

Determination of Ascensional Difference in the *Lagnaprakaraṇa*

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Abstract

The ascensional difference or the *cara* is a fundamental astronomical concept that is crucial in determining the durations of day and night, which are a function of the observer's latitude and the time of the year. Due to its importance, almost all astronomical texts prescribe a certain procedure for the determination of this element. The text *Lagnaprakaraṇa*—a hitherto unpublished manuscript attributed to Mādhava, the founder of the Kerala school of astronomy and mathematics—however discusses not one, but a number of techniques for the determination of the *cara* that are both interesting and innovative. The present paper aims to discuss these techniques.

Key words: *Arkāgragūṇa*, Ascensional difference, *Cara*, *Carajyā*, *Carāsu*, *Dyugūṇa*, *Dyujyā*, Earth-sine, *Kujyā*, *Lagnaprakaraṇa*, Mādhava, *Mahīgūṇa*

1. INTRODUCTION

The ascensional difference (*cara* henceforth) is an important astronomical element that is involved in a variety of computations related to diurnal problems. It is essentially the difference between the right ascension and the oblique ascension of a body measured in time units. At the time of rising, the *cara* gives the time interval taken by a body to traverse between the horizon and the six o' clock circle or vice-versa depending upon whether the declination of the body is positive or negative. It is most commonly determined for the Sun because it is this quantity that helps one in finding the time of sunrise and sunset at a given location, for a given time of the year. It is also used in the determination

of the rising times of the different zodiacal signs (*rāśis*) at a given latitude.

The *Lagnaprakaraṇa*¹ (Treatise for the Computation of the Ascendant) is a work comprised of eight chapters, dedicated to the determination of the ascendant (*udayalagna* or orient ecliptic point), and discusses numerous techniques for the same. However, as a necessary precursor to determining the ascendant, the text first discusses various methods to obtain the *prāṇakalāntara*,² as well as the *cara*. While the procedure for determining the *cara* is discussed in other astronomical works, no text treats it and related phenomena as thoroughly and comprehensively as the *Lagnaprakaraṇa*. To demonstrate this, in this paper we excerpt verses from the first chapter of this text which deal with

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¹The authors obtained two manuscripts of the *Lagnaprakaraṇa* from the Prof. K. V. Sarma Research Foundation, Chennai.

²The *prāṇakalāntara* (lit. difference between *kalās* and *prāṇas*) is the difference between the longitude and corresponding right ascension of a body.

five different ways to compute the *cara* (verses 18–24), and explain the technical content of the verses along with their rationale.

2. DETERMINATION OF *CARA* IN THE *LAGNAPRAKARAṆA*

The *Lagnaprakaraṇa* presents five different methods for obtaining the *cara* of the Sun at any latitude, and also discusses how to apply this quantity depending upon the position of the Sun and the time of the day. The first method below is the standard approach to determining the *cara* given in a number of Indian astronomical texts, while the subsequent methods appear to be unique.³

In the following discussion, we employ the symbols λ , α , and δ to denote the longitude, right ascension, and declination of the Sun respectively. The symbols ϕ and ϵ are employed to denote the latitude of the observer and the obliquity of the ecliptic respectively. The radius of the diurnal circle, which measures $R \cos \delta$, is generally referred to by the word *dyujyā*. However, in the following verses, the author employs the term *dyuguna* more frequently.

METHOD 1

पलमौर्विकाभिनिहतादपक्रमात्
अवलम्बकोद्धृतफलं महीगुणः ।
त्रिभमौर्विकाविनिहतात्महीगुणात्
द्युगुणाहतं कृतधनुश्चरं विदुः ॥१८॥

palamaurvikābhinihatādapakramāt
avalambakoddhṛtaphalaṃ mahīguṇah |
tribhamaurvikāvinihatātmahīguṇāt
dyugunāhṛtaṃ kṛtadhanuścaraṃ viduḥ ||18||

The quotient obtained from dividing the product [of the Rsine] of the declination (*apakrama*) and the Rsine of the latitude (*palamaurvikā*)

by the Rcosine of the latitude (*avalambaka*) is the earth-sine (*mahīguṇa*). They (i.e. scholars) know [the result] from the radius (*tribhamaurvikā*) multiplied earth-sine divided by the day-radius (*dyuguna*) converted to an arc to be the ascensional difference (*cara*).

The above verse composed in the *mañjubhāṣiṇī* metre gives the relations to determine (i) the earth-sine (*mahīguṇa*), and (ii) the ascensional difference (*cara*). The given relations can be expressed as follows:

$$mahīguṇa = \frac{apakramajyā \times palamaurvikā}{avalambakajyā}$$

$$\text{or, } R \sin M = \frac{R \sin \delta \times R \sin \phi}{R \cos \phi} \quad (1)$$

and,

$$cara = dhanuṣ \left(\frac{mahīguṇa \times tribhamaurvikā}{dyuguna} \right)$$

or,

$$\Delta\alpha = R \sin^{-1} \left(\frac{R \sin M \times R}{R \cos \delta} \right). \quad (2)$$

Using (1) in (2), we have

$$\Delta\alpha = R \sin^{-1} \left(\frac{R \sin \phi \times R \sin \delta \times R}{R \cos \phi \times R \cos \delta} \right). \quad (3)$$

To derive the *cara*, we refer to Fig. 1, where the Sun has northern declination and therefore rises earlier for an observer in the northern hemisphere compared to an equatorial observer. The measure of this time difference, i.e. *cara*, is given by the great circle arc *FE*, which corresponds to time taken by the Sun to reach the six o'clock circle (point *Y*) after sunrise (point *X*).⁴ The relation for the *cara* given in the verse can be derived from spherical triangle *PZX*, where *ZX* is the vertical through *X*. In this triangle, we have

$$\begin{aligned} \hat{P} &= 90 + \Delta\alpha, & PX &= 90 - \delta, \\ ZX &= 90, & PZ &= 90 - \phi. \end{aligned}$$

³In the Kerala school, Nīlakaṇṭha Somayājī (*Tantrasaṅgraha*, pp. 76–80) gives the same formula as Method 1 for calculating the *cara*. The *Gaṇita-yukti-bhāṣā* does not explicitly show how to calculate the *cara*. However, methods 4 and 5 below are discussed by Putumana Somayājī (*Karaṇapaddhati*, pp. 252–256).

⁴*PE* and *PF* are meridian circles perpendicular to the equator passing through *Y* and *X* respectively.

