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OSCILLATION OF FIRST ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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In this paper, a necessary and sufficient condition for the oscillation of all solutions of a first order nonlinear delay differential equation is established in which the coefficients are periodic functions with a common period and the delays are constants and multiples of this period.

Key Words : Nonlinear Delay Differential Equation; Solution; Oscillation

1. INTRODUCTION

The oscillatory behaviour of solutions of delay differential equations has been the subject of intensive investigation during the last twenty years. Among numerous papers dealing with this subject we refer in particular to [1] and [4-8]. We also refer to the recent books[2 & 3]. In the oscillation theory of linear delay differential equations one of the most important problems is to obtain a necessary and sufficient condition for the oscillation via the characteristic equation. Such a result for linear delay differential equation with constant coefficients was proved.[1, 4 & 5]. For autonomous nonlinear delay differential equation

$$x'(t) + \sum_{i=1}^{m} p_i \prod_{j=1}^{m_i} |x(t-\tau_{ij})|^{\alpha_{ij}} sgn\, |x(t)| = 0, \qquad \dots (E_0)$$

its "characteristic equation" is

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$$-\lambda+\sum_{i=1}^{m} p_i \exp\left(\lambda \sum_{j=1}^{m_i} \alpha_{ij}\tau_{ij}\right)=0 \qquad \dots (1)$$

where $p_i \in (0,\infty)$, $\tau_{ij} \in (0,\infty)$, $\alpha_{ij} \in [0,1]$ and $\sum_{j=1}^{m_i} \alpha_{ij}=1$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, m_i$. The author[8] obtained the necessary and sufficient condition for the oscillation. More precisely, all solutions of (E_0) are oscillatory if and only if its "characteristic equation" has no real roots, or equivalently for all $\lambda > 0$

$$-\lambda+\sum_{i=1}^{m} p_i \exp\left(\lambda \sum_{j=1}^{m_i} \alpha_{ij}\tau_{ij}\right)>0. \qquad \dots (2)$$

It is an important problem to extend the above criterion for the case of first order linear delay differential equations with variable coefficients. The purpose of this paper is to examine the special case where the coefficients are periodic functions with a common period and the delays are constants and multiples of this period.

Consider the nonlinear delay differential equation

$$x'(t)+\sum_{i=1}^{m} p_i(t)\prod_{j=1}^{m_i} |x(t-\tau_{ij})|^{\alpha_{ij}}\, sgn\, |x(t)|=0, \qquad \dots (E)$$

under the following assumptions:

(A_1) $p_i(t)$, $i = 1, 2, \dots, m$ are nonnegative continuous functions on $[0,\infty)$ which are identically zero and are periodic functions with a common period $\omega > 0$.

(A_2) τ_{ij}, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m_i$ and $\sum_{j=1}^{m_i} \alpha_{ij}=1$.

(A_2) τ_{ij}, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m_i$ are nonnegative constants. There exists nonnegative integers n_{ij} such that $\tau_{ij}=n_{ij}\omega$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m_i$.

(A_3) $\alpha_{ij} \geq 0$, i $= 1, 2, \dots, m$, $j = 1, 2, \dots, m_i$ and $\sum_{j=1}^{m_i} \alpha_{ij} = 1$.

Let $t_0 \geq 0$. By a solution on $[t_0,\infty)$ of (E) we mean a continuous function x defined on the interval $[t_0-\tau,\infty)$, where $\tau = \max\{\tau_{ij}, i = 1, 2, \dots, m, j = 1, 2, \dots, m_i\}$, which is differentiable on $[t_0,\infty)$ and satisfies (E) for all $t \geq t_0$. As usual, a solution of (E) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory.

Throughout this paper we will use the notations

$$\tau = \max\{\tau_{ij}, i = 1, 2, \dots, m, j = 1, 2, .., m_i\}$$

$$\sigma = \min\{\tau_{ij},\ i = 1, 2, ..., m,\ j = 1, 2, .., m_i\}$$

and
$$P_i = \frac{1}{\omega}\int_0^{\omega} p_i(s)ds,\ i = 1, 2, ..., m.$$

2. Main Results

In order to obtain the main theorem of this paper we need the following two lemmas. The first lemma is

Lemma 1 [3, Lemma 1.6.3] — Assume that $P \in C((0, \infty), [0, \infty))$, τ is positive constant and

$$\liminf_{t \to \infty} \int_{t-\tau}^{t} P(s)ds > 0.$$

Moreover, suppose that $u \in C([T-\tau, \infty), (-\infty, 0])$ satisfies the inequality

$$u(t) + P(t)\exp\left[-\int_{t-\tau}^{t} u(s)ds\right] \le 0,\ t \ge T.$$

Then
$$\liminf_{t \to \infty}\left[-\int_{t-\tau}^{t} u(s)ds\right] < \infty.$$

Lemma 2 — Assume that $(A_1) - (A_3)$ hold. Then the following statements are equivalent.

