Project on *Grahagaṇitādhyāya* of Bhāskarācārya's *Siddhāntaśiromaṇi*

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1 Introduction

Bhāskarācārya (b. 1150 CE) is one of the greatest names in the history of ancient and medieval Indian mathematics and astronomy [Gupta, 2008]. *Siddhāntaśiromaṇi* composed in 1150 CE by him is one of the most comprehensive treatises on Indian astronomy [*Siddhāntaśiromaṇi*, 1861, 2005 ; 1981; Satyadev Sharma, 2011]. It was a canonical textbook for students of astronomy in India for the next few centuries, and is taught in the Sanskrit institutes in India, even now.

Siddhāntaśiromaṇi has two parts, namely, *Grahagaṇita* and *Golādhyāya*. *Grahagaṇita* expounds on all the standard calculations and algorithms in astronomy of Bhāskara's times [*Siddhāntaśiromaṇi*, 2005; Arkasomayaji, 2000]. It has 460 verses in 12 chapters. The *Golādhyāya* has the definitions, more fundamental issues (like the nature of the earth, the placement of stars and planets around it and so on), whereas the *Grahagaṇita* gives the principles and theoretical details of the calculations [*Siddhāntaśiromaṇi*, 2005; Wilkinson, 1861]. While the source verses of these two parts of *Siddhāntaśiromaṇi* present the basic results and procedures, Bhāskara himself has written a commentary called the *Vāsanābhāṣya* or *Mitākṣara* on his own work, which gives a detailed exposition, almost like classroom lectures, on the entire subject. This includes details of proofs and justifications along with diagrams etc., in the *upapatti*s (rationales), discussion on how theoretical concepts are linked with observations, constructions and use of instruments etc.

Many scholars have worked on both the parts of *Siddhāntaśiromaṇi* in the past. The verses of *Grahagaṇitādhyāya* have been translated into English by Arkasomayaji, nearly 40 years back. However, the translation needs improvement. Moreover, the *vāsanā* for the verses have not been translated. This is a serious lacuna, as Bhāskara has laid out his methodology only in the commentary. Hence, there is a crying need to translate the *Vāsanābhāṣya* systematically, to understand Bhāskara's astronomy fully. I was the prinicipal investigator of an INSA project on *Grahagaṇitādhyāya* in two phases to undertake this task. Dr. Sita Sundar Ram was the Research Associate, and Dr. R. Venketeswara Pai was a Consultant in the first phase, and both were informal consultants in the second phase of the project. The first phase was operational at the Department of Theoretical Physics of the University of Madras, Chennai during December 2011 – March 2015. The second phase was taken up at the Prof. K.V. Sarma Research Foundation, Chennai during October 2016 – March 2018.

We have translated all the verses of *Grahagaṇitādhyāya*, with the accompanying *vāsanā*. We have also provided detailed explanations of the algorithms in modern notation with a large number of diagrams, using the *Vāsanābhāṣya* mainly.

Grahagaṇita has 460 verses in 12 chapters : (1) *Madhyamādhikāra* (Mean longitudes), (2) *Spaṣṭādhikāra* (True longitudes), (3) *Tripraśna* (Three problems: Time, direction and space), (4) *Parvasambhava* (Possibility of eclipses), (5) *Candragrahaṇa* (Lunar eclipse), (6) *Sūryagrahaṇa* (Solar eclipse), (7) *Grahacchāyādhikāra* (Shadows of planets), (8) *Grahodayāstādhikāra* [(Heliacal) rising and setting of planets], (9) *Śṛṅgonnati* (Elevation of Lunar cusps), (10) *Grahayuti* (Conjunctions of planets), (11) *Bhagrahayuti* (Conjunctions of planets with stars), (12) *Pātādhyāya* (*Vyatīpāta* : Equality of declinations of the Sun and the Moon).

DOI: 10.16943/ijhs/2020/v55i2/154678
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In the following we provide a summary of each chapter, and discuss some specific important topics in some detail.

2 *Madhayamādhikāra* (Mean longitudes)

This chapter has seven sections : *Kālamāna*, *Bhagaṇa*, *Grahānayana*, *Kakṣādhyāya*, *Pratyabdaśuddhi*, *Adhimāsadinirṇaya* and *Bhūparidhi*, with 120 verses. This is long compared to the chapter on mean longitudes in other texts. It gives the various measures of time, the revolution numbers and other parameters of the planets, dimensions of their orbits, continued fraction method for the longitudes and the frequency of *adhimāsa*s (intercalary months) and *kṣayamāsa* (a lunar month which has two *rāśi* transits of the sun within it, which is a very rare occurence). It also gives the method for determining the planetary periods and other parameters like apogee etc. using a *Golayantra* (armillary sphere).

2.1 Sidereal period of the Moon and its revolution number

The number of revolutions of a planet in a *kalpa* cannot be determined directly, as the sidereal periods of the apsides and nodes, and even that of the planet Saturn itself are long. Bhāskara discusses the strategy for finding the revolution numbers in the *vāsanā* for verses 1–6 in the *bhagaṇādhyāya* part of the *madhyamādhikāra*. We will describe his method for the sideral period of the the Moon in the following.

Bhāskara prescribes the use of a *golayantra* or an armillary sphere to find the revolution numbers of the planets and the associated points (like apsides and nodes), beginning with the Moon. A sketch of the *golayantra* is given in Figure 1. It has a fixed celestial equator and an ecliptic which can be rotated around the polar axis. There is also a *vedhavalaya* to locate the planet, which is a moveable ring and is a secondary to the ecliptic. There is a sight at the centre of the sphere.

