. The other based on formulae: F_1: $J_{\frac{1}{2}} = \frac{1}{2}\sqrt{J_i^2 + V_i^2}$ (or $J_{\frac{1}{2}} = \sqrt{\frac{D \times V_i}{4}}$, $D = 2R$), $V_i = R - J_{l-i} = R - \sqrt{R^2 - J_i^2}$ and F_2: $J_{l-\frac{1}{2}} = \sqrt{R^2 - J_{\frac{1}{2}}^2}$ is given by:

Step 1: Apply F_1 and F_2 for $i = k, \frac{k}{2}, \frac{k}{4}, ..., \frac{k}{2^{m-1}}$

Step 2: Repeat the same on J_n for every even n starting from $J_k = \frac{R}{2}, R = 3270', k = \frac{l}{3}$, $l = 3 \times 2^m$, $m = 0, 1, 2, ...$ ($m = 3$ for standard 24 tabular values so that $k = 8$)

Step 3: With $J_i = R$ apply the same procedure for $i = l, \frac{l}{2}, \frac{l}{4}, ..., \frac{l}{2^{m-1}}$.

Brahmagupta's standard table with R = 3270′ (Sharma, 1996, II. pp. 140–141, vs ii.1-9) gives Rsines: 214′, 427′, 638′, 846′, 1051′, 1251′, 1446′, 1635′, 1817′, 1991′, 2156′, 2312′, 2459′, 2594′, 2719′, 2832′, 2933′, 3021′, 3096′, 3159′, 3207′, 3242′, 3263′, 3270′ and Rversines: 7′, 28′, 63′, 111′, 174′, 249′, 337′, 438′, 551′, 676′, 811′, 985′, 1114′, 1299′, 1453′, 1635′, 1824′, 2019′, 2219′, 2424′, 2632′, 2843′, 3056′, 3270′. Brahmagupta's shorter table in degrees at bits of 15° with R = 150 (Sengupta 1934, I. iii. 6) gives Rsines: 39, 75, 106, 130, 145,150; with first differences: 39, 36, 31, 24, 15 and 5; and second differences: –3, –5, –7, – 9, and –10.

2.5. Govindasvāmin (800-850 AD)

Govindasvāmin's commentary on the Mahābhāskariya of Bhāskara I gives certain rules for computing accurate values and applies corrections to Āryabhaṭa's Rsine differences: -9′37″, -7′30″, -2′42″, + 4′57″, +16′22″, +32′26″, -5′34″, -36′12″, +2′09″, -8′33″, -7′02″, +12′10″, -13′11″, -17′14″, +2′02′′, -12′22′′, +2′42′′, -9′28′′, +14′31″, +18′08′′, +4′59′′, -21′19′′, +3′00′′, +21′37″ (Kuppanna Sastri, 1957, pp.200-201). Table 4 shows corrected values J_i' with modern values.

2.6. Vaṭeśvara (b. 880 AD)

Instead of equating 24th part (i.e) of quadrantal arc to its chord, Vaṭeśvara equated a fourth part of the 24th part to the chord. He divided quadrantal arc into 96 equal bits of angular measure 56′.5″. Being much smaller than the 24th part, it is more qualified to be considered straight and so first tabular sine can be more accurately equated to the angular measure of the first tabular arc. Construction of table of 96 Rsines and versed Rsines is a special feature of the *Vaṭeśvara siddhānta* (Shukla 1986, pp.81–92, Part I ch.ii. vs i.2–51). He used R^2 = 1181804 7′35″ and R = 3437′44″. Following algorithm for l = 96 gives Vaṭeśvara's values (Mallayya, 2008b).

Step 1: Divide quadrantal arc into l bits of size $h = \frac{5400'}{l}$. Take $J_0 = 0$, $J_1 = h$, $\Delta J_0 = J_1$.

Step 2: With $R^2 = 11818047'35''$ compute

$$K_1 = \sqrt{R^2 - J_1^2}, \ V_1 = R - K_1, \ \rho = \frac{R}{2V_1}$$

Step 3: For $i = 1, 2 ..., l - 1$, compute

$$\Delta^2 J_{i-1} = \frac{-\ J_i}{\rho} \text{ and } \Delta\ \ J_i = \Delta\ \ J_{i-1} + \Delta^2 J_{i-1}$$

Step 4: For $i = 1, 2 ..., l - 1$, compute $J_{i+1} = J_i + \Delta J_i$, $K_{i+1} = \sqrt{R^2 - J_{i+1}^2}$ and $V_{i+1} = R - K_{i+1}$

Vaṭeśvara's 96 Rsines along with modern values and versed Rsines are given in Table 5.

2.7. Āryabhaṭa II (c.950 AD)

Rsines by Āryabhaṭa II (Sudhakara Dvivedi 1995, p. 55, verses *iii*. 4-8) are 225, 449, 671, 890, 1105, 1315; 1520, 1719; 1910, 2093; 2267, 2431; 2585, 2728; 2859, 2977; 3084, 3177; 3256, 3321; 3371, 3409; 3431, 3438. *R*versines are 7, 29, 66, 117, 182, 261, 354, 461, 579, 710, 853, 1007, 1171, 1345, 1528, 1719, 1918, 2123, 2333, 2548, 2767, 2989, 3213, 3438. His Method is:

Step 1: $R = 3438'$, $J_k = \frac{R}{2}$, $J_i = R$, $J_{1/2} = 2431'$,

Table 4: Govindasvāmin's corrected values and modern values

i	Arc ih'	$\Delta J'_i$	J_i'	Modern value $\sin\theta \times (10800/\pi)$	i	Arc ih'	$\Delta J'_i$	J_i'	Modern value $\sin\theta \times (10800/\pi)$
0	0		0	0	12	2700		2430′50″54‴	2430′51″14‴35.6⁗
		224′50″23‴					153′46″49‴		
1	225		224′50″23‴	224′50″21‴49.6⁗	13	2925		2584′37″43‴	2584′38″05‴32.0⁗
		223′52″30‴					142′42″46‴		
2	450		448′42″53‴	448′42″57‴35.1⁗	14	3150		2727′20″29‴	2727′20″52‴22.9⁗
		221′57″18‴					131′02″02‴		
3	675		670′40″11‴	670′40″16‴02.9⁗	15	3375		2858′22″31‴	2858′22″55‴06.5⁗
		219′04″57‴					118′47″38‴		
4	900		889′45″08‴	889′45″15‴36.7⁗	16	3600		2977′10″09‴	2977′10″33‴43.6⁗
		215′16″22‴					106′02″42‴		
5	1125		1105′01″30‴	1105′01″38‴56.5⁗	17	3825		3083′12″51‴	3083′13″16‴56.1⁗
		210′32″26‴					92′50″32‴		
6	1350		1315′33″56‴	1315′34″07‴26.5⁗	18	4050		3176′03″23‴	3176′03″49‴58.0⁗
		204′54″26‴					79′14″31‴		
7	1575		1520′28″22‴	1520′28″35‴27.6⁗	19	4275		3255′17″54‴	3255′18″21‴34.9⁗
		198′23″48‴					65′18″08‴		
8	1800		1718′52″10‴	1718′52″24‴11.2⁗	20	4500		3320′36″02‴	3320′36″30‴12.3⁗
		191′02″09‴					51′04″59‴		
9	2025		1909′54″19‴	1909′54″35‴11.3⁗	21	4725		3371′41″01‴	3371′41″29‴09.0⁗
		182′51″27‴					36′38″41‴		
10	2250		2092′45″46‴	2092′46″03‴29.5⁗	22	4950		3408′19″42‴	3408′20″10‴56.0⁗
		173′52″58‴					22′03″00‴		
11	2475		2266′38″44‴	2266′39″50‴12.5⁗	23	5175		3430′22″42‴	3430′23″10‴38.9⁗
		164′12″10‴					7′21″37‴		
.... ↑					24	5400		3437′44″19‴	3437′44″48‴22.5⁗

Table 5: Vaṭeśvara's 96 tabular values

i	**Arc, ih**	**Rsine, J_i**	**Modern value $\sin\theta \times (10800/\pi)$**	**Versed Rsines V_i**	i	**Arc, ih**	**Rsine, J_i**	**Modern value $\sin\theta \times (10800/\pi)$**	**Versed Rsines V_i**
1	56′15″	56′15″	56′14″50‴57.9⁗	0′27″	49	2756′15″	2470′18″	2470′18″05‴53.4⁗	1046′59″
2	112′30″	112′29″	112′28″47‴43.0⁗	1′50″	50	2812′30″	2509′05″	2509′05″16‴17.6⁗	1087′43″
3	168′45″	16841	168′40″56‴03.6⁗	4′8″	51	2868′45″	2547′12″	2547′12″08‴25.1‴	1129′5″
4	225′0″	224′50″	224′50″21‴49.6⁗	7′21″	52	2925′0″	2584′38″	2584′38″05‴32.0⁗	1171′5″
5	281′15″	280′56″	280′56″10‴53.5⁗	11′30″	53	2981′15″	2621′22″	2621′22″31‴33.4⁗	1213′40″
6	337′30″	336′57″	336′57″29‴11.3⁗	16′33″	54	3037′30″	2657′25″	2657′24″51‴05.0⁗	1256′51″
7	393′45″	392′53′	392′53′22′43.3⁗	22′31″	55	3093′45″	2692′44″	2692′44″29‴22.4⁗	1300′38″
8	450′0″	448′42″	448′42″57‴35.1⁗	29′24″	56	3150′0″	2727′21″	2727′20″52‴22.9⁗	1344′58″
9	506′15″	504′25″	504′25″19‴58.4⁗	37′12″	57	3206′15″	2761′13″	2761′13″26‴45.1⁗	1389′52″
10	562′30″	559′59″	559′59″36‴11.7⁗	45′55″	58	3262′30″	2794′21″	2794′21″39‴50.6⁗	1435′20″
11	618′45″	615′24″	615′24″52‴41.5⁗	55′32″	59	3318′45″	2826′45″	2826′44″59‴41.5⁗	1481′19″
12	675′0″	670′40″	670′40″16‴02.9⁗	66′3″	60	3375′0″	2858′23″	2858′22″55‴06.5⁗	1527′50″
13	731′15″	725′44″	725′44″53‴00.4⁗	77′29″	61	3431′15″	2889′15″	2889′14″55‴35.8⁗	1574′51″
14	787′30″	780′37″	780′37″50‴29.0⁗	89′48″	62	3487′30″	2919′20″	2919′20″31‴24.4⁗	1622′22″
15	843′45″	835′18″	835′18″15‴35.0⁗	10′31″	63	3543′45″	2948′39″	2948′39″13‴32.0⁗	1670′23″

Contd...