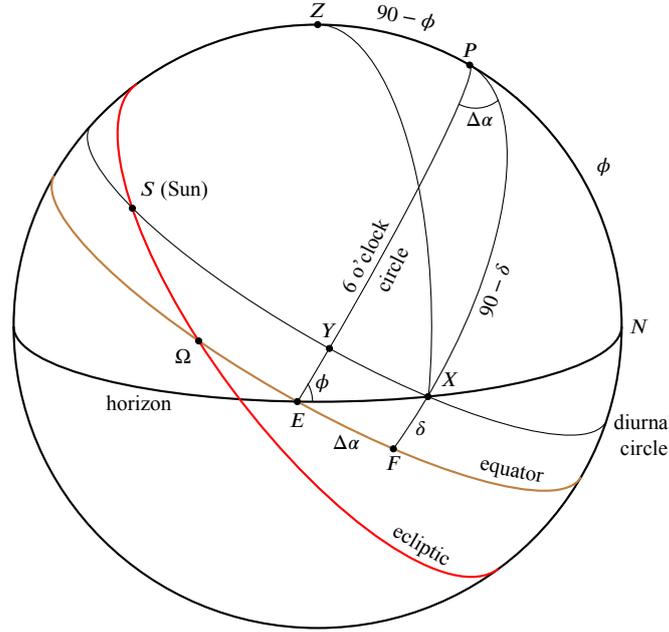


Fig. 1. Determining the earth-sine and the ascensional difference.

Applying the cosine rule of a spherical triangle, we have:

$$\cos 90 = \cos(90 - \phi) \cos(90 - \delta) + \sin(90 - \phi) \sin(90 - \delta) \cos(90 + \Delta\alpha),$$

or,

$$\Delta\alpha = \sin^{-1} \left(\frac{\sin \phi \sin \delta}{\cos \phi \cos \delta} \right),$$

which is the same as (3).⁵

The *mahīguṇa* and the Rsine of the *cara* are the semi-chords of corresponding arcs of the diurnal circle (XY) and the equator (FE) respectively, and are therefore proportional to their respective radii $R \cos \delta$ and R . Therefore, we have the *mahīguṇa*

$$\begin{aligned} R \sin M &= \frac{R \cos \delta \times R \sin \Delta\alpha}{R} \\ &= \frac{R \sin \delta \times R \sin \phi}{R \cos \phi}, \end{aligned}$$

which is the same as (1).

It may be noted that the verse prescribes to first determine the *mahīguṇa*, and then the *cara* using this quantity. However, for convenience, here we first derived the *cara* using spherical trigonometry, and then the *mahīguṇa* from it. In contrast to the technique shown here, Indian astronomers typically first derived the *mahīguṇa* using planar geometry, and then determined the *cara* therefrom, using the fact that they are semi-chords of corresponding arcs of the diurnal circle and the equator respectively. The method of deriving the *mahīguṇa* using planar geometry is shown in our discussion of Method 2.

METHOD 2

अन्त्यापयानाहतदोर्गुणाद्वा
त्रिज्याहतादिष्टभुजापमाद्वा ।
लम्बाहतोऽर्काग्रगुणः स कर्णः
क्रान्तिश्च कोटीह भुजा तु भूज्या ॥१९॥
भास्कराग्रपलमौर्विकयोर्वा
दोर्गुणान्तिममहीगुणयोर्वा ।

⁵The relation for the *cara* can also be derived from the spherical triangle EFX in Fig. 1. Here, we have $\hat{E} = 90 - \phi$, $FE = \Delta\alpha$, $\hat{F} = 90$, and $XF = \delta$. Applying the cotangent four-part formula using these values also gives (3).

घातमत्र विभजेद् दिनमौर्व्या
तत्फलं खलु भवेच्चरजीवा ॥२०॥

antyāpayānāhatadorguṇādvā
trijyāhatādiṣṭabhujāpamādvā |
lambāhṛto 'rkāgragūṇaḥ sa karṇaḥ
krāntīśca koṭīha bhujā tu bhūjyā ||19||
bhāskarāgrapalamaurvikayorvā
dorguṇāntimamahīguṇayorvā |
ghātamatra vibhajed dinamaurvyā
tatphalaṃ khalu bhavēccarajīvā ||20||

The quotient of either the Rsine (*dorguṇa*) [of the Sun's longitude] multiplied by [the Rsine of] the last (maximum) declination (*apayāna*), or [the Rsine of] the declination (*apama*) corresponding to the desired longitude multiplied by the radius (*trijyā*), divided by the Rcosine of the latitude (*lamba*), is the Rsine of the Sun's amplitude (*arkāgragūṇa*). That [Rsine of the Sun's amplitude] is the hypotenuse (*karṇa*). [The Rsine of] the declination (*krānti*) is the upright (*koṭi*) here, and indeed the earth-sine (*bhūjyā*) is the lateral (*bhujā*).

Here, one should divide either the product of [the Rsine of] the amplitude of the Sun (*bhāskarāgra*) and the Rsine of the latitude (*palamaurvikā*), or of the Rsine (*dorguṇa*) [of the Sun's longitude] and the last earth-sine (*antimamahīguṇa*), by the day-radius (*dinamaurvī*). The quotient indeed would be the Rsine of the ascensional difference (*carajīvā*).

The above two verses, composed in the *indra-vajrā* and *svāgatā* metres respectively, together present two expressions for the *cara*. As these expressions involve the Rsine of the Sun's amplitude⁶ (*arkāgragūṇa*) at the time of rising or setting, the verses first present two relations to determine the same, and then describe a right-angled triangle having the *mahīguṇa* and the Rsine of the declination as sides, and the *arkāgragūṇa* as the

hypotenuse.

Expressions for *arkāgragūṇa*

Denoting the Sun's amplitude at the moment of rising or setting as A' , the two expressions for *arkāgragūṇa* given in the verse are:

$$arkāgragūṇa = \frac{dorguṇa \times antyāpayānajyā}{lambajyā}$$

$$\text{i.e., } R \sin A' = \frac{R \sin \lambda \times R \sin \epsilon}{R \cos \phi} \quad (4)$$

and,

$$arkāgragūṇa = \frac{iṣṭabhujāpamajyā \times trijyā}{lambajyā}$$

$$\text{i.e., } R \sin A' = \frac{R \sin \delta \times R}{R \cos \phi}. \quad (5)$$

The above relations can be derived by applying the sine rule to the spherical triangle EXF in Fig. 1. In this triangle, we have

$$\begin{aligned} EX &= A', & X\hat{E}F &= 90 - \phi, \\ XF &= \delta, & E\hat{F}X &= 90. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{\sin A'}{\sin 90} &= \frac{\sin \delta}{\sin(90 - \phi)} \\ \text{or, } R \sin A' &= \frac{R \sin \delta \times R}{R \cos \phi}, \end{aligned}$$

which is the same as (5). Using the well known relation connecting the longitude and declination of the Sun

$$\sin \delta = \sin \lambda \sin \epsilon,$$

the above relation can be written as

$$R \sin A' = \frac{R \sin \lambda \times R \sin \epsilon}{R \cos \phi},$$

which is the same as (4).