(a) Equation (E) has a nonoscillatory solution.

(b) There exists $T \ge \tau$ such that the integral equation

$$u(t) = \begin{cases} \sum_{i=1}^{m} p_i(t)\exp\left(\sum_{j=1}^{m_i} \alpha_{ij}\int_{t-\tau_{ij}}^{t} u(s)ds\right), & t \ge T, \\ \bar{u}(t), & T-\tau \le t < T, \end{cases} \qquad \text{... (3)}$$

has a nonnegative continuous solution $u(t)$ on $[T, \infty)$ for some initial function $\bar{u} \in C([T-\tau, T), [0, \infty))$.

(c) The sequence of functions as follows

$$u_0(t) = \begin{cases} \sum_{i=1}^{m} p_i(t), & t \ge T > \tau, \\ 0, & T-\tau \le t < T. \end{cases}$$

$$u_k(t) = \begin{cases} \sum_{i=1}^{m} p_i(t) \exp\left(\sum_{j=1}^{m_i} \alpha_{ij} \int_{t-\tau_{ij}}^{t} u_{k-1}(s)\, ds \right), & t \geq T > \tau, \\ & k = 1, 2, \ldots, \\ 0, & T - \tau \leq t < T, \end{cases} \quad \ldots (4)$$

is convergent on $[T - \tau, \infty)$.

PROOF : $(a) \Rightarrow (b)$ Suppose that (E) has a nonoscillatory solution $x(t)$ and $x(t) \neq 0$ on $[T - \tau, \infty), T - \tau \geq 0$. Define $u(t) = -[x'(t)/x(t)], t \geq T - \tau$. Then $x(t) = x(T) \exp\left(-\int_{T}^{t} u(s)ds \right)$. Thus from (E) we obtain

$$u(t) = \begin{cases} \sum_{i=1}^{m} p_i(t) \exp\left(\sum_{j=1}^{m_i} \alpha_{ij} \int_{t-\tau_{ij}}^{t} u_{k-1}(s)\, ds \right), & t \geq T, \\ \bar{u}(t) & T - \tau \leq t < T, \end{cases}$$

where $\bar{u}(t) = u(t), t - \tau \leq t < T$, which proves (b).

$(b) \Rightarrow (c)$ Clearly $u_0(t) \leq u(t), t \geq T - \tau$. From (4) we obtain $u_1(t) \leq u(t), t \geq T - \tau$. By a simple induction, it is easy to prove that for all $t \geq T - \tau$,

$$u_k(t) \leq u_{k+1}(t) \leq u(t), t \geq T - \tau, \; k = 1, 2, \ldots . \quad \ldots (5)$$

Thus the sequence $\{u_k(t)\}$ has a pointwise limiting function $\tilde{u}(t)$ with

$$\lim_{n \to \infty} u_n(t) \leq \tilde{u}(t) \leq u(t), \text{ for all } t \geq T - \tau.$$

$(c) \Rightarrow (a)$ Let $\tilde{u}(t) = \lim_{k \to \infty} u_k(t)$, and $u_k(t) \leq \tilde{u}(t), t \geq T - \tau$. As the sequence (4) is uniformly bounded on $[t - \tau_{ij}, t], t > T - \tau, \; i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, m_i$. By the Lebesque's Dominated Convergence Theorem we obtain that

$$\tilde{u}(t) = \begin{cases} \sum_{i=1}^{m} p_i(t) \exp\left(\sum_{j=1}^{m_i} \alpha_{ij} \int_{t-\tau_{ij}}^{t} u(s)ds \right), & t \geq T, \\ 0, & T - \tau \leq t < T, \end{cases}$$

which implies that (3) has a nonnegative solution $\tilde{u}(t)$. Let $x(t) = \exp\left(-\int_{T}^{t} \tilde{u}(s)\, ds \right)$.

It is easy to check that $x(t)$ is a positive solution of (E) on $[T, \infty)$. The proof of Lemma 2 is complete.

Remark : Lemma 2 is true if constants $\tau_{ij} \geq 0$ are replaced by $\tau_{ij}(t) \in C([0, \infty), [0, \infty))$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m_i$.

The following result is main theorem in this paper.

Theorem 1 — *Assume that (A_1) - (A_3) hold. Then the following statements are equivalent.*

(d) All solutions of (E) are oscillatory.

