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SUBCLASSES OF *K*-UNIFORMLY CONVEX AND STARLIKE FUNCTIONS DEFINED BY GENERALIZED DERIVATIVE, I

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Recently, Kanaas and Wisniowska introduced the class of k-uniformly convex, and the related class of k-starlike functions $(0 \le k < \infty)$, denoted by $k-\mathcal{UCV}$ and k-$\mathcal{ST}$, respectively. Including the concept of generalized derivative, due to Sālāgean, we define and investigate the class $\mathcal{T}(k, n)$.

Key Words : Convex Functions; Uniformly Convex Functions; *k*-Uniformly Convex Functions; Jacobian Elliptic Functions

1. INTRODUCTON

Denote by $\mathcal{H}$ the collection of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad \text{... (1.1)}$$

analytic in the open unit disk $\mathcal{U}$ and by $\mathcal{S}$ its subclass consisting of all *univalent* functions. Following Goodman's notation[2], let $\mathcal{CV}$ be the class of *convex* univalent functions and $\mathcal{UCV}$ be the class of *uniformly convex* univalent functions in $\mathcal{U}$. Further, denote by k-$\mathcal{UCV}$, where $0 \le k < \infty$, the class of *k-uniformly convex* univalent functions in $\mathcal{U}$, introduced and investigated by Kanas and Visniowska in[4 & 5]. Recall here the definition.

Definition 1.1[4] — Let $k \in [0, \infty)$. A function $f \in \mathcal{S}$ is said to be k-uniformly convex in $\mathcal{U}$, if the image of every circular are γ contained in $\mathcal{U}$, with centre ζ, where $|\zeta| \le k$, is convex.

An analytical, and more applicable one-variable characterization, of the class k-$\mathcal{UCV}$, and the related class k-$\mathcal{ST}$, due to Kanas *et al.*[4 & 6] is stated below.

Theorem 1.1[4] — *Let* $f \in \mathcal{S}$, *and* $k \in [0, \infty)$. *Then* $f \in k$-$\mathcal{UCV}$ *iff*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|, \; z \in U. \qquad \text{... (1.2)}$$

Theorem 1.2[6] — *Let* $f \in \mathcal{S}$ *and* $k \in [0, \infty)$. *Then* $f \in k\text{-}\mathcal{ST}$ *iff*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathcal{U}. \quad \dots (1.3)$$

It is clearly seen that $f \in k\text{-}\mathcal{UCV}$ if and only if $zf' \in k\text{-}\mathcal{ST}$ and observed that the family of k-uniformly convex functions (and $k\text{-}\mathcal{ST}$, respectively) is characterized by the property that the expression $1 + zf''(z)/f'(z), z \in \mathcal{U}$ (and $zf'(z)/f(z)$, respectively) lies in an ellipse, a parabola or a hyperbola, depending on the value of the parameter k, $(0 \le k < \infty)$. Besides, this family generalizes the idea of convexity in the sense that when $k = 0$ the class $k\text{-}\mathcal{UCV}$ reduces to the class $\mathcal{CV}$, and the case $k = 1$ corresponds to the class $\mathcal{UCV}$, introduced by Goodman[2], and studied extensively by Ronning[9], and independently by Ma and Minda[7 & 8].

We mention also a sufficient condition for functions to be in $k\text{-}\mathcal{ST}$.

Theorem 1.6 — *Let* $f \in \mathcal{S}$ *be the form (1.1). If for some* k, $k \in [0, \infty)$ *the inequality*

$$\sum_{n=2}^{\infty} [n + k(n-1)] \, |a_n| < 1 \quad \dots (1.4)$$

holds, then $f \in k\text{-}\mathcal{ST}$.

Recall now the Sālāgean derivative $D^n f(z)$ $(n \in N \cup \{0\})$, of f from the class $\mathcal{H}$ defined recursively[10]

$$D^0 f(z) = f(z),$$

$$D^1 (f(z) = Df(z) = zf'(z), \quad \dots (1.5)$$

$$D^n f(z) = D(D^{n-1} f(z)) \; z \in \mathcal{U}.$$

Corresponding to this differential operator we also have an integral operator I^n defined as follows

$$I^0 f(z) = f(z), \; I^1 f(z) = \int_0^z \frac{f(t)}{t} dt,$$

$$I^n f(z) = I(I^{n-1} f(z)). \quad \dots (1.6)$$

It is clear that we have

$$I^n (D^n f(z)) = f(z) = D^n (I^n f(z)). \quad \dots (1.7)$$

Directly from the definition of $D^n f$ for the function of the form (1.1) we have the following power series of $D^n f(z)$

$$D^n f(z) = z + \sum_{m=2}^{\infty} m^n a_m z^m, z \in \mathcal{U}. \qquad \dots (1.8)$$

Using the notation (1.5) we may rewrite the condition (1.2) in the from

$$Re \left(\frac{D^2 f(z)}{D^1 f(z)} \right) > k \left| \frac{D^2 f(z)}{D^1 f(z)} - 1 \right|, z \in \mathcal{U}.$$

and the condition (1.3) as

$$Re \left(\frac{D^1 f(z)}{D^2 f(z)} \right) > k \left| \frac{D^1 f(z)}{D^0 f(z)} - 1 \right|, z \in \mathcal{U}.$$

We now want to generalize the classes k-$\mathcal{UCV}$ by using the Sălăgean derivative in the following way.