i	Arc, *ih*	Rsine, J_i	Modern value sinθ × (10800/π)	Versed Rsines V_i	*i*	Arc, *ih*	Rsine, J_i	Modern value sinθ × (10800/π)	Versed Rsines V_i
16	900′0″	889′45″	889′45″15‴36.7⁗	117′8″	64	3600′0″	2977′10″	2977′10″33‴43.6⁗	1718′52″
17	956′15″	943′58″	943′52″11‴55.4⁗	132′8″	65	3656′15″	3004′53″	3004′54″04‴29.8⁗	1767′49″
18	1012′30″	997′55″	997′55″30‴45.8⁗	148′1′	66	3712′30″	3031′49″	3031′49″19‴07.4⁗	1817′12″
19	1068′45″	1051′37″	1051′37″01‴37.8⁗	164′47″	67	3768′45″	3057′55″	3057′55″51‴39.4⁗	1867′1″
20	1125′0″	1105′1″	1105′01″38‴56.5⁗	182′26″	68	3825′0″	3083′13″	3083′13″16‴56.1⁗	1917′16″
21	1181′15″	1158′8″	1158′08″31‴13.2⁗	200′57″	69	3881′15″	3107′41″	3107′41″10‴35.0⁗	1967′55″
22	1237′30″	1210′56″	1210′56″47‴16.4⁗	220′20″	70	3937′30″	3131′19″	3131′19″09‴01.3⁗	2018′57″
23	1293′45″	1263′25″	1263′25″36‴12.4⁗	240′35″	71	3993′45″	3154′6″	3154′06″49‴28.4⁗	2070′23″
24	1350′0″	1315′34″	1315′34″07‴26.5′′′′′	261′41″	72	4050′0″	3176′3″	3176′03″49‴58.0⁗	2122′10″
25	1406′15″	1367′21″	1367′21″30‴43.3⁗	283′38″	73	4106′15″	3197′9″	3197′09″49‴20.8⁗	2174′19″
26	1462′30″	1418′47″	1418′46″56‴07.9⁗	306′25″	74	4162′30″	3217′24″	3217′24″27‴16.7⁗	2226′48″
27	1518′45″	1469′49″	1469′49″34‴06.6⁗	330′3″	75	4218′45″	3236′47″	3236′47″24‴15.0⁗	2279′36″
28	1575′0″	1520′28″	1520′28″35‴27.6⁗	354′31″	76	4275′0″	3255′18″	3255′18″21‴34.9⁗	2332′43″
29	1631′15″	1570′43″	1570′43″11‴21.8⁗	379′49″	77	4331′15″	3272′57″	3272′57″01‴25.5⁗	2386′7″
30	1687′30″	1620′32″	1620′32″33‴23.7⁗	405′55″	78	4387′30″	3289′43″	3289′43″06‴46.6⁗	2439′49″
31	1743′45″	1669′55″	1669′55″53‴32.2⁗	432′51″	79	4443′45″	3305′36″	3305′36″21‴28.4⁗	2493′46″
32	1800′0″	1718′52″	1718′52″24‴11.2⁗	460′34″	80	4500′0″	3320′36″	3320′36″30‴12.3⁗	2547′59″
33	1856′15″	1767′21″	1767′21″18‴10.5⁗	489′5″	81	4556′15″	3334′43″	3334′43″18‴30.6⁗	2602′26″
34	1912′30″	1815′22″	1815′21″48‴46.4⁗	518′24″	82	4612′30″	3347′56″	3347′56″32‴47.2⁗	2657′6″
35	1968′45″	1862′53″	1862′53″09‴42.7⁗	548′29″	83	4668′45″	3360′15″	3360′16″00‴17.6⁗	2711′59″
36	2025′0″	1909′54″	1909′54″35‴11.3⁗	579′21″	84	4725′0″	3371′41″	3371′41″29‴09.0⁗	2767′4″
37	2081′15″	1956′25″	1956′25″19‴52.8⁗	610′59″	85	4781′15″	3382′12″	3382′12″48‴20.8⁗	2822′19″
38	2137′30″	2002′24″	2002′24″38‴57.4⁗	643′23″	86	4837′30″	3391′49″	3391′49″47‴44.6⁗	2877′45″
39	2193′45	2047′52″	2047′51″48‴05.8⁗	676′31″	87	4893′45″	3400′32″	3400′32″18‴04.2⁗	2933′19″
40	2250′0″	2092′46″	2092′46″03‴29.5⁗	710′23″	88	4950′0″	3408′20″	3408′20″10‴56.0⁗	2989′1″
41	2306′15″	2137′06″	2137′06″41‴51.8⁗	745′0″	89	5006′15″	3415′13″	3415′13″18‴49.2⁗	3044′51″
42	2362′30″	2180′53″	2180′53″00‴28.2⁗	780′19″	90	5062′30″	3421′11″	3421′11″35‴05.4⁗	3100′47″
43	2418′45″	2224′4″	2224′04″17‴07.6⁗	816′22″	91	5118′45″	3426′14″	3426′14″53‴59.4⁗	3156′48″
44	2475′0″	2266′39″	2266′39″50‴12.5⁗	853′6″	92	5175′0″	3430′23″	3430′23″10‴38.9⁗	3212′54″
45	2531′15″	2308′39″	2308′38″58‴39.7⁗	890′32″	93	5231′15″	3433′36″	3433′36″21‴04.5⁗	3269′3″
46	2587′30″	2350′01″	2350′01″02‴01.42⁗	928′39″	94	5287′30″	3435′54″	3435′54″22‴10.2⁗	3325′15″
47	2643′45″	2390′45″	2390′45″20‴25.3⁗	967′26″	95	5343′45″	3437′17″	3437′17″11‴42.8⁗	3381′29″
48	2700′0″	2430′51″	2430′51″14‴35.6⁗	1006′53″	96	5400′0″	3437′44″	3437′44″48‴22.5⁗	3437′44″

$k = \frac{l}{3}$, $l = 3 \times 2^m$, $m = 0, 1, 2....$ (m=3, k = 8 for l =24)

Step 2: Compute $J_{\left(\frac{l \pm i}{2}\right)} = \sqrt{\frac{R}{2}\left(R \pm J_i\right)}$ for $i = k, k_1, k_2, k_3$ where $k_n = \frac{l \pm k_{n-1}}{2}$, $k_0 = k$

Step 3: Proceed as in Step 2 for $i = l_1, l_2$ where $l_n = \frac{l \pm l_{n-1}}{2}$, $n = 2, 3;$ $l_1 = \frac{l}{2}$. The scheme is shown in Table 6.

