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Āryabhaṭa-II and his Concept of Concave Quadrilateral

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Abstract

The aim of this article is to throw light on the concept of concave quadrilateral introduced for the first time in the 10th century CE by Āryabhaṭa-II (c.950 CE) in his *Mahāsiddhāntaḥ* (abridged as *Mahāsi.*) erstwhile unidentified by any other mathematician in the world. The validity of the date of Āryabhaṭa-II has also been discussed.

Key words: *Āyāma, Ayanāṃśa,* Āryabhaṭa-II, Bhāskara-I, *Bālendu*, Brahmagupta, *Mahāsiddhāntaḥ*, Nīlkakaṇṭha Somayāyajī, *Pātarekhā,* Śrīdharācārya, *Śulbasūtra*, *Śṛṅgāṭaka caturasra*, *Viṣamacaturasra*

1. Introduction and Historical Background

According to the modern concept and convention, a quadrilateral in a school geometry is known to be a plane figure bounded by four line segments. 'The elements' of *Euclid* does not give the definition in this way. Here, first rectillineal figures have been defined as 'figures which are contained by straight lines' and 'quadrilaterals are those contained by four lines. The definition 22 of 'the Elements' advances the classification of quadrilaterals in the following manner (Wikipedia Free Encyclopedia):

> 'Of quadrilateral figures, a square is that which is both equilateral and right-angled, an oblong (is) that which is right-angled but not equilateral, a rhombus (is) that which is equilateral but not right-angled, a rhomboid (is) that which has opposite sides and angles equal to one-another; but is neither equilateral nor right-angled. And let quadrilaterals other than these be called 'trapezia'.

This definition does not specifically and categorically classify quadrilaterals in two classes namely, (i) 'convex' and (ii) 'concave' following present-day practice, though in the conventional definition (given above) both the classes are included; where by a 'convex quadrilateral' is meant the one whose both the diagonals lie within the region enclosed and a 'concave quadrilateral' has one diagonal lying outside the plane region enclosed *i.e.*, one interior angle subtended by one pair of adjacent sides is more than two right angles. The concept of quadrilateral (Mukhopadhyay & Adhikari, 1997, pp.53-68) in the age of *Śulbasūtra* was confined within the limited scope of some specific cyclic quadrilaterals like square (*samacaturaśra*), rectangle (*āyatacaturaśra*), isosceles trapezium (*caturaśra ubhayata prauga*). Āryabhaṭa-I (b.476 CE) made no new addition to this list. Bhāskara-I (b.576 CE) (Amma, 1979, p.9) was the first to put forward a solitary example of a quadrilateral with unequal sides and called it a *viṣamacaturaśra* which was later identified by Mahāvīrācārya (c. 850 CE) as a quadrilateral whose vertices lie on the circumference of a circle (usually called a 'cyclic quadrilateral') and known by the name '*vṛttagatacaturaśra* in the sixteenth century CE in India (Mukhopadhyay & Adhikari, 1997). Brahmagupta (c.650 CE) generalized the

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concept of a cyclic quadrilateral and advanced the rule for finding circum-radius of such quadrilaterals. All ancient Indian scholars save Āryabhaṭa-II were concerned with convex quadrilaterals only. It was Āryabhaṭa-II who threw light on the topics related to the two types of quadrilaterals namely, (i) general convex quadrilaterals (which may or may not be cyclic) (verses 63-65, 67,68, 70-73) and (ii) concave quadrilaterals (verses 74,75,79) (*Mahāsid*, Sans. com., Dvivedi, 1910, pp.162-167).

2. Āryabhaṭa-II and his concept on Concave Quadrilateral

Āryabhaṭa-II (c.950 CE) is famous for his only mathematical-cum astronomical treatise *Mahāsiddhāntaḥ*. Nowhere of the work he has mentioned about his parentage or anything about his personal testimony. A thorough study of this work reveals that he had an extraordinary mathematical wit of which the present article is a specimen.

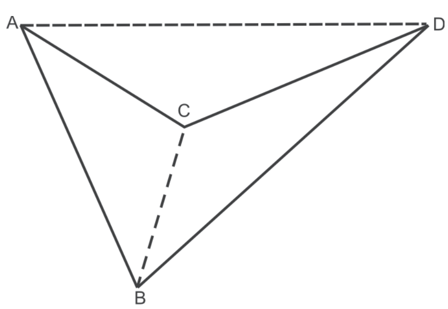

Fig. 1. Concave quadrilateral ABDC; AD is the external diagonal

The Geometrical part of *Mahāsiddhāntaḥ* (*Mahāsi.* Sans. com. by Sudhakara Dvivedi, 1910, pp. 160-177) is contained in the chapter *paṭigaṇitaṃ*. Here Āryabhaṭa-II allotted only three verses on concave quadrilateral. He prescribed the nomenclature *Śṛṅgāṭaka caturasraḥ.* The word *Śṛṅga* means a 'peak' and the word *āṭaka* (√*aṭ* + *ṇaka*) means 'going' (to the peak). The verse 74 is,

śṛṅgaṭakacaturasre bahyaḥ karṇastu no kalpyaḥ |
dakṣiṇbahormūladyadvāmabhujāgragaṁ sūtraṃ (*karṇaḥ syāt*) ||74,

But in a *śṛṅgāmakacaturasra* (concave quadrilateral ABCD), we should consider the external diagonal as the thread joining the end-point (D) of the side (BD) on the right hand and the end-point (A) of the side (AB) lying to the left.