The zero point of Indian zodiac which is the beginning point of the *Aśvinī nakṣatra*, or the end point of the *Revatī nakṣatra* is located in the sky, and the point R corresponding to that is marked on the ecliptic. Let the Moon be located at M at some instant on a particular day. Consider the point X, where the *vedhavalaya* on which the Moon is located, intersects the ecliptic. Then the arc RX is the (*nirayana* (without precession) longitude of the Moon, that is, with respect to a fixed stellar background, and the arc XM is the *vikṣepa*, or the latitude of the Moon. Let X_1 and X_2 be the true longitudes of the Moon at the same *ghaṭī* or instant on two successive days. Then $X_2 - X_1$, which is the difference in the true longitudes is the true daily motion of the Moon. Now, the mean longitudes X_{10} and X_{20} on the two consecutive days can be found from the true longitudes, by an inverse process, as described in verse 45 in the chapter on "*spaṣṭādhikāra*" or the "true longitudes"[1]. Then $X_{20} - X_{10}$ is the daily motion of the mean Moon. From this, the sideral period of the Moon can be computed. The number of revolutions of the Moon in a *kalpa* can then be determined from the rule of proportions. The method is described in the commentary in detail.

3 *Spaṣṭādhikāra* (True longitudes)

The *Spaṣṭādhikāra* or the chapter on true longitudes has many new features. The sine table is discussed at length. Bhāskara discusses the instantaneous rates of motion (*tātkālikagati*) of planets (Sun, Moon and the actual planets) in this chapter, where the derivative of the sine function is taken to be the cosine function. This is remarkable considering the importance of the concept of instantaneous velocity in the development of calculus. For the actual planets, his expression for the rate of motion is based on an ingenious geometrical construction. This expression can be used to discuss the retrograde motion of the planets straight away. The second order interpolation formula of Brahmagupta for finding the sine or cosine of an arbitrary angle is explained in this chapter. This indicates that Bhāskara had a rough understanding of the second derivative of a function. The "*Udayāntara*" , which is essentially the part of the equation of time due to the obliquity of the ecliptic is discussed in this chapter and a simple and accurate approximate expression for the difference between the longitude and the right ascension of the Sun is also given.

[1] The true longitude θ_{ms} can be found from the mean longitude θ_0, using the expression for θ_{ms} in the next subsection. The mean longitude, θ_0 can be obtained from the true longitude, θ_{ms} by an inverse process. Here, as the argument of the inverse sine function in the expression for θ_{ms} itself depends upon θ_0, one has to use an iterative procedure to find θ_0, where θ_{ms} is substituted for θ_0 in the argument, in the first approximation.

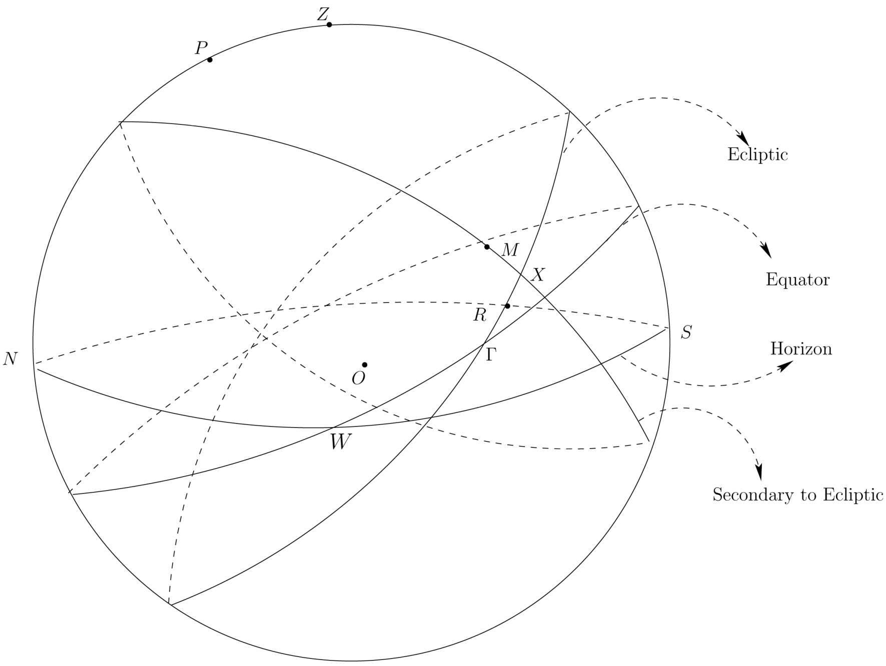

Figure 1 Finding the true longitude of the Moon using the *golayantra*.

3.1 The instantaneous rate of motion of the Sun, the Moon, and the planets

The instantaneous rate of motion of the *mandasphuṭa* (planet corrected for the equation centre)

Let θ_0 be the mean longitude (*madhyamagraha*) , θ_M be the longitude of the apside (*mandocca*), and $M = \theta_0 - \theta_M$ be the mean anomaly (*mandakendra*) . Then the longitude of the *mandasphuṭa* (planet corrected for the equation of centre) denoted by θ_{ms} is given by

$$\theta_{ms} = \theta_0 - \sin^{-1}\left(\frac{r_0}{R}\sin M\right),$$

where r_0 is the radius of the epicycle, and R is the radius of the deferent.

The *mandocca* (apside) is the apogee, and the *mandasphuṭa*, θ_{ms} is the 'true longitude' in the case of the Sun and the Moon (neglecting the second correction for the latter, which is essentially the 'evection' term, included in some texts). For the *tārāgraha*s (actual planets), Mercury, Venus, Mars, Jupiter and Saturn, the *mandocca* is the aphelion, and the *mandasphuṭa* is essentially the heliocentric true longitude.