2.8. Bhāskara II (b.1114 AD)

Bhāskara II gives several methods in the *Siddhāntaśiromaṇi* (Joshi 1988, *Chedyakādhikaraḥ* 2-6, *Jyotpattivāsana*) including those by Brahmagupta, Āryabhaṭa II along with some new methods.

a) Methods based on the following formulae:

1) $J_{\frac{p-q}{2}} = \frac{1}{2}\sqrt{\left(J_p - J_q\right)^2 + \left(K_p - K_q\right)^2}$ where $K_i = J_{l-i} = \sqrt{R^2 - J_i^{\,2}}$. Given $J_l = R$ and $J_k = R/2$,

Table 6: Scheme for finding tabular values by Āryabhaṭa II

$J_0 = 0$	$J_{24} = R$	Known	J_6	J_{18}	Step 3 on J_{12}
J_1	J_{23}	Step 2 on J_{22}	J_7	J_{17}	Step 2 on J_{10}
J_2	J_{22}	Step 2 on J_{20}	J_8	J_{16}	$J_8 = \frac{R}{2}$ known, Step 2 on J_8
J_3	J_{21}	Step 3 on J_{18}	J_9	J_{15}	Step 3 on J_6
J_4	J_{20}	Step 2 on J_{16}	J_{10}	J_{14}	Step 2 on J_4
J_5	J_{19}	Step 2 on J_{14}	J_{11}	J_{13}	Step 2 on J_2
*****→	*******→	************* *****↑	J_{12}	****	*known*, 2431 or *compute using* $J_{\left(\frac{l\pm 0}{2}\right)} = \sqrt{\frac{R}{2}\left(R \pm J_0\right)}$

all tabular values can be computed. (for tables of 24 values,

$l = 3\times 2^m = 24,\ m = 3;\ k = \frac{l}{3} = 8$)

2) $J_{\frac{l}{2}-i} = \sqrt{\frac{1}{2}\left(K_i - J_i\right)^2}$ where $K_i = \sqrt{R^2 - J_i^2} = J_{l-i}$.

3) $K_i = R - \frac{2}{R}J_{\frac{1}{2}}^2$, $J_{l-i} = K_i$ Not all values can be found. But square roots are not involved.

4) $J_{i\pm 1} = \left(J_i - \frac{J_i}{467}\right) \pm \frac{100}{1529}K_i$, i = 1, 2, 3, …, 24; $J_1 = 225 - \frac{1}{7}'$, arc bit being 225′.

5) Tabular Rsines for every degree increase of arc are given by

$$J_{(i\pm 1)^0} = \left(J_{i^0} - \frac{J_{i^0}}{6567}\right) \pm \frac{10}{573}K_{i^0}$$, i = 1, 2, ..., 89 where $J_1 = 60'$

b) Method based on refining tabular differences - Munīśvara's iterative procedure for extracting finer ('*second generation*') tables from known coarser tables

The *Siddhāntaśiromaṇi* (Joshi 1964, *Grahagaṇitādhyāya, Spaṣṭādhikāraḥ*, vs 16) gives a method for refining functional differences for interpolating using a table at arc bits of 10°.

$$d = \begin{cases} \frac{1}{2}\left(d_b + d_a\right) - \left(d_b - d_a\right)\frac{\rho}{20} & \text{for } \textit{kramajyā} \\ \frac{1}{2}\left(d_b + d_a\right) + \left(d_b - d_a\right)\frac{\rho}{20} & \text{for } \textit{utkramajyā} \end{cases}$$

where d is the refined difference applicable in the interior of the interval (qh, $qh + h$) containing the desired argument $\alpha = qh + \rho$, $0 < \rho < h$ where $d_b = \Delta J_{q-1}$ and $d_a = \Delta J_q$ are the tabular differences just before and after that interval. Earlier it was used by Brahmagupta (Sengupta 1934, p.141, ix.8). Taking the value of d for *kramajyā*, the desired Rsine = the *tabular Rsine just before* + $\frac{\rho}{h}d$.

Bhāskara's tabular values for every increase of h = 10° of arc with R = 120° (Joshi 1964, *Spaṣṭādhikāraḥ*, verse. 13) are displayed in Table 7.

Using this table, $R\sin\left(qh + \rho\right) = J_q + \frac{\rho}{h}d$ can be computed. The refined difference d is given by $d = m - \frac{\rho}{h}\left(\frac{\Delta J_{q-1} - \Delta J_q}{2}\right)$ where $m = \left(\frac{\Delta J_{q-1} + \Delta J_q}{2}\right)$. Commentator Munīśvara

Table 7: *Siddhāntaśiromaṇi* tabular values

i	0	1	2	3	4	5	6	7	8	9
Arc *ih*	0	10	20	30	40	50	60	70	80	90
Rsine diff ΔJ_i		21	20	19	17	15	12	9	5	2
J_i = *Rsin*(*ih*)	0	21	41	60	77	92	104	113	118	120

(1653 AD) gives an iterative procedure for further refining the difference *d* (Mallayya 2008a).

