Indian J. pure appl. Math., **32**(9) : 1269-1274, September 2001

OSCILLATION FOR CERTAIN DELAY HYPERBOLIC EQUATIONS SATISFYING THE ROBIN BOUNDARY CONDITION

LIHU DENG* AND WEIGAO GE**

*Department of Applied Mathematics, Dongguan Institute of Technology, Dongguan, 523106, P. R. China

**Department of Applied Mathematics, Beijing Institute of Technology, Beijing, 100081, P. R. China

(*Received* 4 *April* 2000; *accepted* 17 *November* 2000)

In this paper we give some new oscillatory criteria for a class of nonlinear functional hyperbolic equations with the Robin boundary condition, using a new method.

Key Words : Nonlinear Functional Hyperbolic Equations; Robin Boundary Condition; Oscillation

1. INTRODUCTION

The literature[1] has studied the oscillation of the following functional hyperbolic equation with several boundary conditions

$$\frac{\partial^2}{\partial t^2}[u + \lambda(t)\, u(x, t-\tau)] = a(t)\, \Delta u - c(x, t, u) - \int_a^b q(x, t, \zeta)\, u[x, g(t, \zeta)\, d\,\sigma(\zeta) \qquad \ldots (E)$$

and have got some oscillatory criteria. About Robin boundary value problem (i.e, *(E), (B3))*, however, literature[1] have got only a result (see Theorem 5)[1] and results of Theorem 5 are not meticulous. In this paper, we will study continuously the problem (*E*), (*B*3), using a new method, we use the following Robin eigenvalue problem.

$$\Delta\,\Phi + \lambda\,\Phi = 0 \quad x \in \Omega$$

$$\frac{\partial\,\Phi}{\partial n} + \beta(x)\,\Phi = 0 \quad x \in \partial\,\Phi$$

to study the oscillation of the eq. (*E*) satisfying the following Robin boundary condition

$$\frac{\partial u}{\partial n} + \beta(x)\, u(x,t) = 0 \quad x \in \partial\Omega \qquad \text{... (B)}$$

where Δu is the Laplacian in R^n, τ is a positive constant, $(x, t) \in \Omega \times R_n = G, R_+ = [0, +\infty), u = u(x, t)$, Ω is a bounded domain in R^n with a piecewise continuous smooth boundary $\partial\Omega$, n denotes the unit exterior vector normal to $\partial\Omega$.

We assume throughout this paper that the following conditions (H) hold :

(H1) $a(t), \lambda(t) \in C(R_+, R_+); q(x, t, \zeta) \in C(\overline{\Omega} \times R_+ \times [a, b], R_+)$

$$Q(t, \zeta) = \min_{x \in \Omega} \{q, (x, t, \zeta)\}$$

(H2) $g(t, \zeta) \in C(R_+ \times [a, b], R); g(t, \zeta) \leq t, \zeta \in [a, b]; g(t, \zeta)$

are nondecreasing with respect to t and ζ, respectively; and $\lim_{t \to +\infty} \min \{g(t, \zeta)\} = +\infty$.

(H3) $c(x, t, u) \in C(\overline{\Omega} \times R_+ \times R, R); c(x\, t, \zeta) \geq p(t) f(\zeta)$,

$p(t) \in C(R_+, R_+), f(\zeta) \in C([a, b]), R); f(\zeta)$

is a positive and convex function in $(0, +\infty)$ and $c(x, t, -\zeta) = -c(x, t, \zeta)$.

(H4) $\sigma(\zeta) \in C([a, b], R)$,

is nondecreasing, integral if eq. (E) is Stieltjes integral.

(H5) $\beta(x) \in C(\partial\Omega, (0, +\infty))$

Definition 1 — A solution $u(x, t)$ of the problem (E), (B) is called oscillatory in the domain G if for each positive number T there exists a point $(x_0, t_0) \in \Omega \times [T, +\infty)$ such that the condition $u(x_0, t_0) = 0$ holds.

2. Main Results

Lemma 1 — Assume (H5) holds and λ_0 is the smallest eigenvalue of the Robin eigenvalue problem

$$\Delta\Phi + \lambda\Phi = 0 \quad x \in \Omega \qquad \text{... (1)}$$

$$\frac{\partial \Phi}{\partial n} + \beta(x)\, \Phi = 0 \quad x \in \partial\Omega \qquad \text{.... (2)}$$

and $\Phi(x)$ is the corresponding eigenfunction, then $\lambda_0 > 0$ and $\Phi(x) > 0$. $(x \in \Omega)$ (see Theorem 3.3.22 of [2]).

Lemma 2 — Assume condition (H3) holds, then there exists positive $t_1 > 0$ such that

$$\int_{\Omega} c(x, t, u)\, \Phi(x)\, dx \geq p(t) f\left[\int_{\Omega} u\, \Phi(x)\, dx \left(\int_{\Omega} \Phi(x) dx\right)^{-1}\right] \int_{\Omega} \Phi(x) dx, \quad t \geq t_1$$

(see formula (2.7) of [1]).