⁶Amplitude (A') refers to the angular distance of separation from the east or west point of the horizon. It is related to the azimuth (A) by the relation $A = 90 \pm A'$, with the positive sign employed when the *arkāgragūṇa* is south, and negative sign employed when it is north.

Expression for *mahīguṇa*

Having defined *arkāgragūṇa* in the first three quarters of verse 19, the author proceeds to identify what *mahīguṇa* or *bhūjyā* is geometrically in the last quarter of the verse. It is said that the right-angled triangle having *arkāgragūṇa* as its hypotenuse, has the *mahīguṇa* and the Rsine of the declination as its sides. This can be understood from Fig. 2a, which is the same as Fig. 1, but rotated about its axis for ease of visualisation. This figure is depicted in greater detail in Fig. 2b, where X' is the projection of the point X on the equatorial plane. The planar right-angled triangle XOX' lies in the meridian plane passing through X along which the declination is measured. Therefore, we have $X\hat{O}X' = \delta$, and hence

$$XX' = R \sin \delta, \quad OX' = R \cos \delta.$$

The right-angled⁷ triangle OGX lies on the horizon, and $G\hat{O}X = A'$. Therefore,

$$XG = R \sin A', \quad OG = R \cos A'.$$

The right-angled triangle XGX' lies in a plane perpendicular to the equator, and parallel to the plane of the prime meridian. Here, we have already determined XG and XX' as the Rsines of the amplitude and declination respectively. As in this triangle

$$\hat{G} = \phi' = 90 - \phi,$$

we have

$$GX' = R \sin A' \sin \phi.$$

This third side GX' is nothing but the earth-sine or the *mahīguṇa*. This can be understood as follows. Consider the diurnal circle passing through X in Fig. 2a. A perpendicular dropped from X onto the radius of this circle, which is parallel to the line OE , will be the earth-sine. GX' is the image of this line on the equatorial plane. Therefore,

⁷The triangle does not appear right-angled in the figure due to the difficulty in depicting the three-dimensional celestial sphere on a two-dimensional surface.

XGX' is the right-angled triangle which the author alludes to in this verse. It may also be noted that upon substituting (5) in the above expression for the *mahīguṇa* (GX'), the expression reduces to (1).

Expressions for *cara*

In verse 20, the author gives the following relations to determine the Rsine of the ascensional difference, in terms of the relations described above:

$$\begin{aligned} \text{carajīvā} &= \frac{\text{bhāskarāgragūṇa} \times \text{palamaurvīkā}}{\text{dinamaurvī}}, \\ R \sin \Delta\alpha &= \frac{R \sin A' \times R \sin \phi}{R \cos \delta}, \end{aligned} \quad (6)$$

and,

$$\begin{aligned} \text{carajīvā} &= \frac{\text{dorgūṇa} \times \text{antimamahīguṇa}}{\text{dinamaurvī}} \\ \text{i.e., } R \sin \Delta\alpha &= \frac{R \sin \lambda \times R \sin M_l}{R \cos \delta}, \end{aligned} \quad (7)$$

where $R \sin M_l$ is the last earth-sine, or the earth-sine when the declination of the Sun is maximum and equal to the obliquity of the ecliptic. Therefore, substituting $\delta = \epsilon$ in (1), we have

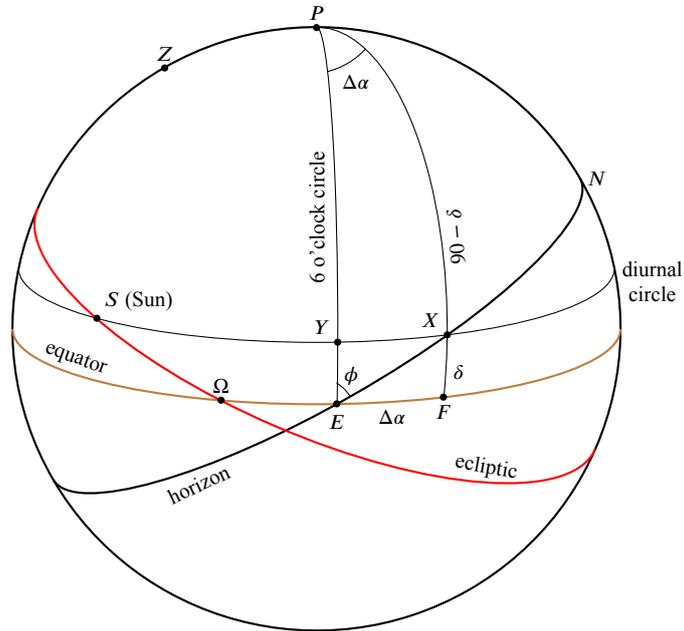
$$R \sin M_l = \frac{R \sin \epsilon \times R \sin \phi}{R \cos \phi}. \quad (8)$$

The rationale behind (6) can be understood with the help of Fig. 3. This figure is the same as Fig. 2b, only modified to highlight the similar triangles OGX' and OQF . From Fig. 2a, we know that EF is the arc corresponding to the ascensional difference, which means that in the planar right-angled triangle OQF we have the angle $Q\hat{O}F = \Delta\alpha$. As the side $OF = R$ in this triangle, we obtain