Definition 1.2 — Let $f \in \mathcal{S}$, $k \in [0, \infty)$ and $n \in N \cup \{0\}$. We say that a function f is of the class $\mathcal{T}(k, n)$ if

$$Re \left(\frac{D^{n+1} f(z)}{D^n f(z)} \right) > k \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right|, z \in \mathcal{U}. \qquad \dots (1.9)$$

We remark that this definition means that $\mathcal{T}(k, 0) = k$-$\mathcal{ST}$ and $\mathcal{T}(k, 1) = k$-$\mathcal{UCV}$.

Directly from the definition of $\mathcal{T}(k, n)$ we deduce

Lemma 1.1 — Let $k \in [0, \infty)$. Then

$$f \in \mathcal{T}(k, n) \Leftrightarrow D^n f \in k\text{-}\mathcal{ST} \Leftrightarrow D^{n-1} f \in k\text{-}\mathcal{UCV}.$$

In the sequel let $\mathcal{P}$ denote the well-known class of Carathèodory functions. Following the notation used in[4, 5] we denote by $\mathcal{P}(p_k)$ $(0 \le k < \infty)$, the family of functions p, such that $p \in \mathcal{P}$, and $p \prec p_k$ in $\mathcal{U}$, where the function p_k maps the unit disk conformally onto the region Ω_k such that $1 \in \Omega_k$ and

$$\partial\Omega_k = \{u + iv : u^2 = k^2 (u-1)^2 + k^2 v^2\}.$$

The domain Ω_k is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic for $k = 1$. and the right half plane when $k = 0$, (for complete information we refer to[4].

Hence, by putting $p(z) = D^{n+1} f(z)/D^n f(z)$ we obtain the relation of equivalence between the class $\mathcal{T}(k, n)$ and $\mathcal{P}(p_k)$, namely

$$f \in \mathcal{T}(k, n) \Leftrightarrow p(z) = \frac{D^{n+1} f(z)}{D^n f(z)} \in \mathcal{P}(p_k), z \in \mathcal{U}. \qquad \dots (1.10)$$

Functions which play the role of extremal functions of the classes $\mathcal{P}(p_k)$ were presented in[4]. Evidently $p_0(z) = (1+z)/(1-z)$, and (compare[6 or 8])

$$p_1(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 = 1 + \frac{8}{\pi^2}z + \frac{16}{3\pi^2}z^2 + \frac{184}{45\pi^2}z^3 + \dots, \qquad \dots (1.11)$$

and when $0 < k < 1$ (see[3, 4, 5]),

$$p_k(z) = \frac{1}{1-k^2}\cos\left\{Ai\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2}{1-k^2} \qquad \dots (1.12)$$

$$= 1 + \frac{1}{1-k^2}\sum_{n=1}^{\infty}\left[\sum_{l=1}^{2n} 2^l \binom{A}{l}\binom{2n-1}{2n-l}\right]z^n,$$

where $A = \frac{2}{\pi}\arccos k$.

Finally when $k > 1$, the function p_k has the form (cf. [4], [5])

$$p_k(z) = \frac{1}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{u(z)}{\sqrt{k}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2t^2}}\right) + \frac{k^2}{k^2-1} \qquad \dots (1.13)$$

$$= 1 + \frac{\pi^2}{4\sqrt{\kappa}(k^2-1)K^2(\kappa)(1+\kappa)} \times \left\{z + \frac{4K^2(\kappa)(\kappa^2+6\kappa+1)-\pi^2}{24\sqrt{\kappa}K^2(\kappa)(1+\kappa)}z^2 + \dots\right\},$$

with $$u(z) = \frac{z-\sqrt{\kappa}}{1-\sqrt{\kappa}z}, \ 0 < \kappa < 1, \ z \in \mathcal{U},$$

and κ is chosen such that

$$k = \cosh\frac{\pi K'(\kappa)}{4K(\kappa)}.$$

Here $K(\kappa)$ denotes Legendre's complete elliptic integral of the first kind, and $K'(\kappa)$ is the complementary integral of $K(\kappa)$.