Here, the conjunction 'but' has been used in the context of the former versified rules 71- 73 on the diagonals of a convex quadrilateral. For a ready reference, the verses are mentioned hereunder:

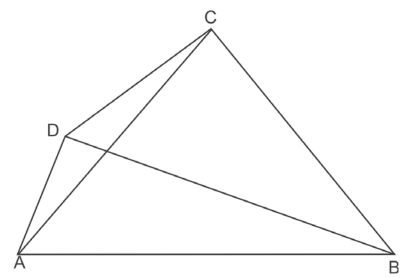

Fig. 2. General convex quadrilateral ABCD with the diagonals AC and BD

dharaṇīvāmabhujaikyaṁ kuryānmu-khayāmyabāhuyogaṁ ca |
anayoralpasamānaḥ paramo yāmyagragaḥ karṇaḥ || 71
dakṣiṇabahukuyogaṁ kuryadvāma-mānanaikyaṁ ca |
anayoralpasamānaḥ paramo vāmāgragaḥ karṇaḥ —— 72
yogavadantarake ye tadadhikato ' lpo na karṇaḥ syāt |
evaṁ jñātvā' bhīṣṭe caturasre kalpayet karṇaṃ —— 73

Meanings of essential words with reference to the figure and interpretation : *dharaṇī /ku* – base (AB), *vāmabhuja* (AD), *aikyaṁ* sum, *mukha*-face (DC), *yāmya-* literally, southern, here the side (BC) on the right, *yogaṁ*-the sum, *anayoḥ alpasamānaḥ* of these (two sums) smaller than (or) equal to, *paramaḥ* the extreme, limiting value / at the most, *yāmyāgragaḥ karṇaḥ* the diagonal passing through the end-point (C) of the diagonal on the right-hand side, *mānana /mānanīya* – literally means 'desrving honour', here it means 'face' (DC) *i.e.*, of the sums AB+AD and DC+BC, the diagonal BD ≤ the smaller one, the equality holds in the limiting case (when the quadrilateral will reduce to a triangle). (The same is the case with AB+BC and AD+DC).

Translation:

The base (AB) and the side (AD) on the left are added, (likewise), the face (DC) and the side (BC) on the right-hand are added (to get respectively AB+AD and DC+BC).The diagonal (BD) through the right-hand corner (B) is less than or at the most equal to the smaller of these two sums. The side (BC) on the right-hand is added to the base (AB) (and also) the side (AD) on the left-hand side is added to the face (DC) (to get respectively BC+AB and AD+DC). The diagonal (AC) through the corner lying to the extreme left is less than or equal to the smaller of the two sums. Just as in the cases of the sums similarly, no diagonal is smaller than the larger of the differences (of the base and the side on the left or the face and the side to the right). Knowing this, (length of a) a diagonal in a desired quadrilateral should be considered.

Discussion: In the article 'Indian concept of a concave quadrilateral' Jha (1999, pp.80-86) referred to Encylopaedia Indica, *Vālmīkīya Rāmāyaṇaṃ* instead of giving the meaning of the word '*Śṛṇgāṭaka*'.

The (above quoted) verses 71, 72 and 73 of *paṭigaṇitaṃ* (*Mahāsi.* Sans. commentary by Dvivedi*,*1910, p.165) are related to convex quadrilateral. In the interpretation of the verses 71-73, Jha has deviated much from the interpretation as well as from the comment: *paramo yāmyagragaḥ karṇaḥ iti vi. pustake prāmādikaḥ pāṭhaḥ* (in the foot note of the concerned page) of the commentator, where *vi. pustaka* refers to the book in the collection of *Vinayaka Śāstrī.* He has also deviated from the elementary geometric rule in the sense that the smaller of the sums of pairs of adjacent sides of a quadrilateral cannot equal a diagonal, not to talk of the maximum length of a diagonal, because, if the smaller of the two sums happens to be equal to the length of the concerned diagonal, the figure fails to assume the shape of a quadrilateral. The gist of the verses 71-73 has been given by the commentator under the heading "*atropapattiḥ*"

The verse 74 differentiating a concave quadrilateral from a convex one from the stand-point of diagonals has been not been mentioned.

tribhuje bhujadvayayogastritīya-bhujādadhiko bhujāntaraṁ ca rekhāgaṇitasiddhāntena karṇamanaṁ tritīyabhujaṁ parikalpya sugamena bodhyeti—

i.e., the sum of two sides of a triangle is greater than the third side and the difference of them is less than the third one.

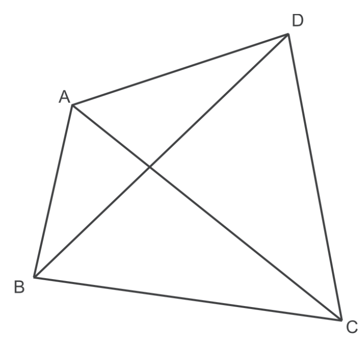

Fig. 3. Convex quadrilateral

The next verse is,

(*karṇaḥ syāt*) *sa tribhuje dakṣiṇabahustadagrakīllambaḥ* |
yāmyabhujāgraśravaṇo vāmabhujo vā tadagrakāllambaḥ —— 74, p.166.

The first pair of words form a part of the former verse. *Tadagrakīllambaḥ* follows no rule of *sandhiprakaraṇaṃ* it should be *tadagrakāllambaḥ* ⇐ *tadagrakāt* +*lambaḥ* follwing the rule '*torli*' (8|4|60) of *Paṇini*. Here, of course, the joint of the sides AC and CB resembles an elbow (*kīla*)

Translation

In the triangle (formed by the external diagonal AD and the nearest pair of adjacent sides AC, CD) the perpendicular (CP) on the external diagonal (is drawn) from the extremity (C) of the side (DC) on the right-hand. (Also) the perpendiculars (AQ and DR) are drawn on the internal diagonal BC (produced) from the end-point (D) the side on the right-hand and also from the end-point (A) of the side to the left.