Now r_0 is much smaller than R. When θ_0, θ_{ms} and M are all in minutes, the arcsine should also be in minutes. Then, $\sin^{-1}\psi \approx R\psi$, when the argument of the arcsine, ψ is small, and we have,

$$\theta_{ms} = \theta_0 - \sin^{-1}\left(\frac{r_0}{R}\sin M\right) \approx \theta_0 - \frac{r_0}{R}\, R\sin M.$$

With our present knowledge, we can **now** say that

$$\frac{\Delta\theta_{ms}}{\Delta t} = \frac{\Delta\theta_0}{\Delta t} - \frac{r_0}{R}(\cos M)\frac{\Delta M}{\Delta t},$$

where $\Delta\theta_0$, ΔM and $\Delta\theta_t$ are the (small) changes in θ_0, M and θ_{ms} respectively in a small time-interval Δt, as

$$\Delta(R\sin M) = (\cos M)\Delta M,$$

when ΔM is in minutes. Here, $\frac{\Delta\theta_{ms}}{\Delta t}$ is the rate of motion of the *mandasphuṭa*, and $\frac{\Delta\theta_0}{\Delta t}$ is the mean rate of motion.

Choosing Δt to be one day, and denoting

(i) the daily motion of the mean longitude , $\frac{\Delta\theta_0}{\Delta t}$ by n_0 ,

(ii) the daily motion of the *mandasphuṭa* , $\frac{\Delta\theta_{ms}}{\Delta t}$ by n_{ms} ,

and (iii) the daily motion of the anomaly, $\frac{\Delta M}{\Delta t}$ by n_M ,

we have

$$n_{ms} = n_0 - \frac{r_0}{R}(\cos M)n_M.$$

This is essentially stated by Āryabhaṭa-II in his *Mahāsiddhānta*. It is stated in a slightly different form even earlier in *Laghumānasa* of Munjālācārya, which appears to be the first text to consider the instantaneous rate of motion and use the cosine function as the derivative of the sine function, though they are not stated as such.

In verses 36b, 37 and 38 of the chapter on *spaṣṭādhikāra* (true longitudes) in the *Grahagaṇitādhyāya* part of *Siddhāntaśiromaṇi*, Bhāskara points out the need for finding an instantaneous (*tātkālikī*) rate of motion which varies from moment to moment, and gives the explicit expression for it, which is the same as the above equation for n_{ms}. This is explained in far greater detail in the *vāsanā* for the verses 36b–38 in *spaṣṭādhikāra* by Bhāskara himself [Ramasubramanian and Srinivas, 2010; Sriram, 2014].

Bhāskara's discussion of the instantaneous rate of motion

Now, apart from a sign, the correction term in the true longitude is $\frac{r_0}{R}\, R\sin M$. He takes the

$$\begin{aligned}\text{Rate of change of } R\sin M \text{ (wrt time)} =& \\ &\text{Rate of change of } M \text{ wrt time } (n_M) \\ &\times \text{ Rate of change of } R\sin M \text{ wrt angle } M\end{aligned}$$

The rate of change of $R\sin M$ wrt angle M is taken as the difference of Rsines at the beginning and the end of a 225-minute interval around M and divided by 225. **It is in this sense that what is understood as the 'derivative' of the sine function in modern times, is conceptualised in this part of *Siddhāntaśiromaṇi*.** It is correctly stated to be equal to the ratio of $R\cos M$ and R, that is $\cos M$. Putting these things together, the magnitude of the correction term in the instantaneous rate of motion is $\frac{r_0}{R}$ times the rate of change of $R\sin M$, which is $\frac{r_0}{R}(\cos M)\ n_M$, as per verse 37. Bhāskara could have arrived at the result based on a geometrical reasoning similar to the ones he uses in various contexts in the work [Sriram, 2014].

The true instantaneous rate of motion of the planets

We now consider the instantaneous rate of motion of the *tārāgraha*s, that is, Mars, Jupiter, Saturn (exterior planets), and Mercury and Venus (interior planets). For these planets, a second correction, namely, the *śīghra-saṃskāra* has to be applied, apart from the equation of centre, to obtain the true geocentric longitude. The *śīghra-saṃskāra* is equivalent to a conversion from the heliocentric to geocentric coordinates. Again, an epicycle or eccentric model is used to determine the correction. Just as the *mandocca* (apside) plays a major role in the application of *manda-saṃskāra*, so too, the *śīghrocca* plays a key role in the application of the *śīghra-saṃskāra*.[2] The *śighrakendra* is the difference between the *śighrocca* and the mean planet. For the planets, the computation of the true rate of motion would involve finding the derivative of the arcsine function, whose argument is a ratio of two functions. Bhāskara did not have the machinery to compute such a quantity.

In verse 39 of *spaṣṭādhikāra* Bhāskara gives an alternate expression for the 'true velocity' which would involve finding only the derivative of the sine function. In the accompanying *vāsanā*, he gives an ingenious geometrical method for obtaining that expression [Sriram, 2019]. In verse 41 of the *spaṣṭādhikāra*, Bhāskara discusses the retrograde motion and states the values of the *śīghrakendra* corresponding to the stationary points of the five planets. At the stationary points, the true rate of motion is zero. We can compute the values of the *śighrakendra* corresponding to this. There is a remakable agreement between the computed and the stated values of the stationary points [Sriram, 2019].