2. THE CASE $\mathcal{T}(k, n)$

Let $n \in N \cup \{0\}$. Using (1.10) and properties of the domain Ω_k we obtain that if $f \in \mathcal{T}(k, n)$ with $0 \le k < \infty$, then

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) > \frac{k}{k+1}, \ z \in \mathcal{U},$$

and $$\left| Arg\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)\right| < \begin{cases} \arctan\frac{1}{k}, & \text{for } 0<k<\infty. \\ \frac{\pi}{2}, & \text{for } k=0. \end{cases}$$

By the fundamental relation between p_k and f_k, where f_k denotes the extremal functions for the class $\mathcal{T}(k, n)$, namely $p_k(z) = D^{n+1} f_k(z)/D^n f_k(z)$, and in view of (1.8) we have, that if $f_k(z) = z + A_2 z^2 + A_3 z^3 + \ldots$, then

$$m^n (m-1) A_m = \sum_{\mu=1}^{m-1} \mu^n A_\mu P_{m-\mu}, A_1 = 1. \qquad \ldots (2.1)$$

In particlar, by straightforward calculations we obtain

$$A_2 = \frac{P_1}{2^n},\ A_3 = \frac{P_2 + P_1^2}{2 \cdot 3^n},\ A_4 = \frac{P_3 + (3/2) P_1 P_2 + (1/2) P_1^3}{3 \cdot 4^n}. \qquad \ldots (2.2)$$

Then

$$f_0(z) = z + \frac{1}{2^{n-1}} z^2 + \frac{1}{3^{n-1}} z^3 + \frac{1}{4^{n-1}} z^4 + \ldots \qquad .. (2.3)$$

and $$f_1(z) = z + \frac{8}{2^n \pi^2} z^2 + \frac{8\pi^2 + 96}{3^{n+1}\pi^4} z^3 + \frac{1}{3 \cdot 4^n}\left[\frac{184}{45\pi^2} + \frac{64}{\pi^4} + \frac{256}{\pi^6}\right] z^4 + \ldots . \qquad \ldots (2.4)$$

In the case when $k \in [0, 1)$ we have $(A = (2/\pi) \arccos k)$

$$f_k(z) = z + \frac{2A^2}{2^n (1-k^2)} z^2 + \frac{1}{2 \cdot 3^n}\left[\frac{2A^2 (A^2+2)}{9(1-k^2)} + \frac{4A^4}{3(1-k^2)^2}\right] z^3 \qquad \ldots (2.5)$$

$$+ \frac{1}{3 \cdot 4^n}\left[\frac{A^2 (2A^4 + 20A^2 + 23)}{45(1-k^2)} + \frac{2A^4 (A^2+2)}{(1-k^2)^2} + \frac{4A^6}{(1-k^2)^3}\right] z^4 + \ldots,$$

and for $k > 1$ the extremal functions have the form

$$f_k(z) = z + \frac{\pi^2}{2^{n+2}\sqrt{\kappa}(k^2-1) K^2(\kappa)(1+\kappa)} z^2 \qquad \ldots (2.6)$$

$$+ \frac{\pi^2}{3^n\, 32 (k^2-1)\kappa K^4(\kappa)(1+\kappa)^2}\left[\frac{4K^2(\kappa)(\kappa^2 + 6\kappa + 1) - \pi^2}{6} + \frac{\pi^2}{k^2-1}\right] z^3 + \ldots$$

Applying the relations from Lemma 1.1 we easily get

Lemma 2.1 — Let $k \in [0, \infty)$, and f_k and h_k be the extremal functions of the classes. $\mathcal{T}(k, n)$ and k-$\mathcal{UCV}$, respectively. Moreover let $f_k(z) = z + A_2 z^2 + A_3 z^3 + \ldots$, and $h_k(z) = z + B_2 z^2 +$

$B_3 z^3 + \dots$. Then

$$B_m = m^{n-1} A_m,\ m = 2,3,\ \dots \qquad \dots (2.7)$$

Theorem 2.1 — *Let $k \in [0, \infty)$, and let f of the form (1.1) belongs to the class $\mathcal{T}(k, n)$. Then*

$$|a_2| \le A_2,\ |a_3| \le A_3,\ \textit{for } k \in [0, \infty),\ \textit{and } |a_4| \le A_4,\ \textit{when } k \in [0, 1].$$

PROOF : In view of Lemma 1.1 and applying the sharp inequalities for the coefficients of functions in the class k-$\mathcal{UCV}$, the result follows immediately. Equalities in all inequalities occur only when the function f is a rotation of f_k.

Theorem 2.2 — *Let $k \in [0, \infty)$, and let f of the form (1.1) belong to the class $\mathcal{T}(k, n)$. Then*

$$|a_m| \le \frac{(P_1)_{m-1}}{(1)_{m-1}\, m^n},\quad m = 2\ 3,\ \dots,\ n \in N \cup \{0\}, \qquad \dots (2.8)$$

where $(\lambda)_n$ is the Pochhammer symbol, defined as

$$(\lambda)_0 = 1,\ (\lambda)_n = \lambda(\lambda + 1) \dots (\lambda + n - 1),\ n \in N.$$

PROOF: The result follows immediately from Lemmas 1.1, 2.1 and by using the estimates in the class k-$\mathcal{UCV}$ (cf.[5]).