Step 1: Divide the arc α by *h* (*h* = 10°) and note the quotient *q* and remainder ρ

Step 2: Note the tabular Rsines J_q and J_{q+1}, tabular Rsine differences ΔJ_{q-1} and ΔJ_q and compute the mean Rsine difference $m = \dfrac{\Delta J_{q-1} + \Delta J_q}{2}$

Step 3: With, $d^{(0)} = \Delta J_q$, compute $d^{(r+1)} = m - \dfrac{\rho}{h}\left(\dfrac{\Delta J_{q-1} - d^{(r)}}{2}\right)$ for *r* = 0, 1, 2, …

Step 4: Stop when *d* attains desired level of stability. (When *r* = 0 we get Bhāskara's value)

Step 5: Compute the desired Rsine using

$$Rsin\alpha = J_q + \theta d \text{ where } \theta = \frac{\rho}{h}.$$

Using this on Table 7, a refined Rsine table (Table 8) is extracted for every 1° increase of arc (Joshi 1964, p.140, *Spaṣṭādhikāraḥ*, vs. 16).

2.9. Nīlakaṇṭha Somayāji (1444-1545 AD)

a) *Tantrasaṅgraha*: Citing Mādhava (1340-1425AD) Nīlakaṇṭha gives the following methods on computation of Rsines, versed Rsines and Rsine differences (Sarma 1977, verses ii. 2-21).

I) Step 1: First Rsine difference is $\Delta J_0 = J_1 - J_0 = J_1$.

Step 2: For *i* = 1, 2, ..., *l* – 1; (*l* = 24) compute

$\Delta J_i = \Delta J_{i-1} - \dfrac{J_i}{233\frac{1}{2}}$ and $J_{i+1} = J_i + \Delta J_i$ or,

Table 8: *Siddhāntaśiromaṇi - Marīci* refined tabular values

ih	1^0	2^0	3^0	4^0	5^0	6^0	7^0	8^0	9^0	10^0
J_i	2;5;40	4;11;16	6;17;10	8;22;14	10;27;30	12;33;36	14;37;2	16;42;2	18;46;20	20;50;36
ih	11^0	12^0	13^0	14^0	15^0	16^0	17^0	18^0	19^0	20^0
J_i	22;53;50	24;56;56	26;59;38	29;1;50	31;3;30	33;4;56	56;5;4	37;4;56	39;04;06	41;02;32
ih	21^0	22^0	23^0	24^0	25^0	26^0	27^0	28^0	29^0	30^0
J_i	43;0;16	44;57;12	46;53;18	48;48;30	50;42;52	52;36;16	54;30;40	56;20;16	58;10;20	60;00;00
ih	31^0	32^0	33^0	34^0	35^0	36^0	37^0	38^0	39^0	40^0
J_i	61;48;16	63;25;24	65;21;24	67;06;12	68;49;24	70;32;02	72;13;44	73;52;46	75;31;06	77;08;04
ih	41^0	42^0	43^0	44^0	45^0	46^0	47^0	48^0	49^0	50^0
J_i	79;23;30	80;17;44	81;50;22	82;20;32	84;50;50	86;19;34	87;45;44	89;10;38	90;33;54	91;55;30
ih	51^0	52^0	53^0	54^0	55^0	56^0	57^0	58^0	59^0	60^0
J_i	93;15;26	94;33;40	95;50;10	97;04;54	98;17;52	99;29;04	100;38;26	101;46;56	102;51;36	103;55;23
ih	61^0	62^0	63^0	64^0	65^0	66^0	67^0	68^0	69^0	70^0
J_i	104;57;16	105;57;14	106;55;10	107;51;20	108;45;26	109;37;32	110;27;38	111;14;26	112;01;46	112;45;48
ih	71^0	72^0	73^0	74^0	75^0	76^0	77^0	78^0	79^0	80^0
J_i	113;27;44	114;05;36	114;45;26	115;21;06	115;14;40	116;26;08	116;55;28	117;22;40	117;47;42	118;10;38
ih	81^0	82^0	83^0	84^0	85^0	86^0	87^0	88^0	89^0	90^0
J_i	118;31;22	118;49;56	119;06;22	119;20;34	119;32;18	119;42;28	119;50;08	119;55;36	119;58;54	120;00;00

compute $J_{i+1} = J_i + J_1 - \left\{ \frac{J_1 + J_2 + J_3 + ... + J_i}{233\frac{1}{2}} \right\}$;

$$\Delta \ J_i = J_1 - \left\{ \frac{J_1 + J_2 + J_3 + ... + J_i}{233\frac{1}{2}} \right\}$$

II) Step 1: With $d = \frac{21600 \times 113}{355}$, $R = \frac{d}{2}$ and $J_i = R$ find $J_{l-1} = \sqrt{R^2 - J_1{}^2}$ and $M = 2\ (R - J_{l-1})$

Step 2: Compute $\Delta \ J_{i-1} - \Delta \ J_i = \left(\frac{M}{R}\right) J_i$ and $J_{i+1} = \Delta \ J_{i-1} + \left(1 - \frac{M}{R}\right) J_i$ or in other terms $J_{i+1} = 2 \ J_i - \left(\frac{M}{R}\right) J_i - J_{i-1}$ or $J_{i+1} = \left(\frac{2J_{l-1} \times J_i}{R}\right) - J_{i-1}$ for i = 1, 2, ..., l – 1 (l = 24).