4 *Tripraśnādhikāra* (The three questions)

The third chapter on the "three questions" (of direction, location and time) is a long one with 109 verses, where the geometrical insights of Bhāskara come into full play. Here he uses the '*trairāśika*', or 'the rule of three' very effectively to solve diurnal problems. The topics include: fixing the east-west direction taking into account the variation of Sun's declination over a day, discussion of various latitudinal triangles which are helpful in diurnal problems, arriving at the formula for the zenith distance in a general situation, and so on. Elegant alternate methods are proposed, wherever possible. As an example of his methods, we present his derivation of the expression for the zenith distance of the Sun in terms of the hour angle and the declination for a place with a given latitude, in the next subsection.

[2] For the exterior planets, the *śighrocca* is the mean Sun. For the interior planets, the *śighrocca* is the mean heliocentric planet.

4.1 Zenith distance, z as a function of the hour angle, H

In Figure 2, O is the centre of the celestial sphere of radius R. Let the declination of the Sun be δ on some day. $S_r S_2 US' S_1 S_t$ is the diurnal circle of the Sun with C as the centre , whose radius is $R\cos\delta$, termed *dyujyā*. The plane of the diurnal circle is inclined to the plane of the horizon at an angle, $90° - \phi$. To be specific, we consider a northern declination, and an instant after the noon, when the Sun is at S', when its zenith distance is z (angle corresponding to the arc ZS'), and the hour angle is H. Draw $S'F$ perpendicular to the horizon meeting it at F. $S'F = R\cos z$ and is known as the *śaṅku* or the gnomon.

***Dinārdhaśaṅku* or Mid-day gnomon**

The Sun would have crossed the meridian at U. Draw UG perpendicular to the horizon meeting it G. This is clearly the mid-day gnomon, termed the '*Dinārdhaśaṅku*'. The line passing through U and C intersects the horizon at T. OC is perpendicular to UT. Then OCT is a right-angled triangle with $OC = R\sin\delta$, and $C\hat{T}O = 90° - \phi$. Hence $CT = R\sin\delta\frac{\sin\phi}{\cos\phi}$, and is known as *kśitijyā* (Earth-sine). $CU = R\cos\delta$, and is the *dyujyā* (Day-radius). Then, in verse 34 of the chapter on three questions, a quantity called *hṛti* is defined as the sum of the day-radius and the earth sine. This is just UT. So,

$$hṛti = UT = R\cos\delta + R\sin\delta\frac{\sin\phi}{\cos\phi}.$$

UGT is also a right-triangle with $U\hat{T}G = \phi$. Then,

$$\textit{Dinārdhaśaṅku} = UG = \cos\phi \times hṛti = R\cos\delta\cos\phi + R\sin\delta\sin\phi,$$

as stated in verse 36.

The desired gnomon, $R\cos z$

From S', draw a perpendicular, $S'C'$ on UT. The arc US' is along the diurnal circle, and $S'\hat{C}C' = H$, the hour angle. As $CS' = CU = R\cos\delta$, we have $CC' = R\cos\delta\cos H$, and $C'U = CU - CC' = R\cos\delta(1-\cos H)$. Draw $C'V$ perpendicular to UG. It can be seen that S', C' and V are in a plane parallel to the horizontal plane. UV is called the *ūrdhva* (upwards), as it is the upper portion of the *dinārdhaśaṅku*, UG. As $U\hat{C'}V = 90° - \phi$, it is clear that

$$\textit{Ūrdhva}\text{ (upwards)} = UV = R\cos\delta\cos\phi(1-\cos H).$$

This is the expression for *ūrdhva*, as described in verses 58 and 59; then in verse 60, it is stated that :

Desired gnomon, $R\cos z = S'F = VG = UG - UV =$ *Dinārdhaśaṅku – Ūrdhva*.

Substituting for the expressions for *dinārdhaśaṅku* (mid-day gnomon) and *ūrdhva* (upwards), we have :

$$R\cos z = R\sin\phi\sin\delta + R\cos\phi\cos\delta\cos H.$$

This is the same as the relation obtained using the cosine formula.

The hour angle H can be determined in terms of z, δ, and ϕ, by rewriting the relation as :

$$R\cos H = \frac{R\cos z}{\cos\phi\cos\delta} - \frac{R\sin\phi\sin\delta}{\cos\phi\cos\delta}.$$

The *Phalakayantra* discussed in *Golādhyāya* is based on this relation [Sriram, 2016].

5 *Parvasambhava* (Possibility of the occurence of an eclipse)

This is a small chapter with only 5 verses, but the explanations in the *vāsanā* are quite detailed. First, simple arithmetical expressions for the number of lunar months between the beginning of the *kaliyuga* and the specified new moon, and the 'Sun – node' (*sapātasūrya*) are given. Then the criterion for the occurence of an eclipse is given. If $\Delta\lambda$ is the difference between the longitudes of the Sun and one of the nodes of the Moon at the '*parvānta*', $\Delta\lambda < 14°$ for a lunar eclipse, and $\Delta\lambda < 7°$ for a solar eclipse. The rationale for these limits are provided in the commentary. In the case of a solar eclipse, the effect of parallax has also to be taken into account.