***Theorem* 1** — *Assume that the condition (H) holds, and the following differential inequalities*

$$\frac{d^2}{dt^2}[V(t) + \lambda(t)\, V(t-\tau)] + \lambda_0\, V(t)$$

$$+ P(t) f(V(t)) + \int_a^b Q(t, \zeta)\, V[g(t, \zeta)]\, d\sigma(\zeta) \leq 0 \qquad \text{... (3)}$$

have no eventually positive solution, then the every solution of the Robin problem (E), (B) is oscillatory in G.

PROOF : Assume that there exists a nonoscillatory solution $u(x, t)$ of the problem (E), (B).

Let $u(x, t)$ is a eventually positive solution of the Robin boundary value problem (E), (B) in $\Omega \times [t_0, +\infty)$, for $t_0 \geq 0$, by the condition (H2),

$$\lim_{t \to +\infty} \min \{g(t, \zeta)\} = +\infty$$

there exists a $t_1 \geq t_0$ such that

$$g(t, \zeta) \geq t_0 \quad (t, \zeta) \in [t_1, +\infty) \times [a, b]$$

and $\qquad t - \tau \geq t_0 \quad t \geq t_1$

then $\qquad u(x, g(t, \zeta)) > 0 \quad (x, t, \zeta) \in \Omega \times [t_1 + \infty) \times [a, b]$

$$u(x, t-\tau) > 0 \quad (x, t) \in \Omega \times [t_1, \infty)$$

Multiplying both side of eq. (E) by eigenfunction $\Phi(x)$ of Robin eigenvalue problem (1) and integrating with respect to x over the domain Ω, we have

$$\frac{d^2}{dt^2}\left[\int_{\Omega} u\, \Phi(x)\, dx + \lambda(t) \int_{\Omega} u(x, t-\tau)\, \Phi(x)\, dx\right]$$

$$= a(t) \int_{\Omega} \Delta u\, \Phi(x) dx - \int_{\Omega} c(x, t, u)\, \Phi(x) dx$$

$$- \int_{\Omega} \int_a^b q(x, t, \zeta)\, u[x, g(t, \zeta)]\, \Phi(x)\, d\sigma(\zeta)\, dx \quad t \geq t_1 \qquad \text{... (4)}$$

using the Green formula, boundary condition (B) and Robin eigenvalue problem (1), we have

$$\int_{\Omega} u(x,t)\,\Phi(x)dx = \int_{\partial\Omega}\left(\Phi(x)\frac{\partial u}{\partial n} - u\frac{\partial \Phi(x)}{\partial n}\right)ds + \int_{\Omega} u\,\Delta\,\Phi(x)dx$$

$$= \int_{\partial\Omega} (-\Phi(x)\beta(x)u + u\beta(x)\Phi(x))ds - \lambda_0 \int_{\Omega} u\Phi(x)ds$$

$$= -\lambda_0 \int_{\Omega} u\Phi(x)dx \quad t \geq t_1 \qquad \dots (5)$$

Therefore, from (2)-(4), and Lemma 2, we have

$$\frac{d^2}{dt^2}\left[\int_{\Omega} u\Phi(x)dx + \lambda(t)\int_{\Omega} u(x,t-\tau)\,\Phi(x)dx\right]$$

$$\leq -\lambda_0 \int_{\Omega} u\,\Phi(x)\,dx - P(t)f\left[\int_{\Omega} u\,\Phi(x)dx\left(\int_{\Omega}\Phi(x)dx\right)^{-1}\right]\int_{\Omega}\Phi(x)dx$$

$$-\int_a^b Q(x,\zeta)\int_{\Omega} u[x, g(t,\zeta)]\,\Phi(x)dx\,d\sigma(\zeta) \quad t \geq t_1 \qquad \dots (6)$$

Let $V(t) = \left(\int_{\Omega}\Phi(x)dx\right)^{-1}\int_{\Omega} u\,\Phi(x)dx$, we have

$$\frac{d^2}{dt^2}[V(t) + \lambda(t)\,V(t-\tau)] + \lambda_0\,a(t)V(t)$$

$$+ P(t)f(V(t)) + \int_a^b Q(t,\zeta)\,V[g(t,\zeta)]\,d\,\sigma(\zeta) \leq 0.$$

Therefore, $V(t)$ (> 0) is an eventually positive solution of differential inequality (3), which contradicts to the condition of Theorem 1.

If $u(x,t) < 0$, let $v(x,t) = -u(x,t)$, then using the above-mentioned method, we can also get that

$$\bar{V} = \left(\int_{\Omega}\Phi(x)dx\right)^{-1}\int_{\Omega} v(x,t)\,\Phi(x)dx$$

is also an eventually positive solution of the inequality (4), which contradicts the condition of Theorem 1 as well. This completes the proof of Theorem 1.