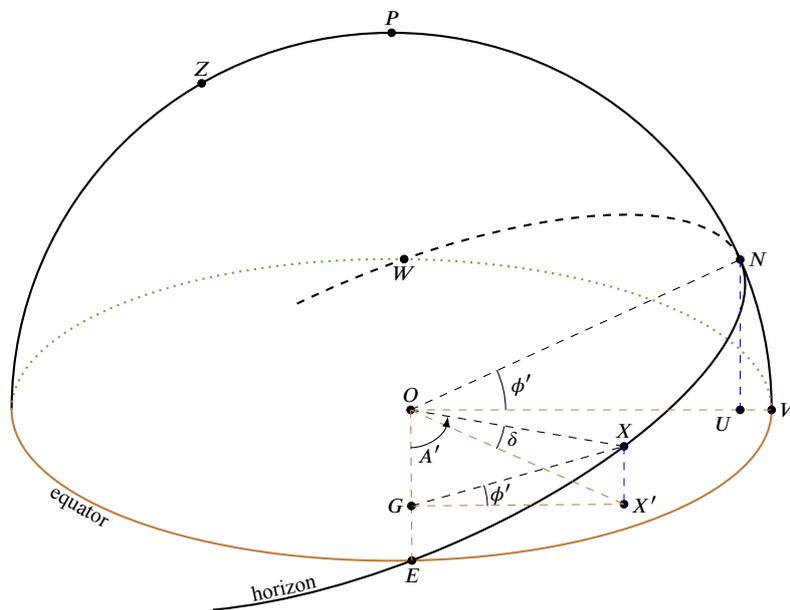
$$FQ = R \sin \Delta\alpha.$$

We have already shown that $GX' = R \sin A' \sin \phi$, and $OX' = R \cos \delta$. Therefore, considering the similar triangles OGX' and OQF , we have

$$\frac{R \sin \Delta\alpha}{R} = \frac{R \sin A' \sin \phi}{R \cos \delta},$$



(a) Rotated celestial sphere.



(b) Planar right-angled triangle formed by the amplitude, the declination, and the earth-sine.

Fig. 2. Visualising the triangle which has the amplitude, the declination and the earth-sine as its sides.

or,

$$R \sin \Delta\alpha = \frac{R \sin A' \times R \sin \phi}{R \cos \delta},$$

which is the same as (6). Note that (6) can also be validated by simply substituting the relation for $R \sin A'$ from (5) into it, upon which the given relation reduces to (3). Similarly, (7) also reduces to (3) upon substituting for $R \sin M_l$ using (8) in it.

METHOD 3

दोर्ज्यावर्गाद् भुजापक्रमकृतिरहितात्
 मूलमस्यैव कोटिः
 दोर्ज्याघ्न्यन्त्यद्युजीवा त्रिभगुणविहता
 वापमस्यास्य कोटिः ।
 तद्वर्गादत्र कोटीगुणकृतिसहितात्
 मूलमाहुर्द्युजीवाम्
 इष्टापक्रान्तिकोटिश्चरमचरहता
 द्युज्ययाप्ता चरज्या ॥२१॥

dorjyāvargād bhujāpakramakṛtirahitāt
mūlamasyaiva koṭiḥ
dorjyāghnyantyaadyujīvā tribhaguṇavīhātā
vāpamasyāsya koṭiḥ |
tadvargādatra koṭīguṇakṛtisahitāt
mūlamāhurdyujīvām
iṣṭāpakrāntikoṭiścaramacarāhatā
dyujyayāptā carajyā ॥21॥

The square-root taken from—the square of the Rsine (*dorjyā*) [of the Sun's longitude] decreased by the square of [the Rsine of] the declination corresponding to the longitude (*bhujāpakrama*)—is the *koṭi* of this [declination] itself. Or, the last day-radius (*antyaadyujīvā*) multiplied by the Rsine (*dorjyā*) [of the Sun's longitude] and divided by the radius (*tribhaguṇa*) is the *koṭi* corresponding to this declination (*apama*). Here, [scholars] state the square root taken from—its (*koṭi*) square increased by the square of the Rcosine (*koṭīguṇa*) [of the Sun's longitude]—to be the day-radius (*dyujīvā*). The

koṭi of the desired declination multiplied by the maximum ascensional difference (*caramacara*) and divided by the day-radius (*dyujyā*) is the Rsine of the ascensional difference (*carajyā*).

The above verse is composed in the long *sragdharā* metre, consisting of twenty-one syllables per quarter. The choice of the long metre for this verse stems from the fact that the author gives several relations here. The first two relations are meant for defining a quantity called the *koṭi*. Next, he defines a relation for determining the radius of the diurnal circle (*dyujīvā*) in terms of *koṭi*. This is followed by an expression for the Rsine of the ascensional difference (*carajyā*) in terms of *koṭi* and *dyujīvā*. The expressions for *koṭi* along with their equivalent in mathematical notation are:⁸

$$\begin{aligned} koṭi &= \sqrt{(dorjyā)^2 - (bhujāpakramajyā)^2} \\ &= \sqrt{(R \sin \lambda)^2 - (R \sin \delta)^2}, \end{aligned} \quad (9)$$

or,

$$\begin{aligned} koṭi &= \frac{dorjyā \times antyaadyujīvā}{tribhaguṇa} \\ &= \frac{R \sin \lambda \times R \cos \epsilon}{R}. \end{aligned} \quad (10)$$

The *dyujīvā* and *carajyā* are given as

$$\begin{aligned} dyujīvā &= \sqrt{(koṭi)^2 + (koṭīguṇa)^2} \\ \text{or, } R \cos \delta &= \sqrt{(koṭi)^2 + (R \cos \lambda)^2}, \end{aligned} \quad (11)$$

and

$$\begin{aligned} carajyā &= \frac{koṭi \times caramacarajyā}{dyujyā} \\ \text{or, } R \sin \Delta\alpha &= \frac{koṭi \times R \sin \Delta\alpha_m}{R \cos \delta}, \end{aligned} \quad (12)$$

where $\Delta\alpha_m$ is the maximum ascensional difference which occurs at the maximum declination of the Sun. Therefore, substituting $\delta = \epsilon$ in (1) and (2), from the two equations we have

$$R \sin \Delta\alpha_m = \frac{R \sin \epsilon \times R \sin \phi}{R \cos \phi} \times \frac{R}{R \cos \epsilon}. \quad (13)$$

⁸The phrase 'koṭi corresponding to this declination' in the verse is not to be confused with the Rcosine of the declination here. Instead, it is to be taken as a technical term denoting the given mathematical expressions.