6 *Candragrahaṇa* (Lunar eclipse)

This is a long chapter with 39 verses. The dimensions of the orbits of the Sun and the Moon (689377 and 52566 *yojanas* repsectively), and their diameters (6522 and 480 *yojanas* respectively) are stated. The diameter of the Earth was given in an earlier chapter to be 1250 *yojanas*. The correction due to the equation of centre should be applied to find the true distances of the Sun and the Moon. The expression for the diameter of the earth's shadow-disc, D_{sh}

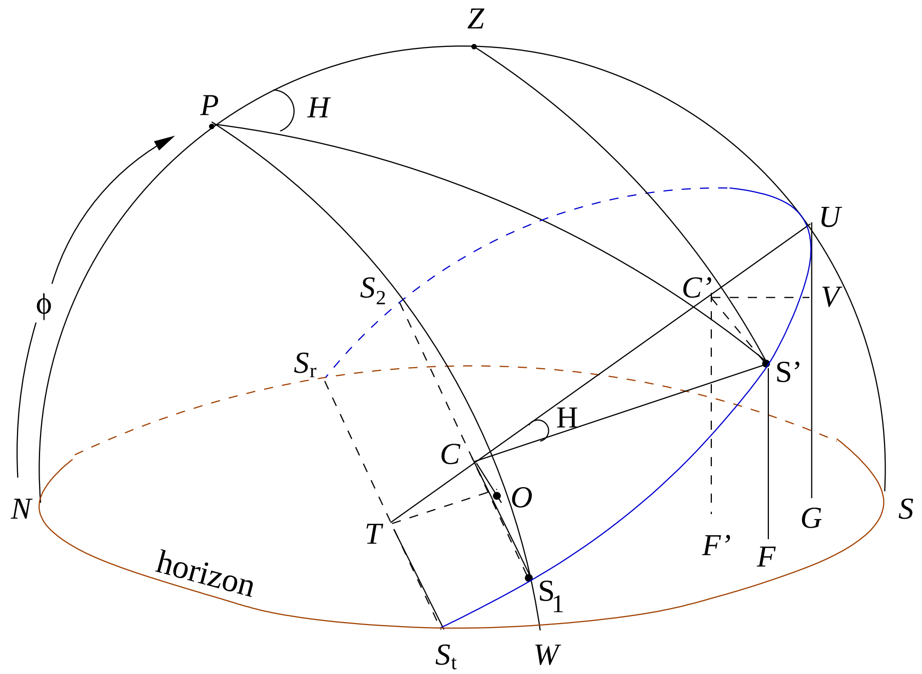

Figure 2 Bhaskara's geometrical construction for finding z in terms of ϕ, δ, and H.

in the plane of Moon's motion is given to be

$$D_{sh} = D_E - \frac{(D_S - D_E)d_m}{d_s},$$

where D_S and D_E are the diameters of the Sun and the Earth, and d_m and d_s are the distances of the Moon and the Sun from the Earth. This can be explained through a diagram described in the *vāsanā* (Figure 3). Simple expressions for the angular diameters of the Sun, the Moon, and the shadow-disc are given.

The standard expression for the latitude of the Moon is given. The magnitude of the eclipse (*sthagita*) is defined. The half-durations of an eclipse as a whole, and totality can be understood from Figure 4.

Here, the bigger disc at the centre of the figure represents the eclipsing body, and the smaller disc, the eclipsed body. The expressions for the first and second half-durations of the eclipse in *nāḍis*, t_1 and t_2, are given by

$$t_1 = \frac{\sqrt{(r_1 + r_2)^2 - \beta_1^2}}{d_{t_m} - d_{t_s}} \times 60 \quad \text{and}$$

$$t_2 = \frac{\sqrt{(r_1 + r_2)^2 - \beta_2^2}}{d_{t_m} - d_{t_s}} \times 60, \qquad (1)$$

where r_1, r_2 are the radii of the eclipsing and eclipsed bodies, d_{t_m} and d_{t_s} are the true rates of motion of the Moon and the Sun in minutes per day, and β_1, β_2 are the latitudes of the Moon at the beginning and the end of the eclipse. t_1, t_2 are calculated using iterative procedures. For the half-durations of totality, r_1+r_2 is replaced by r_1-r_2 and β_1, β_2 represent the latitudes of the Moon at the beginning and the end of totality. The same figure and expressions are applicable for solar eclipses also. Similar expressions and figures can be used for finding the obscuration at any instant, and also the instant corresponding to a specified amount of obscuration.

The topic of *valana* (essentially, the angle between the ecliptic and the local vertical at the eclipsed body) is discussed in detail. A graphical description of the progress of an eclipse is discussed in detail in this chapter. Figure 5 depicts the progress of a lunar eclipse, as described in the *vāsanā*.

7 *Sūryagrahaṇa* (Solar eclipse)

Parallaxes in longitude and latitude play very crucial roles in the computations pertaining to a solar eclipse. To calculate the parallax, one needs to know the *vitribhalagna*,

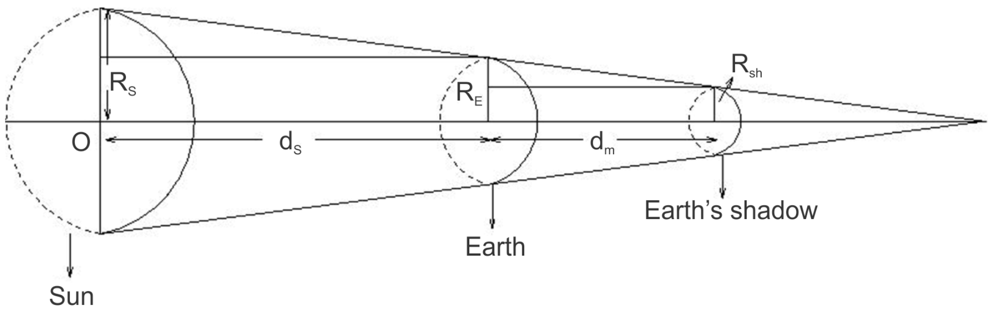

Figure 3 Shadow-cone in a lunar eclipse and the earth's shadow.

