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ON THE APPROXIMATION PROBLEM OF FIXED POINTS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Some iterative approximation theorems of fixed points for asymptotically nonexpansive mappings in Banach spaces are obtained. The results presented in this paper not only improve and extend the corresponding results of Goebel and Kirk[3], Rhoades[5] and Schu[6], but also correct the mistakes in Huang[4].

Key Words : Asymptotically Nonexpansive Mapping; Fixed Point; Ishikawa Iterative Sequence with Errors; Mann Iterative Sequence with Errors

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper we assume that E is a real Banach space, E^* is the dual space of E, D is a nonempty subset of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^*, \langle x, f \rangle = \| x \| \| f \|, \| f \| = \| x \|\}, \; x \in E.$$

Definition 1.1 — Let $T : D \to D$ be a mapping.

1. T is said to be asymptotically nonexpansive[3], if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$\| T^n x - T^n y \| \leq k_n \| x - y \| \text{ for all } x, y \in D, n \in N; \qquad \dots (1.1)$$

2. T is said to be uniformly L-Lipschitzian, where L is a positive constant, if

$$\| T^n x - T^n y \| \leq L \| x - y \| \text{ for all } x, y \in D, n \in N. \qquad \dots (1.2)$$

Remark 1.1 : It is easy to see that if $T : D \to D$ is a nonexpansive mapping then T is an asymptotically nonexpansive mapping with a constant sequence $\{1\}$; if $T : D \to D$ is an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$, then it must be uniformly L-Lipschitzian with $L = \sup_{n \geq 1} \{k_n\}$.

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Definition 1.2[7] — A Banach space E is said to be *uniformly convex*, if the modulus of convexity of E :

$$\delta_E(\varepsilon) = \inf\left\{ 1 - \frac{1}{2}\| x + y \| : \| x \| = \| y \| = 1, \text{ and } \| x - y \| \geq \varepsilon \right\} > 0 \qquad \ldots (1.3)$$

for all $0 < \varepsilon \leq 2$.

Remark 1.2 : It should be pointed out that the normalized duality mapping $J : E \to E^*$ is single-valued, if and only if E^* is strictly convex (see [2 p. 151, Proposition 12.3]).

Definition 1.3 — Let E be a real Banach space, D be a closed subset of E. A mapping $T : D \to D$ is said to be *semi-compact*, if for any bounded sequence $\{x_n\}$ in D such that $\| x_n - Tx_n \| \to 0\ (n \to \infty)$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in D$ $(n_i \to \infty)$.

The iterative approximation problem of fixed points for asymptotically nonexpansive mappings has been studied in Geobel-Kirk[3], Rhoades[5], Schu[6] in the setting of Hilbert space or uniformly convex Banach spaces.

Recently, Huang[4] proved the following two theorems:

Theorem A — *Let E be a real uniformly convex Banach space, D be a nonempty bounded closed convex subset of E and $T : D \to D$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty} (k_n^r - 1) < \infty$ for some $r > 1$. Suppose that $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $0 < a_1 \leq \alpha_n \leq 1 - a_2 < 1$ for all $n \geq 0$, where $a_1, a_2 \in (0, 1)$ are some constants. For any $x_0 \in D$ define the following Mann iterative sequence with errors $\{x_n\}$ in D by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n + u_n,\ n \geq 0 \qquad \ldots (1.4)$$

where $\{u_n\}$ is a sequence in D satisfying

$$\sum_{n=1}^{\infty} \| u_n \| < \infty.$$

Then the sequence $\{x_n\}$ converges strongly to some fixed point of T.