(e) For all $\lambda > 0$,

$$-\lambda + \sum_{i=1}^{m} P_i \exp\left(\lambda \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij}\right) > 0. \qquad \ldots (6)$$

PROOF : $(d) \Rightarrow (e)$ Suppose that (E) has a nonoscillatory solution $x(t)$. By Lemma 2 statement (b) holds. Thus from (4) $u_0(t) = \sum_{i=1}^{m} p_i(t) \leq u(t), t \geq T$. For $t \geq T + \tau$, we have

$$\int_{t-\tau_{ij}}^{t} u_0(s)\,ds = \sum_{n=1}^{m} \int_{t-\tau_{ij}}^{t} p_n(s)ds = \sum_{n=1}^{m} \tau_{ij} \frac{1}{\tau_{ij}} \int_{0}^{\tau_{ij}} p_n(s)\,ds$$

$$= \sum_{n=1}^{m} \tau_{ij} P_n = \tau_{ij} \sum_{n=1}^{m} P_n = \tau_{ij} \lambda_0 \qquad \ldots (7)$$

where $\lambda_0 = \sum_{i=1}^{m} P_i$. Hence from (4) and (7) we obtain

$$u_1(t) = \sum_{i=1}^{m} p_i(t) \exp\left(\lambda_0 \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij}\right) \leq u(t), \ t \geq T + \tau.$$

In general, by induction, we obtain that

$$u_k(t) = \sum_{i=1}^{m} p_i(t) \exp\left(\lambda_{k-1} \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij}\right) \leq u(t), \ t \geq T + k\tau, \ k = 1, 2, \ldots, \qquad \ldots (8)$$

where

$$\lambda_k = \sum_{i=1}^{m} P_i \exp\left(\lambda_{k-1} \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij}\right), k = 1, 2, \ldots. \qquad \ldots (9)$$

From (8), we have

$$\int_{t-\tau}^{t} u_k(s)ds$$

$$= \sum_{i=1}^{m} \tau \left(\frac{1}{\tau} \int_{t-\tau}^{t} p_i(s)\, ds \right) \exp\left(\lambda_{k-1} \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right)$$

$$= \tau \sum_{i=1}^{m} P_i \exp\left(\lambda_{k-1} \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right)$$

$$= \tau \lambda_k, \; k = 1, 2. \, . \qquad \text{... (10)}$$

It is easy to prove that

$$0 < \lambda_k < \lambda_{k+1}, \; k = 1, 2, \ldots .$$

We now claim that $\lim_{k \to \infty} \lambda_k < \infty$. Otherwise, as $k \to \infty, t \to \infty$ thus from (8) and (10), we have

$$\lim_{t \to \infty} \int_{t-\tau}^{t} u(s)ds > \lim_{t \to \infty} \int_{t-\tau}^{t} u_k(s)\, ds = \lim_{k \to \infty} \tau \lambda_k = \infty. \qquad \text{... (11)}$$

On the other hand, from Lemma 2, the integral equation (3) has a nonnegative continuous solution $u(t)$ on $[T, \infty)$ for $T \geq \tau \geq 0$. Thus by (3) we have

$$0 \geq -u(t) + \sum_{i=1}^{m} p_i(t) \exp\left(\sum_{j=1}^{m_i} \alpha_{ij} \int_{t-\sigma}^{t} u(s)\, ds \right)$$

$$= -u(t) + \sum_{i=1}^{m} p_i(t) \exp\left(\int_{t-\sigma}^{t} u(s)ds \right)$$

By Lemma 1

$$\liminf_{t \to \infty} \left[\int_{t-\sigma}^{t} u(s)ds \right] < \infty,$$

which contradicts (11). So $\lim_{k \to \infty} \lambda_k = \lambda < \infty$. Therefore, by (9)

$$\lambda = \sum_{i=1}^{m} P_i \exp\left(\lambda \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right)$$

which contradicts (e)

$(e) \Rightarrow (d)$. Suppose that (6) does not hold. We may then choose $\bar{\lambda} > 0$ such that

$$\bar{\lambda} = \sum_{i=1}^{m} P_i \exp\left(\bar{\lambda} \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right)$$

$$\text{Set } f(t) = \sum_{i=1}^{m} p_i(t) \exp\left(\bar{\lambda} \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right), t \geq 0.$$

Then
$$\int_{t-\tau_{ij}}^{t} f(s) ds$$

$$= \int_{t-\tau_{ij}}^{t} \sum_{i=1}^{m} p_i(s) \exp\left(\bar{\lambda} \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right) ds$$

$$= \tau_{ij} \sum_{i=1}^{m} P_i \exp\left(\bar{\lambda} \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right)$$

$$= \tau_{ij} \bar{\lambda} \; t \geq \tau > 0. \qquad \ldots (12)$$

Let
$$x(t) = \exp\left[-\int_{0}^{t} f(s) ds \right]$$

Then, by using (12) we get for $t \geq \tau$

$$x'(t) + \sum_{i=1}^{m} p_i(t) \prod_{j=1}^{m_i} [x(t-\tau_{ij})]^{\alpha_{ij}}$$