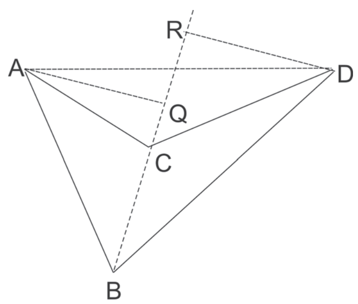

Fig. 4. Concave quadrilateral with perpendiculars from the vertices to the diagonals

From the above construction suggested by the author Āryabhaṭa-II, it is clear that if the lengths of the internal diagonal BC, the perpendiculars AQ and DR be respectively d_i, p_1, p_2 then the area of the concave quadrilateral ABCD is, $\frac{1}{2} d_i \cdot \left(p_1 + p_2\right)$

This has been mentioned in the following verse:

idānīṁ kṣetraphale viśeṣamāh |
śṛṅgāṭake na niyamādviṣamacatur-bāhuke ca na prāyaḥ |
yamyottaralambaikyārdhaṁ kkasyaikyā-rdhatāḍitaṁ nikaṭaṃ | 79, p.167.

Translation:

Now is mentioned the special mode of finding the area (of a concave quadrilateral) because, the general rule (*prāyaḥ* - in all probability/ mostly) for (finding the area of) quadrilaterals with unequal sides, (*na niyamāt*- due to absence of a rule) is not applicable in (the cases of) a *śṛṅgāṭaka* quadrilateral. The half the sum of the perpendiculars (one) from the right (peak) and (the other) from the left (peak) (to the internal diagonal produced) multiplied by the nearer (internal) diagonal (is the required area)

According to Āryabhaṭa-II, the letter *ka* stands for the numeral 'one' and the conjunct letter '*kka*' stands for 'one-one' which is often used to denote 'one by one' ,'each'. *tāḍitaṁ* means *guṇitaṁ.*

In the commentary of the portion *kkāsyaikyārdhatāḍitaṁ* (*ka* + *kasya* + *aikya* + *ardhaṁ tāḍitaṁ- eka-ekasya aikya* + *ardhaṁ tāḍitaṁ*), the commentator states *bhumu-khayogārdhaguṇitaṃ*

i.e., 'multiplied by half the sum of the base and the face' and has thus deviated much from the actual sense carried in the portion of the verse. The **Fig. 4** will attest this. For a ready reference, the commentary of the above verse is given below:

śṛṅgāmake śṛṅgāṭakākāre caturbhuje niyamāt niścayena pūrvavidhinā na phalaṁ bhavati | viṣamacaturbahuke viṣamacaturbhujakṣetre ca prāyo bahulyena phalaṁ bhavati | tatra samānalambacaturbhuje pūrva-prakāreṇa vāstavaṁ phalaṁ bhavatyedarthaṁ prāyaḥ śabdaḥ pryukta iti dhyeyaṃ | atha viṣamacaturbhuje āsanna phalaṁ sādhayati | yāmyottaralambaikyārdhaṁ karṇa-dānena ye tribhuje yayoreko bāhuḥ krameṇa mukhaṁ bhūmiśca tatra karṇopari yau lambau tayoryogārdhaṃ |

[*Mahāsi.* Sans. commentary by Dvivedi*,* 1910), pp.167-168]

i.e., in a *śṛṅgāṭaka* quadrilateral, the area cannot be unerringly found by the earlier rule. In a quadrilateral with unequal sides, the area is often found in various ways. In the case when a quadrilateral has equal lengths of perpendiculars from the end-points of the face to the base, the area is realizable. The word '*prāyaḥ* should be considered to have been applied here in this sense. Now, in a quadrilateral with unequal sides, half of the sum of the right-hand and the left hand perpendiculars associated with the diagonal means half the sum of the altitudes of the triangles (formed by the division of the quadrilateral in to two parts by the diagonal) having the base and the face in order.

Here, considering the meaning given in the translation, the word '*prāya* has been misinterpreted, because *samānalambacaturbhuja* means a quadrilateral having equal perpendiculars dropped from the two end-points of the face to the base (or vice versa) *i.e.*, 'a trapezium /rectangle /square, has a realizable (*vastavam*) area' does not carry any geometric or rather any mathematical sense. It is to be noted that, area being a measure, is always a non-zero real number and the area of a concave quadrilateral also so. Wide deviation starts from *kkāsyaikyārdhatāḍitaṁ* and this has been explained before.

In the last line of the *upapatti* (clarification) he resorts to the actual meaning: *vastuto lambaikārdhaṁ karṇaguṇaṁ vāstavaṁ viṣamacaturbhujaphalmiti dhyeyaṃ* this remark means:

'In fact, the half of the sum of the perpendiculars (to the diagonal from a pair of opposite vertices) multiplied by the diagonal is actually the area (*phalaṃ*) of a quadrilateral with unequal sides (both convex and concave)'.