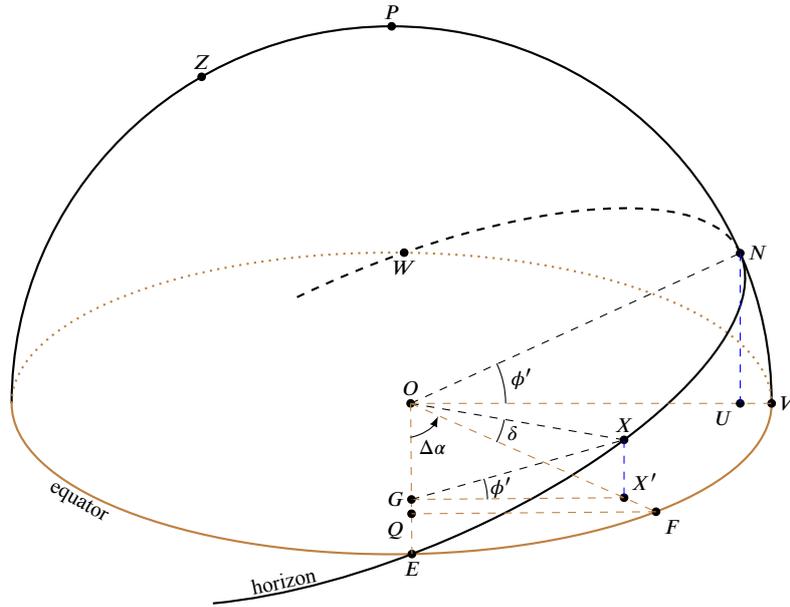


Fig. 3. Determining the relation for the *cara*.

Rationale for the expressions for *koṭi* and *dyujīvā*

We now show that the expression for the *koṭi* described in this verse corresponds to the line segment TS' shown in Fig. 4, which depicts the celestial sphere from the point of view of the equatorial plane. Here, let S be the Sun, having longitude λ and declination δ . In the right-angled triangle STS' , we have the side $SS' = R \sin \delta$, and the hypotenuse $TS = R \sin \lambda$. As $S\hat{T}S'$ is nothing but the angle between the equator and the ecliptic, we have $S\hat{T}S' = \epsilon$. Therefore,

$$TS' = R \sin \lambda \cos \epsilon.$$

This is equivalent to the expression for *koṭi* given in (10). In the right-angled triangle STS' , we also have

$$TS'^2 = TS^2 - SS'^2.$$

Noting $TS' = koṭi$, and substituting the values of the other two sides of the triangle in this expression yields (9).

Similarly, (11) can be easily derived by considering the right-angled triangle OTS' in the same

figure. In this triangle, we have the hypotenuse $OS' = R \cos \delta$, and the sides $OT = R \cos \lambda$, and $TS' = koṭi$. Since $OS'^2 = TS'^2 + OT^2$, we obtain (11).

Rationale for the expression for the *carajyā*

The given expression for the *carajyā* or the Rsine of the ascensional difference can be understood with the help of Fig. 3 as well as Fig. 4. In Fig. 3, noting that $ON = R$, and that GX' is the *mahīguṇa*,⁹ from similar triangles XGX' and NOU , we have

$$R \sin \delta = mahīguṇa \times \frac{R \sin \phi'}{R \cos \phi'}.$$

In Fig. 4, noting that $OS_1 = R$ and TS' is the *koṭi*, from similar triangles STS' and S_1OD , we have

$$R \sin \delta = koṭi \times \frac{R \sin \epsilon}{R \cos \epsilon}.$$

⁹See Method 2.

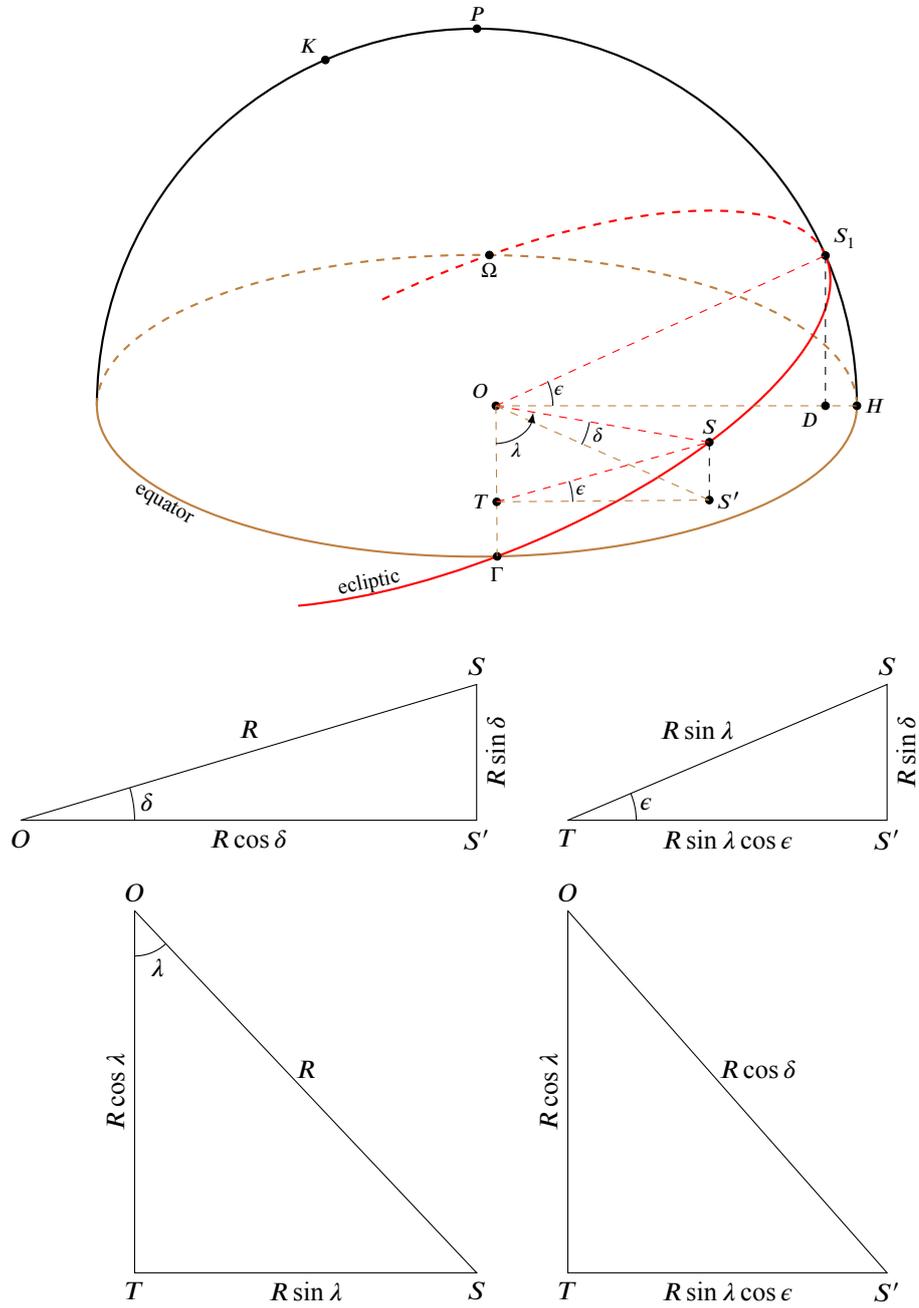


Fig. 4. Understanding the term *koti*.