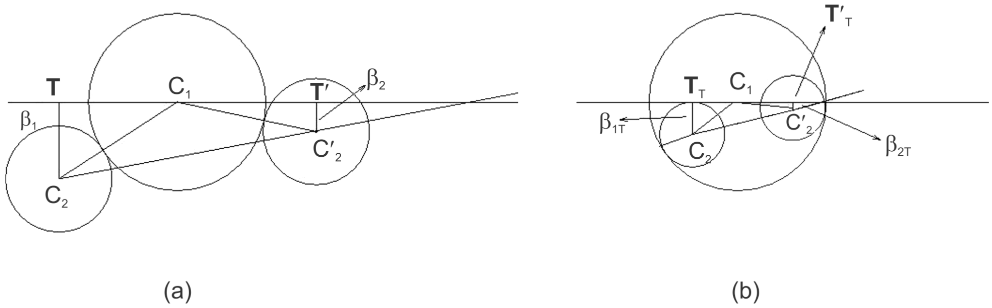

Figure 4 (a) Diagram for half-durations of an eclipse. (b) Diagram for half-durations of totality.

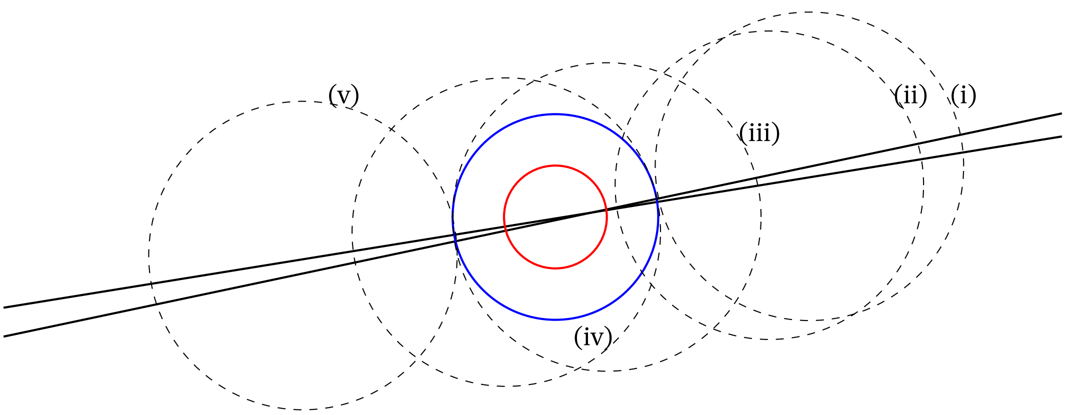

Figure 5 Progress of a lunar eclipse. If r_1 and r_2 are the radii of the eclipsing body (earth's shadow) and the eclipsed body (Moon), the small solid circle has radius $r_1 - r_2$, and the bigger solid circle with radius r_2 represents the Moon. The dashed circles with radius r_1 represent the earth's shadow at various instants: (i) At the beginning of the eclipse (*sparśa*), (ii) at some instant between the beginning and the middle of the eclipse, (iii) at the beginning of totality, (iv) at the end of totality and (v) at the end of the eclipse (*mokṣa*).

or just *vitribha*, V, which is a point on the ecliptic 90° westwards of the *lagna* at any instant. From the *lagna* at the middle of the eclipse (computed for the centre of the Earth), the longitude of the *vitribha*, its declination δ_v and its hour angle H can be found for a location with a latitude, ϕ. Then the *vitribhaśaṅku*, $R\cos z_v$ can be found from the relation:

$$R\cos z_v = R\sin\phi\sin\delta_v + R\cos\phi\cos\delta_v\cos H.$$

The parallax in longitude (called '*lambana*') in time units is then stated as:

$$Lambana\ (ghaṭikās) = 4\ ghaṭikās \times \cos z_v \sin(\lambda_s - \lambda_v),$$

where λ_s and λ_v are the longitudes of the Sun and the *vitribha*. Bhāskara does not derive this, but gives a persuasive argument based on the rule of proportions for this approximate relation. The expression for the parallax in latitude, *nati* is given by

$$nati \quad = P\sin z_v,$$

where P is the maximum parallax of "Sun – Moon".

The Parallax correction is applied to the instant corresponding to the middle of the eclipse, and the corrected middle of the eclipse is found. At this corrected instant, the longitude of the Sun and Moon are calculated again, and the parallax is also found and applied. In this way, the exact middle of the eclipse is found using an iterative process.

Bhāskara also proposes an epicycle-type of model for finding the *lambana* similar to the one for obtaining the *śīghraphala* of a planet, where the radius of the earth plays the role of the epicycle-radius. This is shown in Figure 6.

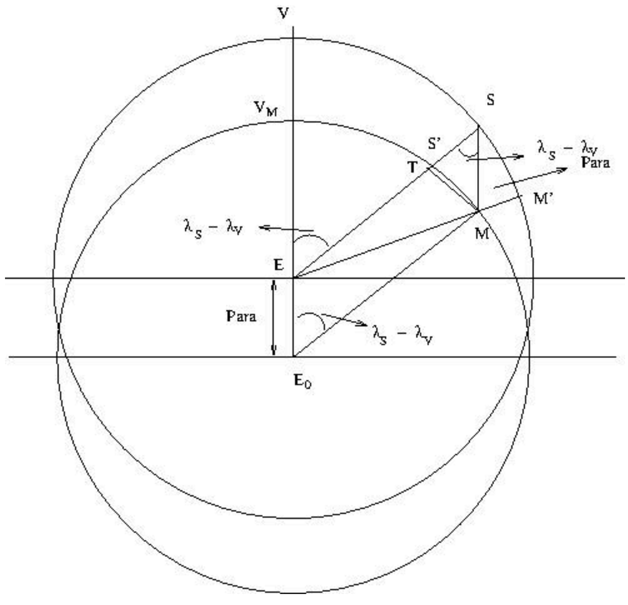

Figure 6 Model for *lambana* similar to the one for obtaining the *śīghraphala* of a planet.