Alternative way of finding the area of a concave quadrilateral taking the external diagonal instead of the internal one: Though this alternative way was not suggested by Āryabhaṭa-II, it was adopted by Śrīdharācārya as will be seen in the example (ii) without specifying the quadrilateral as a *śṛṅgāṭaka caturasra.*

Denoting AD, BQ and CR by d_1, p_1 and p_2, (Fig. 5) we have, the area of the concave quadrilateral ABDC

= Area of ΔABD – area of ΔACD =

$$=\frac{1}{2}d_1.(p_1-p_2).$$

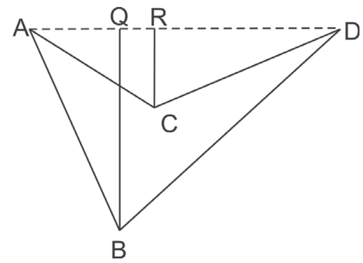

Fig. 5. Concave quadrilateral with perpendiculars BQ, CR from vertices B and C to AD, the external diagonal

2.1 Examples of the usage of the word *śṛṅgaṭaka* in earlier Mathematical works

(i) The term *śṛṅgaṭaka* was not a coinage of *Āryabhaṭa*-II. It was used by *Bhāskara*-I, (c.6th century CE) [*Āryabhaṭīya*, *Bhāskara*-I's Sans com., ed. Shukla (1976),p.58] in the examples 1 and 2 in his commentary on the following verse of the *gaṇitapādaḥ* in the *Āryabhaṭīya*:

ūrdhvabhujātatsaṁvargārdhaṁ sa ghanaḥ ṣaḍaśririti (**2nd line, verse 6).**

Here the word *aśri* (*aśra* = √*aś* +*rak*) means 'edge', a corner (of a room) and also an angle. In-spite of the fact that 'a room with six corners' means that the floor of the room having six vertical walls, has six corners which indicates that base of the solid body is hexagonal and the solid body becomes a prism with hexagonal base with the lateral sides perpendicular to the base (justifying the validity of the rule given in the *Āryabhaṭīya*), we adhere to the commentary of *Bhāskara*-I who has considered a right pyramid with equilateral triangular base for the solid body mentioned in the above rule. Without going through the Geometrical feasibility of the original verse, we go straight to the examples of the commentator as they have a close bearing on the topic under our consideration. We describe here the first example as a specimen:

śṛṅgaṭakaghanagaṇitaṁ dvādaśa-gaṇitāśritasya yaccāsya |
ūrdhvabhujāparimāṇaṁ sphuṭata-ramācakṣva me śīghraṃ ||

Shukla (1976, p. 58)

Translation:

Tell me quickly the height (*ūrdhabhujāparimānaṃ*) of a *śṛṅgāṭaka* solid body (whose three lateral edges converge at a point above the base) with six equal edges (*aśri*) each of length 12 unit.

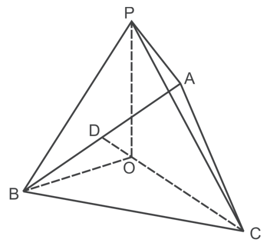

Fig. 6. *Śṛṅgāṭaka ghana* with each edge 12 unit long

In the fig.6, PA=PB=PC=AB=BC=CA =12 unit. Let O be the centroid of ABC. According to *Nīlkaṇṭha*, O is the *pāta* and the *ūrdhabhujā* PO = the height of the pyramid (According to *Nīlkaṇṭha*, each of PO, BO, CO is termed a *pātarekhā* [*Āryabhaṭīya, Nīlkaṇṭha*'s Sans. com. ed. Sambasiva, (1930), p.30]. The calculation shown by *Bhāskara*-I is this: The right bisector CD (called *avalambaka*) of AB in the equilateral ABC (base of the pyramid) $= \sqrt{12^2 - 6^2} = \sqrt{108}$, (written as *ka* 108) when $BD = \frac{1}{2}BA = 6\ \sqrt{36}$

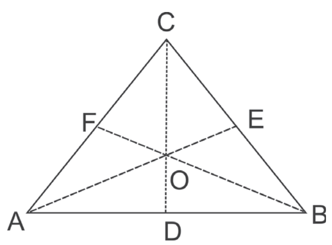

Fig. 7. Length of the *pātarekhā* BO

That the base is an equilateral triangle has been clearly mentioned by *Nīlkaṇṭha Somayajī* in *his* commentary in *Āryabhaṭīya* of the above verse in the line: '*atra tribhujamiti sama tribhujaṁ vivakṣitaṃ* (*Āryabhaṭīya*, *Nīlkaṇṭha*'s Sans. com. ed. Sambasiva, 1930, p.27) and that AE is a median of the triangle ABC (fig.7) is clear from his statement: *savyabhujāmadhyagatamapi sūtramitarayoḥ saṁyoga. i.e.*, the thread (through A) extended up to the mid-point of the side to the right. In fact, AE, BF are also the medians and O is the centroid (*pāta*) of ΔABC. The method followed by *Bhāskara*-I to find the length of BO is described here (as this a special approach applicable to an equilateral triangle only):

> *yadi aṣṭoṭaraśatakaraṇikena (avalambakena) catuścatvāriṁśaduttara-śata karaṇikaḥ karṇo labhyate, tadā ṣaṭtriśatkaraṇikenāvalambakena kiyān karṇa iti | trairaśiko- papattipradarśanārthaṁ kṣetranyāsaḥ* ॥
>
> Shukla (1976, p. 59)
>
> *i.e.*, 'if for the perpendicular (*aṣṭottaraśatakaraṇika*), the diagonal (here, hypotenuse) be $\sqrt{144}$ (*catuścatvāriṁśat uttaraśata-karaṇika*), then what is the diagonal for the perpendicular $\sqrt{36}$?

With a view to explaining by the rule of three, the computation of the area is given below:

The rule of three is applicable here because of the similarity of Δ^s CDB and ODB, which again holds because the ΔABC is equilateral.