Theorem B — *Let E be a real uniformly convex Banach space, D be a nonempty bounded closed convex subset of E and $T : D \to D$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty} (k_n^r - 1) < \infty$ for some $r > 1$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0, 1]$ satisfying $0 < a_1 \leq \alpha_n \leq 1 - a_2 < 1, 0 \leq \beta_n < 1$ for all $n \geq 0$, $\limsup_{n \to \infty} \beta_n \leq b < 1$ for all $n \geq 0$, where $a_1, a_2 \in (0, 1)$ and $b \in [0, 1]$ are some constants. For any $x_0 \in D$ define the following Ishikawa iterative sequence with errors $\{x_n\}$ in D by*

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + u_n, \quad n \ge 0 \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + \upsilon_n, \end{aligned}\right\} \quad \dots (1.5)$$

where $\{u_n\}$, $\{\upsilon_n\}$ are two sequences in D satisfying

$$\sum_{n=1}^{\infty} \| u_n \| < \infty, \quad \sum_{n=1}^{\infty} \| \upsilon_n \| < \infty.$$

Then the sequence $\{x_n\}$ converges strongly to some fixed point of T.

Remark 1.3 : It should be pointed out that these two theorems have no meaning. Although T is a selfmapping of D, $\{x_n\}$, $\{y_n\}$ need not belong to D. Hence Tx_n, Ty_n, $T^n x_n$ and $T^n y_n$ need not be defined.

The purpose of this paper is to use a new method to prove the following Theorem 1.1 and Theorem 1.2, which not only correct the mistakes in[4] but also improve and generalize the corresponding results of [3, 5, 6].

Theorem 1.1 — *Let E be a real uniformly convex Banach space, D be a nonempty bounded closed convex subset of E and $T : D \to D$ be a semi-compact asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be four sequences in $[0, 1]$ satisfying the following conditions:*

(*i*) $\alpha_n + \gamma_n \le 1$, $\beta_n + \delta_n \le 1$, *for all* $n \ge 0$;

(*ii*) *there exist positive integers* n_0, n_1 *and* $\varepsilon > 0, 0 < b < \min\left\{1, \frac{1}{L}\right\}$ (*where* $L = \sup_{n \ge 0} k_n$) *such that*

$$\left.\begin{aligned} 0 < \varepsilon \le \alpha_n \le 1-\varepsilon \quad & \textit{for all } n \ge n_0, \\ 0 \le \beta_n \le b \quad & \textit{for all } n \ge n_1; \end{aligned}\right\} \quad \dots (1.6)$$

(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\sum_{n=0}^{\infty} (k_n^2 - 1) < \infty$.

Then the Ishikawa iterative sequence with errors $\{x_n\}$ defined by

$$\left.\begin{aligned} & x_0 \in D \\ & x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n T^n y_n + \gamma_n w_n, \quad n \ge 0 \\ & y_n = (1-\beta_n-\delta_n)x_n + \beta_n T^n x_n + \delta_n z_n, \end{aligned}\right\} \quad \dots (1.7)$$

converges strongly to some fixed point x^* of T in D, where $\{w_n\}$, $\{z_n\}$ are two sequences in D.

Theorem 1.2 — *Let E be a real uniformly convex Banach space, D be a nonempty bounded closed convex subset of E and $T : D \to D$ be a semi-compact asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Suppose that $\{\alpha_n\}$ and $\{\gamma_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions :*

(i) $\alpha_n + \gamma_n \leq 1$, *for all* $n \geq 0$;

(ii) there exist positive integers n_0 *and* $\varepsilon > 0$, *such that*

$$0 < \varepsilon \leq \alpha_n \leq 1 - \varepsilon \text{ for all } n \geq n_0.$$

(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty$ and $\sum_{n=0}^{\infty} (k_n^2 - 1) < \infty$.

Then the Mann iterative sequence with errors $\{x_n\}$ *defined by*

$$\left.\begin{array}{l} x_0 \in D \\ x_{n+1} = (1 - \alpha_n - \gamma_n)\, x_n + \alpha_n T^n x_n + \gamma_n w_n,\ n \geq 0 \end{array}\right\} \quad \ldots (1.8)$$

converges strongly to some fixed point x^* *of* T *in* D, *where* $\{w_n\}$ *is a sequence in* D.

PROOFS OF THE THEOREMS

The following Lemmas play an important role in proving our main results.