Equating the above two expressions, noting $\phi' = 90 - \phi$, and rearranging, we have the *mahīguṇa*

$$\begin{aligned} R \sin M &= \frac{\text{koṭi} \times R \sin \epsilon \times R \sin \phi}{R \cos \epsilon \times R \cos \phi} \\ &= \frac{\text{koṭi} \times R \sin \Delta\alpha_m}{R}. \quad [\text{using (13)}] \end{aligned}$$

Now, multiplying both sides by $\frac{R}{R \cos \delta}$, and using (2), we have

$$R \sin \Delta\alpha = \frac{\text{koṭi} \times R \sin \Delta\alpha_m}{R \cos \delta},$$

which is the required expression (12).

METHOD 4

परमद्युगुणाहता भुजाज्या
विहतेष्टद्युगुणेन कालजीवा ।
परमेण चरेण ताडिता सा
त्रिभजीवापहताथवा चरज्या ॥२२॥

paramadyugunāhatā bhujājyā
vihṛteṣṭadyuguneṇa kālajīvā |
parameṇa careṇa tāditā sā
tribhajīvāpahṛtāthavā carajyā ||22||

Or, the Rsine (*bhujājyā*) [of the Sun's longitude] multiplied by the extreme (last) day-radius (*paramadyuguna*) and divided by the desired day-radius (*iṣṭadyuguna*) is the Rsine of the right ascension (*kālajīvā*). That (*kālajīvā*) multiplied by [the Rsine of] the maximum ascensional difference (*parama cara*), and divided by the radius (*tribhajīvā*), is the Rsine of the ascensional difference (*carajyā*).

The above verse composed in the *mālabhāriṇī* metre prescribes the formula for obtaining *cara* in terms of the *kālajīvā* or the Rsine of the Sun's right ascension. For this purpose, the verse first gives an expression for the *kālajīvā* as follows:

$$\begin{aligned} \text{kālajīvā} &= \frac{\text{bhujājyā} \times \text{paramadyuguna}}{\text{iṣṭadyuguna}} \\ \text{or, } R \sin \alpha &= \frac{R \sin \lambda \times R \cos \epsilon}{R \cos \delta}. \quad (14) \end{aligned}$$

The *cara* is then given in terms of the *kālajīvā* as:

$$\begin{aligned} \text{carajyā} &= \frac{\text{kālajīvā} \times \text{paramacarajyā}}{\text{tribhajīvā}} \\ \text{or, } R \sin \Delta\alpha &= \frac{R \sin \alpha \times R \sin \Delta\alpha_m}{R}. \quad (15) \end{aligned}$$

The expression for the right ascension given by (14) can be understood using Fig. 5. This figure is the same as Fig. 4, only modified to highlight the similar triangles *OTS'* and *OCB*. Here, the arc ΓB represents the right ascension of the Sun, and therefore $\widehat{BOC} = \alpha$. As $OB = R$, we have the semi-chord

$$CB = R \sin \alpha.$$

We also already know that¹⁰

$$TS' = R \sin \lambda \cos \epsilon, \quad \text{and} \quad OS' = R \cos \delta.$$

Now, from similar triangles *OTS'* and *OCB*, we have

$$\begin{aligned} \frac{R \sin \alpha}{R} &= \frac{R \sin \lambda \cos \epsilon}{R \cos \delta} \\ \text{or, } R \sin \alpha &= \frac{R \sin \lambda \times R \cos \epsilon}{R \cos \delta}, \end{aligned}$$

which is the same (14).

From (10) and (14), we have

$$\frac{\text{koṭi}}{R \cos \delta} = \frac{R \sin \alpha}{R}.$$

One can easily see that substituting the above expression in (12) yields (15).

METHOD 5

त्रिज्याक्षघातलम्बांशगुणिता क्रान्तिमौर्विका ।
द्युज्यासा चरजीवा स्यात् तच्चापं हि चरासवः ॥२३॥

trijyākṣaghātāmbāṅśaguṇitā krāntimaurvikā |
dyujyāptā carajīvā syāt
taccāpaṃ hi carāsavaḥ ||23||

¹⁰See Method 3.

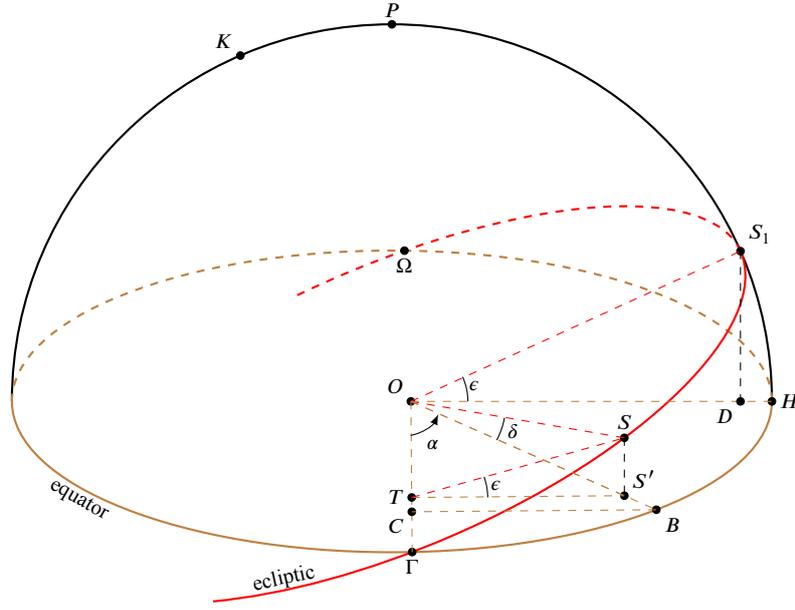


Fig. 5. Derivation of *kālajīvā* or right ascension.

The Rsine of the declination (*krāntimaurvikā*) multiplied by one-*lambath*¹¹ part of the product of the radius (*trijyā*) and [the Rsine of] the latitude (*akṣa*) and divided by the day-radius (*dyujyā*) would be the Rsine of the ascensional difference (*carajīvā*). Indeed, its arc would be the ascensional difference in arc-minutes (*carāsavaḥ*).