By the rule of three,

$$\frac{BO}{BD} = \frac{BC}{CD} \Rightarrow BO = \frac{BC}{CD} \times BD = \frac{\sqrt{144}}{\sqrt{108}} \times \sqrt{36} = \sqrt{48}$$

The computation of the height of the pyramid has been done by the commentator thus:

> *labdho 'ntaḥ karṇaḥ (karaṇyaḥ)* 48 | *ayameva karṇaḥ ūrdhvamavasthita-tribhujakṣetrasya bhujāḥ | karṇakṛteḥ bhujāvargaviśeṣaḥ ūrdhvabhujāvargaḥ | sa ca* 96 |
>
> **Translation:**
>
> (Thus) obtained the internal diagonal BO $= \sqrt{48}$. This diagonal is the base of the vertical (right-angled) triangle (ΔPOB). The square of the base (BO) subtracted from the square of the diagonal (PB) (gives) the square of the diagonal (*karṇakṛteḥ i.e.*, $PB^2 - BO^2 = PO^2 = 12^2 - 48 = 96$; so PO = $\sqrt{96}$ unit.

(ii) Śrīdharācārya (850-950 CE) set the following example (verse 83, p. 33) in his *Triśatikā* (*Triśatikā*, Sans. com. by Dvivedi, 1899, p.33) where the complete figure contemplated is a concave quadrilateral with the pairs of adjacent sides equal; though he did not prescribe any name to the quadrilateral. The concerned verse is,

madhyāyāmaḥ ṣoḍaśa bālendau madhyavistarastrikaraḥ |
tribhujadvaya kalpanayā gaṇitaṁ kiṁ tatra kathayāśu | (Dvivedi, 1899, p 33)

Translation:

The middle-most span of a crescent of the Moon (in the last quarter of the dark half or in the first quarter of the bright half of a lunar month) is 16 cubit and the distance between the horns is 3 cubit. Taking the two halves (of the crescent) for triangles, find quickly the area (of the plane figure of the said shape).

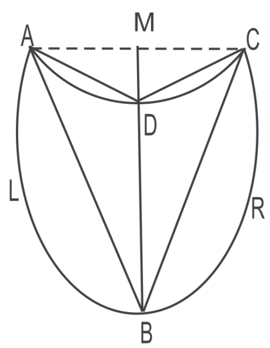

Fig. 8. Crescent of the Moon: a special case of a concave quadrilateral

In the Fig.8, ALBRCDA is the crescent of the Moon on a night in the last quarter of the dark half or in the first quarter of the bright half of a lunar month. Let A and C denote the two horns and $\overline{BDM}$ denote the line of symmetry of the curve ALBRCDA, BD be the middle-most breadth (=16 cubit) of the crescent, AC (=3 cubit) , the distance between the horns, where 1 cubit = I *hasta* (also called one *kara*) = 10 *aṅgula*s = approximately 18 inches of a man of a common height.

Clearly, the quadrilateral ABCD (suggested in the question as equivalent to the crescent of the Moon in her particular phase for the purpose of finding the area of the plane figure bounded by the curvilinear outline) is a concave quadrilateral with pairs of equal adjacent sides (AB, BC) and (AD, DC).

Since, $\overline{BDM}$ is perpendicular to AC and ΔABC , ΔADC stand on the common base AC, therefore, the area of the quadrilateral = $\frac{1}{2}$AC.(BM - DM) = $\frac{1}{2}$AC.BD =24 square unit.

It may be mentioned here that for finding the area of the curvilinear figure resembling the crescent of the Moon, it is to be regarded as a combination of two triangles. This has been stated by Āryabhaṭa-II in an abridged way in the verse 101 *balendu tribhuje dve* [*Mahāsi.*, Sans.com. by Dvivedi (1910), p.177].

The idea of a concave quadrilateral as available in the *Mahāsiddhāntaḥ* was unprecedented in Āryabhaṭa-II's contemporary world of Mathematics.

3. Opinions of Mercier (1993), pp. 1-13 and of Billard (1971), pp.157-161 about the date of *Mahāsiddhāntaḥ* of *Āryabhaṭa*-II

There was raised a controversy regarding the date of *Mahāsiddhāntaḥ* of Āryabhaṭa-II by Mercier. Referring to Billard's astronomical findings, Mercier claimed that Āryabhaṭa-II's *Mahāsiddhāntaḥ* was written after the *siddhanta śiromaṇi* had been written by Bhāskarācārya (b. 1114 CE). Let us examine the actual situation.

The starting pair of verses of the *parāśaramatādhyāyaḥ* [*Mahāsi.*Sans.com., Dvivedi (1910), p.43 are,

kalisaṅge yugapāde pārāśarya ṁataṁ praśastayataḥ |
vakṣye tadahaṁ tanmamamatatulyaṁ madhyamānyatra ||, Verse 1, p. 43

ie., during the first quarter of the *yuga* after the beginning of *kaliyuga* the doctrine/ opinion of *Parāśara* is suitable. I, therefore put forward the mean values here as those tally with those of mine.

etatsiddhāntadvayamīṣadyāte kalau yuge jātaṃ |

svasthāne dṛktulyā anena kheṭāḥ sphuṭāḥ kāryāḥ || Verse 2, p. 43

i.e., The (subjects/materials) of the two *siddhānta*s (the one written by me and the other by *Parāśara*) have the origin at the time shortly after the start of the *kaliyuga.* The positions of planets were determined by piercing holes (for marking the positions of planets) in (the authors') own locality (*svasthāne dṛktulyā*).