b) *Golasāra*: In this (Sarma 1970, pp.17-19, verse iii. 6-14) Nīlakaṇṭha gives a method to construct accurate tabular Rsines and versed Rsines (Mallayya, 2004) starting from an arc of angular measure θ = 30°. Denoting $S_i = R \sin\left(\frac{\theta}{2^i}\right)$, $v_i = R ver \sin\left(\frac{\theta}{2^i}\right)$, $k_i = R \cos\left(\frac{\theta}{2^i}\right)$, $i = 0,1,2,..,m$ (m = 0, 1, 2, … for $l = 3 \times 2^m$ tabular values) then Nīlakaṇṭha's procedure is:

Step 1: Starting from $S_0 = \frac{R}{2}\left(= R \sin 30^0\right)$ determine S_i for i = 1, 2, 3… m using the geometric method based on the formula $S_i = \frac{1}{2}\sqrt{S_{i-1}^2 + v_{i-1}^2}$ where $v_{i-1} = R - k_{i-1}$ and $k_{i-1} = \sqrt{R^2 - S_{i-1}^2}$

Step 2: Take $R = \frac{21600 \times 113}{2 \times 355}$ and with $J_1 = R\sin h = S_m$, $J_l = R\sin lh = R$ as first and last Rsines, find $J_{l-1} = \sqrt{R^2 - J_1^2}$, $K_{l-1} = \sqrt{R^2 - J_{l-1}^2}$ and $V_{l-1} = R - K_{l-1}$

Step 3: Using J_l and J_{l-1} compute $\Delta\ J_{l-1} = J_l - J_{l-1}$ and $\lambda = 2\left(\frac{\Delta \ J_{l-1}}{R}\right)$

Step 4: Compute $\Delta \ J_{l-i} = \lambda \times J_{l-(i-1)} + \Delta \ J_{l-(i-1)}$, $J_{l-i} = J_{l-(i-1)} - \Delta \ J_{l-i}$, $K_{l-i} = \sqrt{R^2 - J_{l-i}^2}$ and $V_{l-i} = R - K_{l-i}$, for i = 2, 3, 4, ..., l – 2.

Step 1 generates the first tabular Rsine for construction of tables of lengths l. It also provides all tabular values directly (same as Āryabhaṭa I's geometrical scheme). Tabular Rsines shown in Table 9.

2.10. Jyeṣṭhadeva (1500-1608 AD)

The *Yuktibhāṣā* (Ramavarma 1948, ch. vii.) gives several methods for determination of tabular Rsine differences, Rsines, Rversines, and Rcosines.

I) One method is the geometrical method of Āryabhaṭa I using which J_1, J_2, J_4, J_5, J_7, J_{10}, J_{11}, J_{13}, J_{14}, J_{16}, J_{17}, J_{19}, J_{20}, J_{22}, J_{23} are derived starting from $J_8 = \frac{R}{2}$ and the remaining J_3, J_6, J_9, J_{12}, J_{15}, J_{18}, J_{21} are derived starting from $J_{24} = R$

II) Another method is by adding the tabular differences. With $\Delta\ J_0 = J_1$, compute $J_i = \Delta\ J_0 + \Delta\ J_1 + \Delta\ J_2 + ... + \Delta\ J_{i-1}$; $V_i = \Delta\ J_{l-1} + \Delta\ J_{l-2} + ... + \Delta\ J_{l-i}$ and $K_i = R - V_i$; for i = 1, 2, 3, ..., l (l = 24). Āryabhaṭa's tabular differences are to be used for the computation.

Table 9: Tabular Rsines using Golasāra method

i	arc	Rsines J_i	Modern Rsine $\sin\theta \times (10800/\pi)$	i	arc J_i	Rsines	Modern Rsine $\sin\theta \times (10800/\pi)$
1	225′	224′ 50″ 21.76‴	224′50″ 21.83‴	13	2925′	2584′ 38″ 04.59‴	2584′ 38″ 05.53‴
2	450′	448′ 42″ 57.27‴	448′ 42‴ 57.58‴	14	3150′	2727′ 20″ 51.44‴	2727′ 20″ 52.38‴
3	675′	670′ 40″ 15.66‴	670′ 40″ 16.04‴	15	3375′	2858′ 22″ 54.16‴	2858′ 22″ 55.11‴
4	900′	889′ 45′ 15.15‴	889′ 45″ 15.61‴	16	3600′	2977′ 10″ 32.77‴	2977′ 10″ 33.73‴
5	1125′	1105′ 01″ 38.42‴	1105′ 01″ 38.94‴	17	3825′	3083′ 13″ 15.97‴	3083′ 13″ 16.94‴
6	1350′	1315′ 34″ 06.85‴	1315′ 34″ 07.44‴	18	4050′	3176′ 03″ 48.99‴	3176′ 03″ 49.97‴
7	1575′	1520′ 28″ 34.79‴	1520′ 28″ 35.46‴	19	4275′	3255′ 18″ 20 59‴	3255′ 18″ 21 58‴
8	1800′	1718′ 52″ 23.45‴	1718′ 52″ 24.19‴	20	4500′	3320′ 36″ 29.20‴	3320′ 36″ 30.20‴
9	2025′	1909′ 54″ 34.41‴	1909′ 54″ 35.19‴	21	4725′	3371′ 41′ 28.13‴	3371′ 41″ 29.15‴
10	2250′	2092′ 46″ 02.66‴	2092′ 46″ 03.49‴	22	4950′	3408′ 20″ 9.90‴	3408′ 20″ 10.93‴
11	2475′	2266′ 39″ 49.34‴	2266′ 39″ 50.21‴	23	5175′	3430′ 23″ 09.60‴	3430′ 23″ 10.65‴
12	2700′	2430′ 51″ 13.69‴	2430′ 51″ 14.60‴	24	5400′	3437′ 44″ 47.32‴	3437′ 44″ 48.37‴