The above verse in the *anuṣṭubh* metre gives the following relations:

$$carajīvā = \frac{\frac{trijyā \times akṣajyā}{lambajyā} \times krāntimaurvikā}{dyujyā}$$

$$\text{or, } R \sin \Delta\alpha = \frac{\frac{R \times R \sin \phi}{R \cos \phi} \times R \sin \delta}{R \cos \delta}, \quad (16)$$

and,

$$carāsu = cāpa(carajīvā) \\ \text{or, } \Delta\alpha = R \sin^{-1}(R \sin \Delta\alpha). \quad (17)$$

The above expressions are a restatement of (2), upon substituting for the *mahīguṇa* from (1).

3. APPLICATION OF THE ASCENSIONAL DIFFERENCE

प्रातश्चरं वणिगजादिवशाद्भ्रनर्णम्
अस्तेऽन्यथा न तदहर्निशयोस्तु मध्ये ।
कार्यं त्रिपञ्चमलवादिषु चानुपातात्
नैवानुपातविधिरस्य घटीप्रसारे ॥२४॥

prātaścaram vaṇigajādivaśāddhanarṇam
aste'nyathā na tadaharniśayostu madhye |
kāryaṃ tripañcamalavādiṣu cānupātāt
naivānupātavidhirasya ghaṭīprasāre ||24||

At sunrise, the ascensional difference is positive or negative depending on [the Sun's position with regard to] Libra or Aries. While setting, it is otherwise. That should not be applied during midst of the day and night. And in case of third, fifth etc. portions [of the day, the *cara*] should be calculated [afresh] successively. The rule of proportion for this (*cara*) will not be applicable in [setting] the motion of the clock (*ghaṭīprasāra*).

¹¹The term *lamba* refers to the Rcosine of the latitude. The compound word लम्बांश is to be understood here as लम्बेन विभज्यासः अंशः। This is an unusual way to express division by a variable.

This verse composed in the *vasantatilakā* metre prescribes rules for the application of the *cara* depending upon the time of the day, and the Sun's position on the ecliptic for an observer in the northern hemisphere. The verse also briefly comments on the nature of the *cara* function.

3.1. Application of *cara* at sunrise and sunset

When the Sun is in the northern hemisphere (i.e. its longitude is in the range of 0° to 180° , referred to as *ajādi* or 'beginning with Aries' in the verse), the *cara* is to be subtracted in the morning, and added in the evening to obtain the time of the local sunrise and sunset respectively. This is because, during this period, the Sun rises earlier and sets later than at the equator, where sunrise and sunset occur at six o'clock throughout the year.

Fig. 6a depicts the *cara* at the time of sunrise when the Sun has northern declination. Here, it can be seen that the Sun reaches the horizon at point *X* before crossing the six o'clock circle at point *Y*. Therefore, the Sun rises before 6 a.m. The *cara* gives the magnitude of this difference and needs to be subtracted from six o'clock to determine the local time of sunrise. Fig. 6b depicts the *cara* at the time of sunset when the Sun has northern declination. Here, it can be seen that the Sun first reaches the six o'clock circle at point *Y*, before it sets at point *X*. Therefore, the Sun sets after 6 p.m., and the *cara* needs to be added to determine the time of local sunset.

On the other hand, when the Sun is in the southern hemisphere, the *cara* is to be applied positively in the morning, and negatively in the evening to obtain the time of local sunrise and sunset respectively. This is because, during this period, the Sun rises later and sets earlier than for observers at the equator, and the duration of the day is shorter for observers in the northern hemisphere. Figures 6c and 6d depict this situation. If t_e is the time of sunrise or sunset at the equator, and t_ϕ is the time of sunrise or sunset at a desired latitude (ϕ), then application of *cara* in different situations to deter-

mine t_ϕ is summarised in Table 1.

3.2. Application of *cara* at noon and midnight

It is clearly stated in the verse that the correction due to ascensional difference is zero at the midst of the day and the night. In other words, the magnitude of the *cara* at the prime meridian is zero at mid-noon and mid-night when the Sun reaches the prime meridian either during the day or the night, irrespective of the latitude of the observer. This is another way of stating that the Sun reaches the prime meridian at the same instant for all observers on a given longitude.

After discussing the application of *cara* at sunrise, sunset, noon, and midnight, the author makes noteworthy remarks about the very nature of the *cara* function itself, not usually found in other astronomical works.

3.3. The nature of the *cara* function

In the latter half of the verse, the author makes the following remarks regarding the nature of the *cara* function from a mathematical viewpoint:

- (i) the *cara* has to be computed successively during every third part, or fifth part etc. of the day, and
- (ii) the variation in *cara* cannot be taken to be proportional to the change in time for setting the clock.

It seems to us that the intent of the author here is to caution against assuming a linear variation in *cara* from sunrise to noon, and also from noon to sunset. This prescription to determine *cara* successively at various intervals during the day, and not to apply the method of proportion while setting the clock is quite interesting and gives us a clue to the fact that the day—irrespective of its duration—was divided into a fixed number of units, for certain purposes. Clearly, the duration of each such unit would vary over the course of the year along with