Clearly, this second verse is at the root of a serious confusion because it claims that the two *siddhāntaḥ*s were originated shortly after the beginning of the *kaliyuga*, which started in 3102 BCE (Bentley, 1825, p.116), though Āryabhaṭa-II flourished in the tenth century CE. A little scrutiny of the language will reveal that Āryabhaṭa-II's astronomical data were based on practical observations made by Parāśara just as Bhāskarācārya did on Brahmagupta. In the second line of the verse it has been mentioned that data were collected on the basis of ocular observation unaided by any sophisticated instrument. Astronomical refraction, aberration etc. were totally out of consideration in those days. As a result, serious error might have crept in, as for example, the case of finding the number of revolutions of the *ayanagraha* (a planet whose longitude is affected by ecliptic deviation) in a *kalpa* has been given as 581709 instead of 181709 in the verse 9 of *Parāśaramatādhyāya* (This may, of course be a simple cryptographic error). Now the question is, who was this Parāśara? The name Parāśara occurs in *Ṛgveda* as the seer of the verses 1.65-73 praising *agni* and also of the verses 31-44, 9.97 for praising *Soma.* He was the son of Śakti muṇi and himself was a sage. As *Ṛgveda* is an work of the period: third million BCE to 800 BCE [Winternitz, (1920),Vol.I, Eng Tr.p.258)][1], this *Parāśara* mentioned may or may not be the same as has been mentioned in the *Ṛgveda*. Recently a book entitled *Parāśara Tantra* (ISBN 9788192099248) has been published by the Jain University, India. In this book it has been claimed that the astronomical tradition carried on in the book dates from 1350-1130 BCE. This deepens the darkness besetting the identification of the astronomer Parāśara. Let us now check the points where Parāśara doctrine differs from other notable *siddhānta*s, namely, *Brāhmasphuṭa-siddhānta* (abridged as *Br.Sp.Si.*) and *Āryabhaṭīya.* Before this, one point must need be made clear that in most of the cases, the number of revolutions of the Sun, the Moon, and other planets including their apogees are given in a *mahāyuga* while those in the *Parāśara*-plan have been given in a *kalpa* = 1000 *mahāyuga* approximately. We therefore reduce the revolutions in the *Parāśara*-plan to a *mahāyuga* to facilitate the necessary comparison:

Planet	*Br.Sp.Si.*	*Āryabhaṭīya*	*Parāśara*-plan
Moon	57,353,300	57,753,336	57,753,334
Sun	4,320,000	same	same
Mars	2,296,828.522	2,296,824	2296831
Jupiter	364,226.455	364,224	364,219.382
Saturn	146,567.298	146,564	146,569
Moon's apogee	nil	488,219	488,108.674
Venus	7,022,389.492	7,022,,338	4,320,000
Mercury	17,936,998.984	17,937,020	4,320,000
Moon's node	232,311.168	232226	232,313.354

As it was a customary hypothesis that the Sun, the Moon and all other planets were in conjunction at the first point of Aries at *Laṅkā* in

[1] History of Indian Literature (1920) by Winternitz M consists of three volumes written in German. Volume I dealing with the Vedic Literature, Volume II, with Buddhist and Jaina Literature and Volume III with post-vedic Sanskrit Literature. The first two volumes have been translated into English by Mrs Shilavati Khetkar in 1927 and 1933 respectively and published by the University of Calcutta in the year 1959 A.D.

the beginning of *kalpa* as well as at the start of *kaliyuga* and then they started revolving in their respective concentric circular orbits with the Earth as the centre, the mean longitude of them and therefore their true longitudes also will vary if their number of revolutions in a *mahāyuga* varies. So, the dependence of Āryabhaṭa-II's calculations on Parāśara-plan might be a factor for the positional variations of some planets. In the present writer's opinion, it is quite possible to detect such positional variations of planets in the astronomical works of many ancient astronomers, irrespective of the geographical boundaries of the places they flourished in.

3.1 The *ayanāṁśa* –formula, the parameters considered by Billard as mentioned by Mercier

Mercier has not mentioned what were the so called 'parameters' considered in Billard's search nor has he mentioned the actual source wherefrom he has obtained the *ayanāṁśa*–formula:*ayanāṁśa*= $\sin^{-1}$ [sin(0.504^r + 578159^r.*t*).sin24°]. He has simply stated 'the *ayanāṁśa* according to *Mahāsiddhānta*', though no verse in the *Mahāsiddhānta* (with the commentary of *Sudhakara dvivedi*) in support of the formula is available. Actually, the concept of precession of equinox was introduced for the first time by *Muñjalācārya* in the first half of the tenth century CE (Shukla, 1990, p.2). The term *ayanāṁśa* used by Āryabhaṭa-II clearly means the portion of the ecliptic passed over by the *ayanagrahaḥ* (which is undoubtedly the Sun, where the word *ayana* means *gamana i.e.*, 'to go' /'to traverse'). It may be added here, that the modern concept of the longitude of a star as an arc of the ecliptic measured from the first point of Aries (in the counter-clock-wise direction) was not in vogue in the ancient India. This will be clear from the followings:

The verse in which the word *ayanāṁśa* occurs in the chapter *spaṣṭādhikāraḥ* of *Mahāsiddhānta* is:

idānīmayanāṁśānāha
ayanagraha-doḥkrāntijyācāpaṁ
kendravaddhanar ṇaṁ syāt |
ayanalavāstatsaṁskṛtakheṭādayanacarār-
dhapalāni ||13, *Mahāsi.*, p.57

Meanings of essential words: *ayanagrahaḥ*- longitude of a planet which is affected by ecliptic deviation.

{Here the *ayanagrahaḥ* has been taken for a heavenly body (hence forth we shall mention it as a planet just to maintain parity with the author's sense) moving along the ecliptic and therefore, it is the Sun because, the Sun alone moves along the ecliptic}, *do-i.e., bhujā*-the base , *krānti* – declination (of a star), (it is angular distance of the planet in the ecliptic from the equator measured along the vertical through the planet), *krānti-jyācāpaṁ* - the arc of the declination, *kendravat dhanarṇaṁ* - (this is) positive or negative like the *kendras*, where by *kendra*s are meant the *mandakendra* (anomaly) and the *śīghrakendra* (180° ± commutation), *ayanalavā-ayanāṁśāḥ* (the portion of the ecliptic passed over) *khetāt*-from a celestial body, *cara*- ascensional difference, *pala*-(1/60) of a *ghaṭi*.