Theorem 2.3 — *If for the function f of the form (1.1) the inequality*

$$\sum_{m=2}^{\infty} [k(m-1) + m]\, m^n\, |a_m| < 1 \qquad \dots (2.9)$$

holds for some $k \in [0, \infty)$ and $n \in N$, then $f \in \mathcal{T}(k, n)$.

PROOF : The condition (1.9) of the definition of the class $\mathcal{T}(k, n)$ is equivalent to

$$S = k \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right| - Re\left(\frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right) < 1.$$

Then
$$S \le (k+1) \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right| = (k+1) \left| \frac{z + \sum_{m=2}^{\infty} m^{n+1} a_m z^m}{z + \sum_{m=2}^{\infty} m^n a_m z^m} - 1 \right|$$

$$= (k+1)\left|\frac{\sum_{m=2}^{\infty} m^n (m-1) a_m z^{m-1}}{1+\sum_{m=2}^{\infty} m^n a_m z^{m-1}}\right| \le (k+1)\frac{\sum_{m=2}^{\infty} m^n (m-1) | a_m |}{1-\sum_{m=2}^{\infty} m^n | a_m |} < 1$$

which holds when the inequality (2.9) is fulfilled.

Corollary 2.1 — For $n = 0$ Theorem 2.3 reduces to Theorem 1.3.

Theorem 2.4 — *Let* $k \in [0, \infty)$. *A function* f *of the form (1.1) is in* $\mathcal{T}(k, n)$ *if and only if there exists a function* $p \in \mathcal{P}(p_k)$ *such that*

$$f(z) = I^n\left[z \exp \int_0^z \frac{p(t)-1}{t}\, dt \right]. \qquad \text{... (2.10)}$$

PROOF : From Definition 1.2 a function f is an element of $\mathcal{T}(k, n)$ if and only if there exists a function $p \prec p_k$ such that

$$\frac{D^{n+1} f(z)}{D^n f(z)} = p(z).$$

Then
$$D^n f(z) = z \exp \int_0^z \exp \int_0^z \frac{[(t)-1}{t}\, dt.$$

By the relation (1.7) the result is established.

Theorem 2.5 — $\mathcal{T}(k, n+1) \subset \mathcal{T}(k, n)$.

PROOF : Let us denote

$$q(z) = \frac{D^{n+1} f(z)}{D^n f(z)}.$$

Then $q(0) = 1$, and

$$\frac{D^{n+2} f(z)}{D^{n+1} f(z)} = q(z) + \frac{zq'(z)}{q(z)}.$$

If a function f is an element of $\mathcal{T}(k, n+1)$ then

$$\frac{D^{n+2} f(z)}{D^{n+1} f(z)} \prec p_k(z), z \in \mathcal{U},$$

and so is $q(z) + zq'(z)/q(z)$. Moreover the function p_k is convex, $p_k(0) = 1$ and Re $p_k(z) > 0$ in U. Then, by the theory of differential subordinations of Briot-Bouquet type (cf. [1]), we get $q \prec p_k$ or equivalently $f \in \mathcal{T}(k, n)$. Thus the proof is complete.

Besides, by the geometric properties of the domains Ω_k we have that

Theorem 2.6 — $\mathcal{T}(k_1, n) \subset \mathcal{T}(k_2, n)$, *when* $k_1 \geq k_2$.

REFERENCES

1. P. J. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, *General Inequalities, 3, International Series of Numerical Mathematics*, vol. **64**, Birkhäuser Verlag, Basel, 1983, 339-48.
2. A. W. Goodman, *Ann. Polon. Math.* **56** (1991) 87-92.
3. S. Kanas, *Mathematica (Cluj)*, Vol. 41 (64), no. 1 (1999) 47-54.
4. S. Kanas and A Wisniowska, *J. Comput. appl. Math.* **105** (1999) 327-36.
5. S. Kanas and A. Wisniowska, *Revue Roum. Math. Pures Appl. t.* **45** No. 3 (2000).
7. W. Ma and D. minda, *Ann. Polon. Math.* **57**(2) (1992) 165-75.
8. W. Ma and D. Minda, *Ann. polon. Math.* **58**(3) (1993) 275-85.
9. F. Ronning, *Proc. Am. Math. Soc.* **118** (1993) 189-96.
10. G. S. Sălăgean, *Complex Analysis, Fifth Romanian-Finnish Seminar, Lect. Notes in Math.* **1013**. Springer Verlag 1983, 362-72.