Now we concern a kind of particular case.

We have the following theorem :

When $\lambda(t) = 0, t \in R_+$ the problem (E) become the following form :

$$\frac{\partial^2 u}{\partial t^2} = a(t)\, \Delta u - c(x, t, u) - \int_a^b q(x, t, \zeta)\, u\, [x, g(t, \zeta)]\, d\, \sigma(\zeta) \qquad \ldots (E1)$$

Lemma 3 — Assume $y(t) \in C^2([t_0, +\infty), R)$ and

$$y(t) > 0,\ y'(t) > 0\ y''(t) < 0\ t \geq t_0,$$

then there exists $t \geq t_0$ for every number $\theta \in (0, 1)$ such that

$$y(t) \geq \theta\, t\, y'(t)\ \ t \geq t_1 \text{ (see refs [3])}.$$

Lemma 4 — Assume $Q(t), Q_i(t) \in C([t_0, +\infty), R_+))$ $i = 1, 2, \ldots m$

$g_i(t) \in C([t_0, +\infty), R)$, $g_i(t)$ is nondecreasing and $g_i(t) \leq t$,

$\lim_{t \to +\infty} g_i(t) = +\infty$, $i = 1, 2, \ldots m$ if $\exists\, i \in \{1, 2, \ldots m\}$, such that

$$\liminf_{t \to +\infty} \int_{g_i(t)}^{t} \left\{ Q_i(s) \exp \int_{g_i(s)}^{s} Q(r)dr \right\} ds > \frac{1}{e} \qquad \ldots (10)$$

then the following differential inequality

$$y'(t) + Q(t)y(t) + \sum_{i=1}^{m} Q(t)\, y(g_i(t)) \leq 0\ \ t \geq t_0 \qquad \ldots (11)$$

have no eventually positive solution. (see refs [4]).

***Theorem* 2** — *Assume that (H) holds,* $\lambda(t) = 0,\ t \in R_+$, *and* $g(t, \zeta) \equiv g_0(t)$ $\zeta \in [a, b]$, *and* $f(\zeta)/\zeta \geq c_0$, $\zeta \in R$. *If the following differential inequality*

$$V''(t) + (\lambda_0\, a(t) + c_0\, tP(t))\, V(t) + V(g_0(t)) \int_a^b Q(t, \zeta)\, d\, \sigma(\zeta) \leq 0 \qquad \ldots (12)$$

has no eventually positive solution, then every solution of the Robin boundary value problem (E1), (B) is oscillatory in G.

The proof of Theorem 4 is also similar to that of Theorem 1, and hence is omitted

***Theorem* 3** — *Assume that (H) holds,* $\lambda(t) = 0, t \in R_+$, *and* $g(t, \zeta) \equiv g_0(t)$ $\zeta \in [a, b]$. *If the following condition hilds*

$$\liminf_{t \to +\infty} \int_{g_0(t)}^{t} \left(\int_a^b \theta g_0(s) Q(s, \zeta) d\sigma(\zeta) \exp \int_{g_0(s)}^{s} \theta [\lambda_0 a(r) + c_0 rP(r)] dr \right) ds > \frac{1}{e} \quad \dots (13)$$

then every solution of the Robin boundary value problem $(E2)$, (B) is oscillatory in G.

PROOF : We need only to proof that the differential inequality (12) has no eventually positive solution. Assume now that the differential inequality (12) has a eventually positive solution, i.e. there exists a $T > 0$ such that $V(t) > 0$ $(t > T)$, from (12), we have

$V''(t) \le 0, t > T$, we can get

$V'(t) > 0, t > T$.

From Lemma 4, $\left(\text{let } \theta = \frac{1}{2}\right)$, we have

$V(t) > \theta t V'(t)$ $t > T$

$V(g_0(t)) > \theta t g_0(t) V'(g_0(t))$ $t > T$.

Let $W(t) = V'(t) > 0$, from (12), we have

$$W'(t) + \lambda_0 \theta t W(t) + \theta g_0(t) W(g_0(t)) \int_a^b Q(t, \zeta) d\sigma(\zeta) \le 0 \quad \dots (14)$$

From condition (13), we know that the inequality (14) has no eventual positive solution, which contradicts $W(t) > 0$, $(t > T)$. This completes the Proof of Theorem 5.

REFERENCES

1. P. G. Wang, *Indian J. pure appl. Math.*, **30**(6) (1999) 557-65.
2. Q. X. Ye, Z. Y. Li, *Beijing science press*, (1990) 134.
3. J. J. Wei, *Ann. diff. eqs.* **4**(4), (1984) 473-78.
4. G. S. Ladde, V. Lashmikanthm, B. G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*. New York and Basel, Marcel Dekker, Inc, (1987).