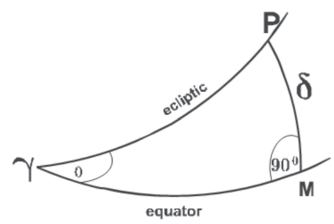

Fig. 9. Spherical triangle showing *ayanagraha* P, declination δ =PM, longitude ϒP

Translation: Now I am telling about *ayanāṁśāḥ*. For the arc ϒP (denoting of the longitude of a planet which moves on the ecliptic) having (ϒM) for the base (of a spherical triangle), the arc (PM= δ) denoting the declination (is the perpendicular of the spherical triangle ϒPM, right-angled at M) is positive or negative like the *kendra*s and provides the correction to be applied to the (longitude of) a planet (moving in the ecliptic) for finding half of the *cara* in minutes.

The commentator clarifies that the declination (= $\frac{\gamma P.\sin\theta}{R}$) {θ has the maximum value of 24^0 as mentioned in the verses 12-15, *spaṣṭādhikāraḥ gaṇitātādhyāyaḥ* of *siddhānta śiromaṇi* of Bhāskarācārya (abbreviated as *Bhā., siśi spaṣṭa, gaṇita.*), and not by Āryabhaṭa-II} is positive in between the starting of the Aries to the end of Virgo (*meṣādau*) on the ecliptic and negative in its remaining portion) and this is the correction to be applied to the (longitude of) a planet (moving in the ecliptic) for finding half of the *cara* in minutes (*i.e.*, the period between the epoch of sun-rise at a place (not on the equator) and the epoch of the same on the equator on the same meridian, vide verse 2, *tripraśnavāsanā, golādhyāyaḥ* (abbreviated as *Bhā., golā*) of *Bhāskarācārya* [*Bhā., golā* Sans. com. by Marīchi, (1988), p.274. In the figure 9 and also in the figure 10, P is the *ayanagraha* moving north-wards along the ecliptic, ♈ is the first point of Aries, ♈P is the arc of the ecliptic denoting the longitude of P), PM is an arc of the vertical through P meeting the celestial equator at M; Z and N denoting the zenith and Nadir respectively.

3.2 Discussion: The correction mentioned in the commentary is the commentator's addition on the basis of the opinion of Bhāskarācārya-II

According to the elucidation and commentary by the commentator Sudhākara Dvivedī, 'in the opinion of the *ācārya* (which is actually the opinion of Bhāskarācārya), the greatest declination is 24° (*jināṁśa*) and the declination of a planet (which is actually a star in modern astronomy) is to be found from the *ayanāṁśāḥ* actually, longitude) at a given epoch'. Now, declination has been termed *krānti* or *apama* (not *krāntipātaḥ*) by Bhāskarācārya in the following line

nāḍikāmaṇḍalāt tiryagtrāpamaḥ
krāntivṛttāvadhḥ krāntivṛttāccharaḥ|
Verse no . 16.

i.e., the arc drawn from the position of a planet (in the ecliptic) perpendicular to the equator is the *krānti* or *apama* of the planet—*golabandhādhikāraḥ*,verse 16 of Bhāskarācārya-II [*Bhā, golā*, Sans. com. by Marīchi, (1988), p.240] .

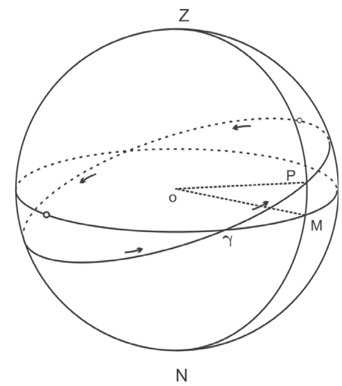

Fig. 10. *Ayanagraha* P and its *krānti* $\widehat{PM}$

This is positive in *meṣādau* (*i.e.*, between the beginning of Aries to the end of Virgo) and negative in *tulādau* (*i.e.*, from the beginning of Libra to the end of Pisces).

It is therefore clear that Āryabhaṭa-II was not aware of the concept of precession of equinoxes, though Muñjalācārya residing at Prakāśapattana in the northern India, wrote (some time in CE 932) his famous *Laghumānasa* wherein he put forward his findings of the rule on Precession of Equinoxes. This is almost in accordance with the corresponding modern findings. It may be noted that Bhāskarācārya (b. 1114 CE) was well-aware of the findings of Muñjalācārya and recorded the matter in the verses 17-19 in *Siddhāntaśiromaṇi, golabandhādhikāraḥ* [*Bhā., golā*, Sans com. by Marīchi, (1988), pp.241-242].

For a ready reference, the verses are mentioned below:

viṣuvavatkrāntivalayoḥ saṁpātaḥ
krāntipātaḥ syāt |
tadbhagaṇāḥ sauroktā vyastā
ayutatrayaṁ kalpe ||17
ayanacalanaṁ yaduktaṁ muñjalādyaiḥ
sa evayaṃ |

tatpakṣe tadbhagaṇāḥ kalpegoṅgartun-andagocandrāḥ (199669)||18
tatsaṁjātaṁ pātaṁ kṣiptā kheṭe 'pamaḥ sādhyaḥ |
krāntivaśāccaramudayaścaradalalagnāgame tataḥ kṣepyaḥ ||19

Translation: The point of intersection of the celestial equator (*viṣuvavat*) and the ecliptic (*krāntivalaya*) is the *krantipātaḥ*. In *saūrasiddhāntaḥ*, it has been stated that the point of intersection moves in the reverse direction (*i.e.*, west-ward) and revolves three thousand times in a *kalpa*. What *Muñjala* said previously about the precession is this : in one *kalpa* the said point of intersection revolves 199669 times, The arc (of the ecliptic) from the initial position of the first point of Aries to its position at the desired epoch (*i.e.*, $\widehat{\gamma\gamma_1}$) is the precession of the equinox and it should added to the position of the planet for finding the declination on which depends the half of the ascensional difference and the epochs of the rising of signs and hence the precession should be added to them to know the actual ascensional difference and the ascendant.