8 Grahacchāyādhikāra (Shadows of planets)

The maximum latitudes of planets which are essentially inclinations of their orbits are stated. The expression for the latitude of a planet, β, as observed from the earth which involves its *śīghrakarṇa* is given. Let δ' be the declination of a point on the ecliptic with the same longitude as the planet. Then, the true declination of a planet is given by

$$\delta = \delta' + \beta\cos\theta,$$

where θ is the angle between the secondaries to the ecliptic and the equator at the planet. This is depicted in Figure 7 which is based on Bhāskara's description. The

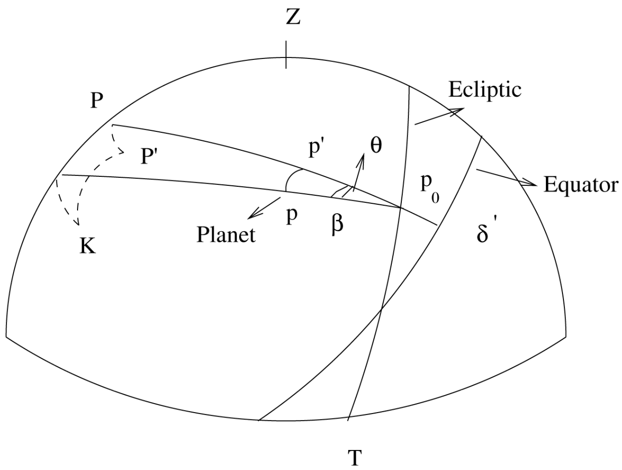

Figure 7 Component of the latitude (β), $\mathcal{P}_0\mathcal{P}'$ along the secondary to the equator.

udayalagna of the planet is the point on the ecliptic which rises on the eastern horizon along with the planet. This can be found using the concepts of '*āyanadṛkkarma*' and '*akṣadṛkkarma*'. From the declination and the *udayalagna* of the planet, its *chāya* can be obtained. The zenith distance of the planet can be found using a '*nalaka*'.

9 *Grahodayāstādhikāra* [(Heliacal) rising and setting of planets]

In this chapter, the heliacal rising and setting of planets is discussed. The minimum angular separations (in longitudes) for the visibility of the five planets are listed. From the given longitudes of a planet and the Sun, one can determine the number of days yet to elapse before heliacal rising, or the number of days elapsed after heliacal setting.

10 *Śṛṅgonnati* (Elevation of Lunar cusps)

Bhāskara points out that the crescent shape of the Moon is observed in the first and fourth quarters of the lunar month, when $|\lambda_{Moon} - \lambda_{Sun}| < 90°$, when the Moon is less than half. The line of cusps will be deflected from the horizontal, and it would have a vertical component. A formula for this deflection is given, which is not entirely satisfactory.

The 'brightness', or the 'phase', or the '*śukla*', s of the Moon is taken to be $6 \times \frac{\rho}{90}$ *aṅgulas*, where $\rho = E'\hat{M}S$ in Figure 8. Note that $\rho = 90°$ corresponds to the half-Moon. This is satisfactory.

The rule for sketching the crescent is given. Bhāskara points out that Brahmagupta's method for sketching and computing the elevation of lunar horns would not agree with his computations based on observations.

11 *Grahayuti* (Conjunctions of planets)

The mean angular dimensions of the five planets are stated. Their true dimensions are to be found using their true distances which are calculated using their *śīghrakarṇas*. If the longitudes of planets 1 and 2 are λ_1, λ_2 at some instant t_0, then their instant of conjunction is T, where

$$T - t_0 = \frac{\lambda_2(t_0) - \lambda_1(t_0)}{(\dot{\lambda_1} - \dot{\lambda_2})},$$

where $\dot{\lambda_1}$ and $\dot{\lambda_2}$ are the rates of motion of the two planets.

It is possible to take conjunction to mean the equality of polar longitudes. Then, the *āyanadṛkkarma* has to be applied to the longitude of a planet to convert it into its polar longitude.

Parallax corrections have to be applied to the longitudes and latitudes of planets before calculating the instant of conjunction and the north-south (that is, the direction perpendicular to the ecliptic) separation. Occultation will happen if this north-south separation is less than the sum of the semi-diameters of the discs of the planets.

12 *Bhagrahayuti* (Conjunctions of planets with stars)

The polar longitudes and latitudes of the 27 zodiacal stars, *Abhijit*, *Agastya* (Canopus) and *Lubdhaka* (Sirius) are stated. The measurement of these using an armillary sphere is also discussed. The heliacal rising times (the minimum angle of separation between the star and the Sun in time units for the star to be observable) of *Agastya* and *Lubdhaka* are stated to be 12 and 13 *nāḍis* respectively. *Āyanadṛkkarma* has to be applied to a planet before finding the instant of its conjunction with a star. This instant is found in the same mannner as the instant of conjunction of two planets discussed in the previous chapter. The visibility of a star vis-a-vis its location with respect to the Sun is discussed in detail.