Lemma 1 (Chang[1]) — Let E be a real Banach space, $J : E \to 2^{E^*}$ be the normalized duality mapping, then for any $x, y \in E$ and for any $j(x + y) \in J(x + y)$

$$\| x + y \|^2 \leq \| x \|^2 + \langle y, j(x + y) \rangle.$$

Lemma 2 (Xu[7], Theorem 2]) — Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g\ (0) = 0$ such that

$$\| \lambda x + (1 - \lambda)\, y \|^p \leq \lambda \| x \|^p + (1 - \lambda) \| y \|^p - \omega_p (\lambda)\, g(\| x - y \|)$$

for all $x, y \in B(0, r)$ and $0 \leq \lambda \leq 1$, where $B(0, r)$ is the closed ball of E with center zero and radius r and

$$\omega_p (\lambda) = \lambda^p (1 - \lambda) + \lambda (1 - \lambda)^p. \quad \ldots (2.1)$$

Lemma 3 — Let E be a real Banach space, D a nonempty bounded closed convex subset of E, and $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n \to \infty} k_n = 1$. Let $\{x_n\}$ be the Ishikawa iterative sequence with errors defined by (1.7), in which $\{\gamma_n\}$, $\{\delta_n\}$ are two sequences in [0, 1] satisfying the following conditions :

$$\sum_{n=0}^{\infty} \gamma_n < \infty, \sum_{n=0}^{\infty} \delta_n < \infty. \quad \ldots (2.2)$$

Then from $\| x_n - T^n x_n \| \to 0$ we can obtain that $\| x_n - Tx_n \| \to 0$.

PROOF : First we rewrite the sequence $\{x_n\}$ defined by (1.7) as follows :

$$\left.\begin{array}{l} x_0 \in D \\ x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n\, T^n y_n + u_n, \quad n \geq 0 \\ y_n = (1-\beta_n)\, x_n + \beta_n\, T^n x_n + v_n, \end{array}\right\} \qquad \text{... (1.7)}'$$

where $u_n = \gamma_n (w_n - x_n)$, $v_n = \delta_n (z_n - x_n)$ for all $n \geq 0$. Since w_n, x_n, z_n all are in D, and D is bounded, $\{w_n - x_n\}$ and $\{z_n - x_n\}$ both are bounded sequences in E. It follows from condition (2.2) that

$$\sum_n \| u_n \| < \infty, \quad \sum_n \| v_n \| < \infty. \qquad \text{...(2.3)}$$

Again since $T : D \to D$ is asymptotically nonexpansive, by Remark 1.1, T is uniformly L-Lipschitzian, with $L = \sup_{n \geq 1} k_n$. Denote $c_n = \| x_n - T^n x_n \|$ for all $n \in N$. By using the same method as given in Huang[4], [Lemma 2], we can obtain that

$$\| x_{n+1} - Tx_{n+1} \| \leq c_{n+1} + L\,(L^2 + 2L + 2)\, c_n + L\,(L+2) \| u_n \| + L^2 (L+2) \| v_n \| \quad .$$

Therefore the conclusion of Lemma 3 is proved.

Now we are in a position to give a detailed proof of Theorem 1.1.

PROOF : From Goebel and Kirk[3], [Theorem 1] T has a fixed point in D. Hence $F(T)$ (the set of all fixed points of T in D) is nonempty.

Next we consider the sequences $\{x_n\}$, $\{y_n\}$ defined by (1.7)', which is an equivalent version of (1.7). From (2.3) we know that $\{u_n\}$ and $\{v_n\}$ both are bounded sequences in E.

Take $q \in F(T)$. Since D is bounded and $x_n, y_n, T^n y_n, T^n x_n$ all are in D, there exists an $r > 0$ such that

$$D \cup \{x_n - q\} \cup \{y_n - q\} \cup \{x_n - q + u_n\} \cup \{T^n y_n - q + u_n\}$$

$$\cup \{T^n x_n - q + v_n\} \cup \{x_n - q + v_n\} \subset B(0, r),$$

where $B(0, r)$ is a closed ball of E with center zero and radius r.