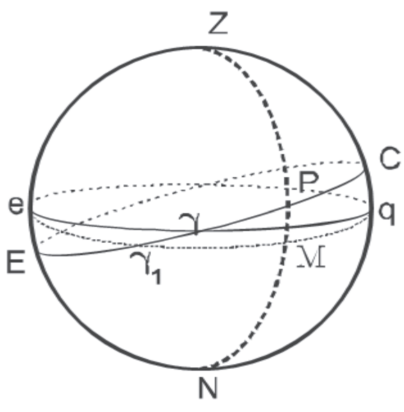

Fig. 11. Precession of equinox from γ to γ_1 and the declination PM of the planet P on the ecliptic

As Bhāskarācārya was a well known Mathematician-cum astronomer throughout India and remembered by all his countrymen even today, had Āryabhaṭa-II been posterior to Bhāskarācārya, he would have been acquainted with *Siddhāntaśiromaṇi* and would have been able to rectify his idea about *ayanāṁśa* (following Bhāskarācārya).

It is known that the motion of the equinox [Smart (1977), p.233] on the ecliptic is given by,

$$at + bsin\Omega + csin\ 2\Omega + lsin\Theta + msin\mu \quad \ldots(\mathrm{I})$$

where, *at* denotes the luni-solar precession in longitude in *t* years, Ω, e, μ denote respectively longitudes of the moon's ascending node, the Sun and the Moon, *a*, *c*, *l*, *m* are constants determinable by dynamical theory. Here, excepting the part *at* all other terms are of periodic nature and the sum of the parts save *at* give the nutation in longitude.

In (modern) *Sūryasiddhāntaḥ* as mentioned by Bhāskarācārya in the above verse, it has been stated that all the asterisms move first westward through 27°, then after returning to the original position, move eastward through 27°. There is no other Indian work on ancient Indian Astronomy which speaks of the oscillatory nature of asterisms.

So the formula quoted by Mercier with reference to Billard does not have any support from the modern Western as well as from the ancient Indian Astronomy.

3.3 Now let us come to the point where Mercier has differed from Dixit [Dixit, (1981), pp. 33 & 96] on the point of anteriority of *Āryabhaṭa*-II to *Bhāskarācārya*

Whenever Bhāskarācārya has mentioned the name of Āryabhaṭa-I, he has used the adjective '*ādya*' (*ādi+ṣṇya*), meaning 'the first' qualifying the name Āryabhaṭa as for example, in the *vāsanābhāṣya* (self commentary) of the verse 52 in *bhuvanakośa*,[*Bhā.*, *golā*, Sans com. by Marīcī, (1988), p.90] he mentions,

ato 'yutadvayavyāse 20000 *dvikāg-nyaṣṭyamaturmitaḥ* 62832
paridhirāryabhaṭādyairaṅgīkṛtaḥ |

Translation: Now, for (a circle) with a diameter 20,000 (unit), the circumference was stated as 62832 by the **first** Āryabhaṭa.

In the self commentary of the verse 65 of the *spaṣṭādhikāraḥ*,[*Bhā.*, *Gaṇit¸ spaṣṭa.*, Sans com. by Girija, (2007), p. 211 Bhāskarācārya mentioned,

ata evāryabhaṭādibhiḥ sūkṣmatvārthaṁ dṛkhānodayā paṭhitāḥ |

Here, the word '*ādi*' has been used to mean 'etcetera', 'so on'. Indeed the difference between the two words *ādyaiḥ* (the singular form of the third case-ending of the word '*ādya*') and '*ādi*' is possibly one of the causes of confusion of Mercier. It is interesting to note that Mercier has used the word 'decan' instead of the Sanskrit word *dṛkkāṇaḥ* (also *drekkāṇaḥ*), most probably to remind the reader of the Greek origin (δε′κανος) of the word.

It is therefore clear that Dixit's opinion (that Āryabhaṭa-II was anterior to Bhāskarācārya) is justified.

In connection with his reference to Apte (1943, p.217), about the mention of the formula in the commentary of *Munīśvara* on the verses 17-19 of *golabandhādhikaraḥ* of Bhāskara's *golādhyāyaḥ*, I would like to point out that in the original *bhāṣya* (*vāsanābhāṣya*) given by Bhāskarācārya, there is not even a hint of any formula or of the mention of Āryabhaṭa-II's name; only the doctrine of the modern *sūryasiddhāntaḥ* and the name of Muñjalācārya along with his findings have been mentioned. So, the insertions of the formula and of the name of Āryabhaṭa-II, if these so happened, was made by Munīśvara in course of his commentary.

From the above discussion it is likely that there is confusion as to the mention of the name of Āryabhaṭa-II in connection with the formula but actually it does not stand and may be cleared.

In conclusion it may be said that the claims of both Billard and Raymond Mercier does not stand and is liable to be disregarded. Late Professor David Pingree rightly criticized it as an unacceptable [Mercier (1993), *ibid.*] one, suggesting rightly the date of *Mahāsiddhānta* of Āryabhaṭa II a work of 10th century CE.

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