III) The method based on '*jīve paraspara nyāya*' is as follows

Step 1: J_1 is the first Rsine, $K_0 = R$, and $K_1 = \sqrt{R^2 - J_1^{\,2}}$ the first Rcosine

Step 2: Compute $J_i = \dfrac{J_{i-1} \times K_1 + K_{i-1} \times J_1}{R}$ and

$K_i = \sqrt{R^2 - J_i^{\,2}}$; for i = 2, 3, 4 ..., l; (l = 24)

IV) Another method for computing the tabular values without using the value of R is:

Step 1: J_1, J_2 are known (or J_2 can be derived without using R from $J_2 = J_1 + \Delta J_1$)

Step 2: Compute $J_{i+1} = \dfrac{J_i^{\,2} - J_1^{\,2}}{J_{i-1}}$ for i = 2, 3, …, $l - 1$

V) The following method is to determine tabular values with desired degree of accuracy.

Step 1: $J_0 = 0$, $V_0 = 0$, $K_0 = R$.

Step 2: Compute $J_i = J_{i-1} + \left(\dfrac{h}{R}\right) K_{i-\frac{1}{2}}$; $V_i = V_{i-1} + \left(\dfrac{h}{R}\right) J_{i-\frac{1}{2}}$; and $K_i = K_{i-1} - \left(\dfrac{h}{R}\right) J_{i-\frac{1}{2}}$

for $i = 1,\ 1\tfrac{1}{2},\ 2,\ 2\tfrac{1}{2},\ \ldots,\ l - \tfrac{1}{2},\ l$.

This gives 2l tabular values are at arc bits of $\dfrac{h}{2}$, of which l tabular values at arc bits of h'. For l = 24, we get 48 values for every increase of $112\tfrac{1}{2}'$.

2.11. Mādhava's Power Series Method for Rsines, Rversines and Rcosines

Mādhava prescribes the following procedure using power series coefficients. Nīlakaṇṭha in his *Āryabhaṭīyabhāṣya* (Sambasiva Sastry 1930, I. p.113) as well as Śaṅkara in the *Yuktidīpikā* (Sarma 1977, p.117) cites Mādhava's enunciation for computation of desired Rsines and versed Rsines. According to this enunciation, if s is any arc (preferably small for greater accuracy, and for computation of tabular values take $s = jh$ for j =1, 2, …, l), then *jīvā*(s)

$$= s - \frac{s^3}{c_q^{\,3}}\left(M_1 - \frac{s^2}{c_q^{\,2}}\left[M_2 - \frac{s^2}{c_q^{\,2}}\left\{M_3 - \frac{s^2}{c_q^{\,2}}\left(M_4 - \frac{s^2}{c_q^{\,2}} M_5\right)\right\}\right]\right)$$

where c_q is the quadrantal arc of measure 5400′, and M_k denote the Mādhava numbers in minutes (*kalā*), seconds (*vikalā*) and sub seconds (*tatpara*) written in descending order. 24 main Rsines obtained are listed in a set of verses attributed to Mādhava (Sambasiva Sastry 1930, I. p.55; Ramavarma 1948, p.198; Nayar 1956, p.191). Mādhava's procedure is given below in two stages.

Stage I (To compute the decreasing sequence of Mādhava numbers M_k)

Step 1: With $R = \frac{12375888'}{60 \times 60}$, and

quadrantal arc c_q = 5400′ compute $\lambda^2 = \frac{c_q^2}{R^2}$

Step2: Taking $M_0 = c_q$, compute the Mādhava numbers M_k successively using

$$M_k = M_{k-1} \frac{\lambda^2}{\left[(2k)^2 + (2k)\right]},$$

$$\left(\text{or using } M_k = M_{k-1} \frac{c_q^2}{\left[(2k)^2 + (2k)\right] R^2} \right)$$

for k = 1, 2, 3, ..., m.

Table 10: Tabular values by power series method with Mādhava's values and modern values