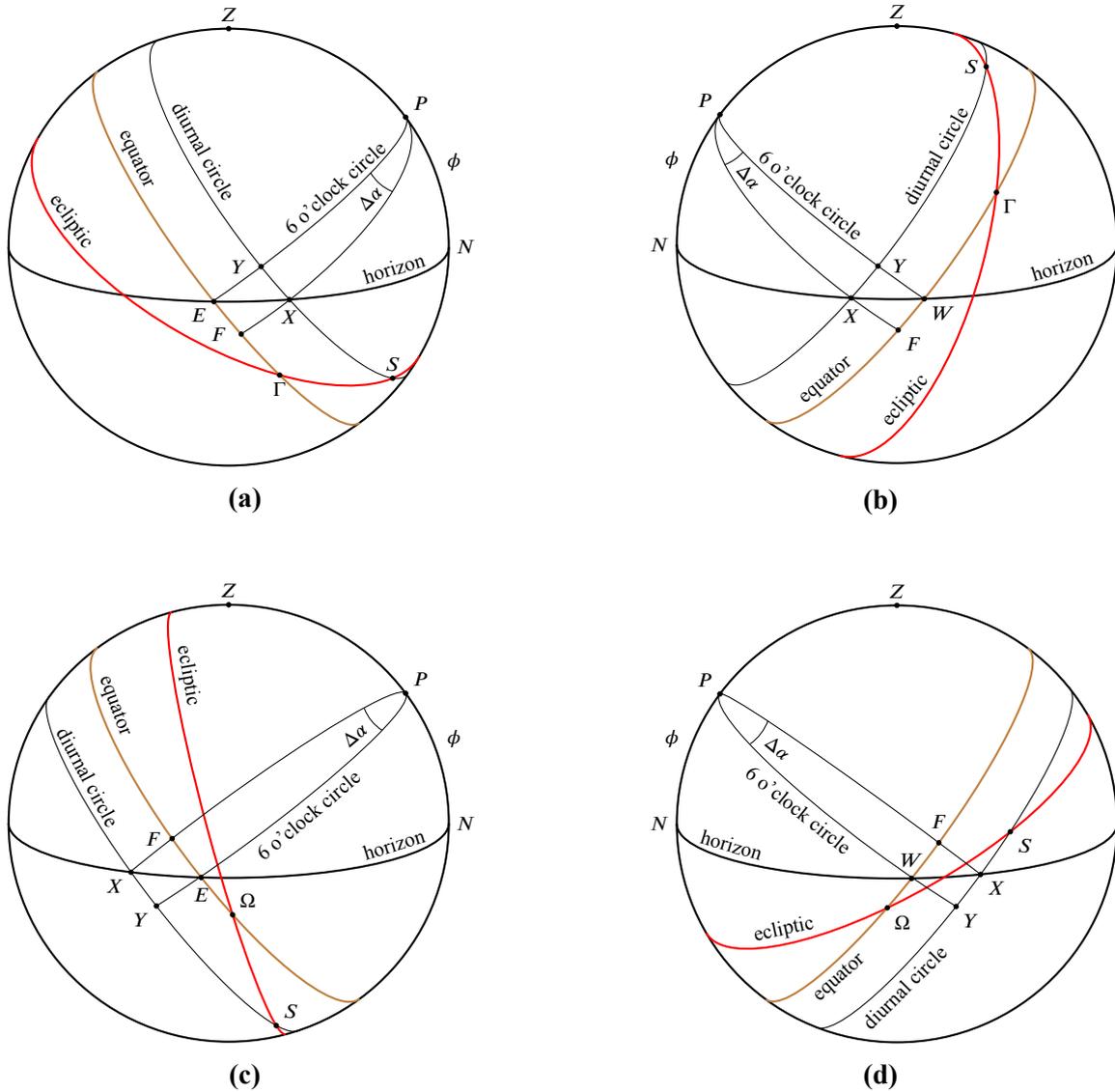


Fig. 6. Application of *cara* at sunrise and sunset for an observer in the northern hemisphere: (a) and (b) Sun having northern declination, (c) and (d) Sun having southern declination.

Table 1. Application of *cara* in determining the time of local sunrise and sunset for an observer in the northern hemisphere.

Sun's declination	Sunrise	Sunset
North	$t_\phi = t_e - \Delta\alpha$	$t_\phi = t_e + \Delta\alpha$
South	$t_\phi = t_e + \Delta\alpha$	$t_\phi = t_e - \Delta\alpha$

day length, as indicated by the following verse of Śrīpati:¹²

दिनस्य यः पञ्चदशो विभागो
रात्रेस्तथा तद्धि मुहूर्तमानम् ।
dinasya yaḥ pañcadaśo vibhāgo
rātrestathā taddhi muhūrtamānam |

Muhūrta is that [span of time] which is [equal to] one fifteenth of a day; and also of the night similarly.

The verse clearly states a *muhūrta* is equal to one-fifteenth of a day, indicating that it is not a fixed time unit, but rather varies with the length of the day (and night) over the course of the year.

Now, in the *Lagnaprakaraṇa*, the author seems to be indicating that the variation in the *cara* is to be considered even while measuring time during a day, and that this variation cannot be considered to be proportional to the change in time. In any case, it is quite interesting and noteworthy that the author presents such discussions in the context of application of *cara*. Obviously, building any such clock which takes into account the non-linear variation of *cara* would be quite complicated. However, it may be noted that in the calculation of the *śaṅku* and the *dr̥ggati* in verses 53–56 of the *Lagnaprakaraṇa*, the author makes use of the *cara* which is determined using the declination of the Sun (or an ecliptic point ninety degrees behind it) at a desired instant, rather than at sunrise. Therefore, it is clear that the author definitely had a theoretical use for this quantity, whether or not it had any practical application.

4. CONCLUSION

Of the various approaches for determining the *cara* described in the article, the first method will

be familiar to most students of Indian astronomy as that which is discussed in standard astronomical texts like the *Siddhāntaśiromaṇi*.¹³ The subsequent methods are more interesting and involve determining the *cara* using parameters like the amplitude of the Sun, the maximum earth-sine, the maximum *cara*, and so on. Methods 2, 3 and 4 in particular showcase a great dexterity and artistry in conceiving of various planar triangles and projections that lead to a variety of formulae and approaches for arriving at the same result. This indeed reveals a strong grasp of mathematical and astronomical concepts.

While method 5 is a modification of method 1, the subsequent discussion on the application of the *cara* at different times of the year and during different times of the day is important to help students understand the practical application of an otherwise abstract astronomical concept. The author deserves praise for succinctly conveying several mathematical results in a single verse, and in one particular case, all the information contained in Table 1 in less than two quarters of a verse.

The variety of approaches presented here to compute the same physical quantity suggests that the purpose of this work was more than just providing users with a way to compute the *cara*. It was fundamentally about showcasing the number of ways in which the same phenomenon could be described mathematically, and thus demonstrating a certain mathematical ingenuity and facility. Taken in conjunction with the thorough exposition on the *prāṇakālāntara* in this same text,¹⁴ the above discussion certainly reveals a deftness and robustness of approach towards solving problems which are signs of a pure mathematician. Being privy to subsequent discussions in the *Lagnaprakaraṇa*, we are confident that our future investigations will only further strengthen this claim.

¹²The given verse is attributed to Śrīpati by the commentator of the *Muhūrtakalpadrūma*. See Viṭṭhala Dīkṣita (*Muhūrtakalpadrūma*, p. 43). The exact origin is unclear. Similar verses are found in many *muhūrta* texts.

¹³See Bhāskarācārya (*Siddhāntaśiromaṇi*, p. 114).

¹⁴See Kolachana, Mahesh, and Ramasubramanian (“Mādhava’s multi-pronged approach for obtaining the *prāṇakālāntara*”).

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