13 *Pātādhyāya* (*Vyatīpāta*: Equality of declinations of the Sun and the Moon)

This chapter has only 21 verses but it has elaborate *vāsanā* for these. *Vyatīpāta* and *vaidhṛta* occur when the magnitudes of the declinations of the Sun and the Moon are equal, and one of them is increasing, while the other is decreasing. In this verse, Bhāskara says that there is confusion in the minds of even great ashonomers of the past regarding their occurance and this chapter is devoted to the clarification of the method for computing them.

Bhāskara describes the drawing of the celestial equator, ecliptic, Moon's orbit which is inclined to the ecliptic at an angle of 4.5°, its nodes, and the *golasandhis* of the Sun and the Moon, as in Figure 9.

The true declination of the Moon has to be found, including the contribution from its latitude. The instant of *vyatīpāta* is found using an iterative procedure, originally due to Brahmagupta. Bhāskara himself gives an example of this procedure in the *vāsanā*.

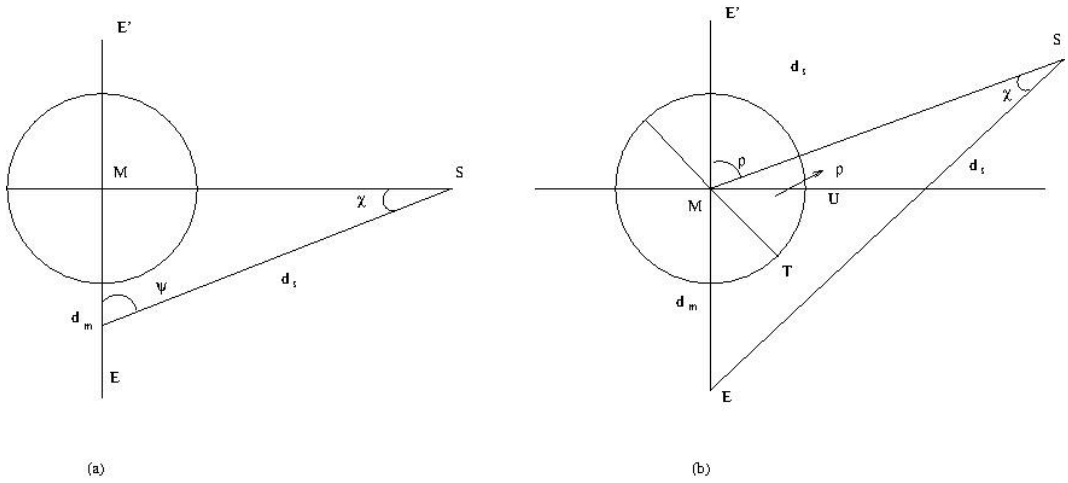

Figure 8 'Brightness' or 'Phase' of the Moon. (a) Half-Moon. (b) Brightness for an arbitrary elongation.

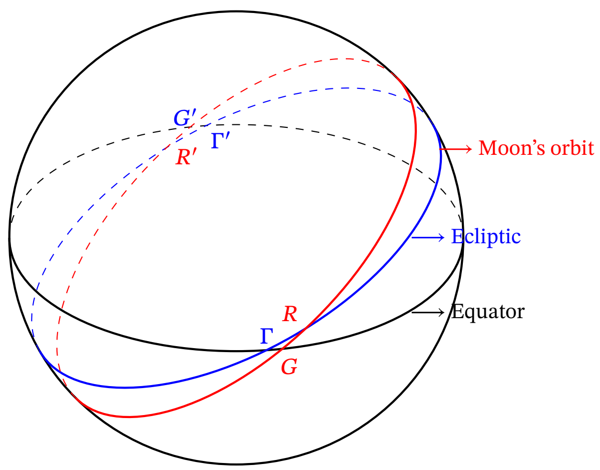

Figure 9 Celestial equator, ecliptic, Moon's orbit, Moon's nodes (R, R'), *Golasandhi*s of the Sun (Γ, Γ'), and those of the Moon (G, G').

Actually, the phenomenon of *vyatīpāta* stretches over the time interval during which the magnitude of declination of any point on the lunar disc is equal to that of some point on the solar disc. Clearly, at the beginning and end of the *vyatīpāta*, the difference between the declinations of the centres of the disks of the Sun and the Moon would be equal to the sum of the semi-diameters of the disks. Bhāskara gives the iterative procedure to find the half-durations of *vyatīpāta*, which is essentially the same as due to Brahmagupta and Lalla.

14 Concluding remarks

In the beginning of *Grahagaṇita*, in verse 4 of *Kālamānādhyāya* of *Madhyamādhikāra*, Bhāskara says:

> Ancient astronomers did write, of course, works abounding in intelligent expression; nonetheless, **this work is started by me to give better expression to (or improve) some of their special/important statements. These (improvements) are given by me here and there in their respective places.** So, I beseech the good-minded mathematicians to go through this entire work of mine also.

Bhāskara lives up to his promise. In this work, most of the standard calculations and algorithms in Indian astronomy of his times are included, mistakes in many of them are rectified, generalisations are made where necessary, and many new results are presented. All these are presented in the source verses of the text, and are explained in detail in his own commentary, *Vāsanābhāṣya*. In this project in two phases, we have translated all the verses, and the accompanying auto-commentary, and have also provided the explanatory notes which include nearly 140 diagrams based on the descriptions in the commentary. We hope that our work will enable any serious reader to understand Bhāskara's astronomy, and his methodology very well.

Acknowledgement

I am grateful to Dr. Sita Sundar Ram of the Kuppuswami Sastri Research Institute, Chennai and Dr. R. Venketeswara Pai of IISER, Pune for very fruitful collaborations during the course of this project. Financial support from the Indian National Science Academy, New Delhi for both the phases of the project is gratefully acknowledged.

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