j	Arc *jh*	Mādhava's Rsines	Computed Rsines	Modern Rsines $\sin\theta \times (10800/\pi)$	Rversnes	Rcosines
0	0	0	0	0	0	3437; 44; 48
1	225	224; 50; 22	224; 50; 22	224;50 ;21.83	7; 21; 38	3430; 23; 10
2	450	448; 42; 58	448; 42; 58	448;42;57.58	29; 24; 37	3408; 20; 11
3	675	670; 40; 16	670; 40; 16	670;40;16.04	66; 03; 19	3371; 41; 29
4	900	889; 45; 15	889; 45; 15.6	889;45;15.61	117; 8; 18	3320; 36; 30
5	1125	1105; 01; 39	1105; 01; 39	1105;01;38.94	182; 26; 27	3255; 18; 21
6	1350	1315; 34; 07	1315; 34; 07	1315;34;07.44	261; 40; 58	3176; 3; 50
7	1575	1520; 28; 35	1520; 28; 35	1520;28;35.46	354; 31; 31	3083; 13; 17
8	1800	1718; 52; 24	1718; 52; 24	1718;52;24.19	460; 34; 15	2977; 10; 33
9	2025	1909; 54; 35	1909; 54; 35	1909;54;35.19	579; 21; 53	2858; 22; 55
10	2250	2092; 46; 03	2092; 46; 03	2092;46;03.49	710; 23; 56	2727; 20; 52
11	2475	2266; 39; 50	2266; 39; 50	2266;39;50.21	853; 6; 43	2584; 38; 5
12	2700	2430; 51; 15	2430; 51; 15	2430;51;14.60	1006; 53; 34	2430; 51; 14
13	2925	2584; 38; 06	2584; 38; 06	2584;38;05.53	1171; 4; 58	2266; 39; 50
14	3150	2727; 20; 52	2727; 20; 52	2727;20;52.38	1344; 58; 45	2092; 46; 3
15	3375	2858; 22; 55	2858; 22; 55	2858;22;55.11	1527; 50; 13	1909; 54; 35
16	3600	2977; 10; 34	2977; 10; 34	2977;10;33.73	1718; 52; 24	1718; 52; 24
17	3825	3083; 13; 17	3083; 13; 17	3083;13;16.94	1917; 16; 13	1520; 28; 35
18	4050	3176; 03; 50	3176; 03; 50	3176;03;49.97	2122; 10; 41	1315; 34; 7
19	4275	3255; 18; 22	3255; 18; 22	3255;18;21.58	2332; 43; 9	1105; 1; 39
20	4500	3320; 36; 30	3320; 36; 30	3320;36;30.20	2547; 59; 33	889; 45; 15
21	4725	3371; 41; 29	3371; 41; 29	3371;41;29.15	2767; 4; 32	670; 40; 16
22	4950	3408; 20; 11	3408; 20; 11	3408;20;10.93	2989; 1; 51	448; 42; 57
23	5175	3430; 23; 11	3430; 23; 10.2	3430;23;10.65	3212; 54; 26	224; 50; 22
24	5400	3437; 44; 48	3437; 44; 48	3437;44;48.37	3437; 44; 48	0; 0; 0

Stage II (To compute Rsines): For each arc $s = jh$ (where j = 2, 3, 4, ..., l),

Step 1: Compute $\rho = \frac{s}{c_q}$, ρ^2 and ρ^3

Step 2: Taking the last Mādhava number M_m as d_m compute successively the product $p_{m-i} = \rho^2 d_{m-(i-1)}$ and the difference $d_{m-i} = M_{m-i} - p_{m-i}$ for i = 1, 2, 3, ..., m –1.

Step 3: Compute the final product $p_0 = \rho^3 d_1$ and final difference $d_0 = s - p_0$. Then $J = d_0$

Mādhava numbers (Sambasiva Sastry 1930, I. p.113: Sarma 1977, p.117) are 2220, 39, 40 ; 273, 57, 47; 16, 05, 41; 0, 33, 06 ; 0, 0, 44 and Mādhava's tabular Rsines (Sambasiva Sastry 1930, p.55;Ramavarma 1948, p.198; Nayar 1956, p.191) are 224,50,22; 448,42,58; 670,40,16; 889,45,15; 1105,1,39; 1315,34,7; 1520,28,35; 1718,52,24; 1909,54,35; 2092,46,3; 2266,39,50; 2430,51,15; 2584,38,6; 2727,20,52; 2858,22,55; 2977,10,34; 3083,13,17; 3176,3,50; 3255,18,22; 3320,36,30; 3371, 41, 29; 3408,20,11; 3430,23,11; 3437, 44, 48.

Similarly for computation of versed Rsines and Rcosines if $s = jh$ are the arcs then *śara*(s)

$$= \frac{s^2}{c_q^2}\left[M_1' - \frac{s^2}{c_q^2}\left\{M_2' - \frac{s^2}{c_q^2}\left[M_3' - \frac{s^2}{c_q^2}\left\{M_4' - \frac{s^2}{c_q^2}\left(M_5' - \frac{s^2}{c_q^2}M_6'\right)\right\}\right]\right\}\right]$$

where $c_q = 5400'$ and M'_k are the Mādhava numbers for *R*versines (in *kalā*s, *vikalā*s, and *tatpara*s) placed one below the other. M'_k are 4241,9,0; 872,3,5; 71,43,24; 3,9,37; 0,5,12; 0,0,6 (Sarma 1977, p.118, vs 438; Ramavarma 1948, p.188, 195; Nayar, 1956, p.192, vs vi.15). Table 10 shows 24 tabular values from power series method along with Mādhava's values and modern values.

3. Concluding Remarks

The above presentation just peeps into some trigonometric tables and their construction methods from a very few select treatises such as *Āryabhaṭīya* of Āryabhaṭa I and *bhṣya* of Nīlakaṇṭha Somayāji, *Sūryasiddhānta*, *Pañcasidhāntikā* of Varāha Mihira, *Khaṇḍakhādyaka* and *Brahmasphuṭasiddhānta* of Brahmagupta, *Mahābhāskarīya bhāṣya* of Govindasvāmin, *Vaṭeśvara Siddhānta* of Vaṭeśvara, *Mahāsiddhānta* of Āryabhaṭa II, *Siddhāntaśiromaṇi* of Bhāskarchārya and *Marīci* of Munīśvara, *Tantrasaṅgraha* and *Golasāra* of Nīlakaṇṭha Somayāji, *Yuktibhāṣā* of Jyeṣṭhadeva and *Karaṇapaddhati* of Putumana Somayāji. The concept and method of extraction of a *second generation* refined table with shorter interval of differencing from a known coarser table deserves special attention. From the vast store of manuscripts on trigonometric tables only a small percentage has been edited and explored so far and several original manuscripts are still lying in various repositories (Sarma KV, 2002). Development of trigonometric tables in India can be fully traced by carrying out explorative studies on the contents of the large mass of such manuscripts also.

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