$$= -f(t) \exp\left[-\int_{0}^{t} f(s)\, ds \right] + \sum_{i=1}^{m} p_i(t) \prod_{j=1}^{m_i} \exp\left[-\alpha_{ij} \int_{0}^{t-\tau_{ij}} f(s)\, ds \right]$$

$$= \left\{ -f(t) + \sum_{i=1}^{m} p_i(t) \prod_{j=1}^{m_i} \exp\left(\alpha_{ij} \int_{t-\tau_{ij}}^{t} f(s)\, ds \right) \right\} \exp\left(-\int_{0}^{t} f(s) ds \right)$$

$$= \left\{ -f(t) + \sum_{i=1}^{m} p_i(t) \exp\left(\lambda \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right) \right\} \exp\left(- \int_0^t f(s)\, ds \right) = 0.$$

which implies that $x(t)$ is a solution on $[\tau, \infty)$ of the differential equation (E). Obviously, x is positive on $[0, \infty)$. The proof of Theorem 1 is complete.

Corollary 1 — Assume that (A_1) – (A_3) hold. Then each of the following conditions is sufficient for the oscillation of all solutions of (E),

$$\sum_{i=1}^{m} P_i \left[\sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right] > \frac{1}{e}; \qquad \text{...(13)}$$

and

$$\left(\prod_{i=1}^{m} P_i \right)^{\frac{1}{m}} \left[\sum_{i=1}^{m} \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right] > \frac{1}{e}. \qquad \text{... (14)}$$

PROOF : For any $c > 0$, one has $\min_{\lambda > 0} (e^{\lambda c}/\lambda) = ec$. Hence by this fact and (13), we have for every $\lambda > 0$ and any $T \geq \tau$

$$-\lambda + \sum_{i=1}^{m} P_i \exp\left(\lambda \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right)$$

$$\geq \lambda \left[-1 + \frac{1}{\lambda} \sum_{i=1}^{m} P_i \exp\left(\lambda \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right) \right]$$

$$\geq \lambda \left(-1 + e \sum_{i=1}^{m} P_i \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right) > 0.$$

By Theorem 1, all solutions of (E) are oscillatory.

Next by using the arithmetic-geometric mean inequality and (14) we find for every $\lambda > 0$ and any $T \geq \tau$,

$$-\lambda + \sum_{i=1}^{m} P_i \exp\left(\lambda \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right)$$

$$\geq \lambda \left[-1 + \frac{1}{\lambda} \sum_{i=1}^{m} P_i \exp\left(\lambda \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij} \right) \right]$$

$$\geq \lambda\left[-1+m\left(\prod_{i=1}^{m} P_i\right)^{\frac{1}{m}} \exp\left(\frac{\lambda}{m}\sum_{i=1}^{m}\sum_{j=1}^{m_i}\alpha_{ij}\tau_{ij}\right)/\lambda\right]$$

$$\geq \lambda\left[-1+\left(\prod_{i=1}^{m} P_i\right)^{\frac{1}{m}} e\sum_{i=1}^{m}\sum_{j=1}^{m_i}\alpha_{ij}\tau_{ij}\right]>0.$$

By Theorem 1, all solutions of (E) are oscillatory. The proof of Corollary 1 is complete. In particular, consider the nonlinear delay differential equation

$$x'(t)+p(t)\prod_{i=1}^{m}|x(t-n_i\,\omega)|^{\alpha_i}\, sgn\,|x(t)|=0, \qquad \ldots (E_1)$$

where $p(t)$ is nonnegative continuous periodic function on $[0,\infty)$ with period $\omega>0, n_i$, $i = 1, 2, \ldots, m$, are nonnegative integers and α_i, $i = 1, 2, \ldots, m$, are nonnegative constants with $\sum_{i=1}^{m}\alpha_i=1$.

Corollary 2 — All solutions of (E_1) are oscillatory if and only if

$$\omega P\sum_{i=1}^{m} n_i\,\alpha_i>\frac{1}{e}, \qquad \ldots (15)$$

where $$P=\frac{1}{\omega}\int_0^{\omega} p(s)ds.$$

PROOF : From (15), for all $\lambda>0$,

$$-\lambda+P\exp\left(\lambda\sum_{i=1}^{m}\omega\, n_i\,\alpha_i\right)\geq\lambda\left(-1+\exp\left(\lambda\,\omega\sum_{i=1}^{m} n_i\,\alpha_i\right)/\lambda\right)$$

$$\geq\lambda\left(-1+eP\,\omega\sum_{i=1}^{m} n_i\,\alpha_i\right)>0.$$

Hence, by Theorem 1, all solutions of (E_1) are oscillatory if and only if (15) hold. The proof of Corollary 2 is complete.

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