By Lemma 2 with $p = 2$ and $\lambda = \alpha_n$ and (1.7)' we have

$$\begin{aligned} \| x_{n+1} - q \|^2 &= \| (1-\alpha_n)(x_n - q + u_n) + \alpha_n (T^n y_n - q + u_n) \|^2 \\ &\leq (1-\alpha_n) \| x_n - q + u_n \|^2 \\ &\quad + \alpha_n \| T^n y_n - q + u_n \|^2 \\ &\quad - \omega_2(\alpha_n)\, g(\| x_n - T^n y_n \|). \end{aligned} \qquad \text{... (2.4)}$$

Since $\{x_n - q + u_n\}$ and $\{T^n y_n - q + u_n\}$ both are contained in $B(0, r)$, by Lemma 1 we have

$$\begin{aligned}\| x_n - q + u_n \|^2 &\le \| x_n - q \|^2 + 2 \langle u_n, J(x_n - q + u_n) \rangle \\ &\le \| x_n - q \|^2 + 2 \| u_n \| \cdot \| x_n - q + u_n \| \qquad \text{... (25)} \\ &\le \| x_n - q \|^2 + 2r \| u_n \|;\end{aligned}$$

Similarly, we also have

$$\| T^n y_n - q + u_n \|^2 \le \| T^n y_n - q \|^2 + 2r \| u_n \|; \qquad \text{... (2.6)}$$

It follows from (2.1) that

$$\omega_2(\alpha_n) = \alpha_n^2 (1 - \alpha_n) + \alpha_n (1 - \alpha_n)^2 = \alpha_n (1 - \alpha_n).$$

Substituting the above expressions into (2.4) and simplifying we have

$$\begin{aligned}&\| x_{n+1} - q \|^2 \\ &\le (1 - \alpha_n) \| x_n - q \|^2 + \alpha_n \| T^n y_n - q \|^2 + 2r \| u_n \| \\ &\quad - \alpha_n (1 - \alpha_n) g(\| x_n - T^n y_n \|) \\ &= \| x_n - q \|^2 + \alpha_n \{ \| T^n y_n - q \|^2 - \| y_n - q \|^2 \} \qquad \text{... (2.7)} \\ &\quad + \alpha_n \{ \| y_n - q \|^2 - \| x_n - q \|^2 \} + 2r \| u_n \| \\ &\quad - \alpha_n (1 - \alpha_n) g(\| x_n - T^n y_n \|).\end{aligned}$$

First, we consider the third term on the right side of (2.7). By Lemma 2 with $p = 2$, we have

$$\begin{aligned}&\| y_n - q \|^2 - \| x_n - q \|^2 \\ &= \| (1 - \beta_n)(x_n - q + v_n) + \beta_n (T^n x_n - q + v_n) \|^2 - \| x_n - q \|^2 \\ &\le (1 - \beta_n) \| x_n - q + v_n \|^2 + \beta_n \| T^n x_n - q + v_n \|^2 \qquad \text{... (2.8)} \\ &\quad - \omega_2(\beta_n) g(\| x_n - T^n x_n \|) - \| x_n - q \|^2 \\ &\le (1 - \beta_n) \| x_n - q + v_n \|^2 + \beta_n \| T^n x_n - q + v_n \|^2 - \| x_n - q \|^2.\end{aligned}$$

Since $x_n - q + \upsilon_n \in B(0, r)$, $T^n x_n - q + \upsilon_n \in B(0, r)$, it follows from Lemma 1 that

$$\| x_n - q + \upsilon_n \|^2 \leq \| x_n - q \|^2 + 2 \langle \upsilon_n, J(x_n - q + \upsilon_n) \rangle$$

$$\leq \| x_n - q \|^2 + 2 \| \upsilon_n \| r; \quad \ldots (2.9)$$

Similarly we have

$$\| T^n x_n - q + \upsilon_n \|^2 \leq \| T^n x_n - q \|^2 + 2r \| \upsilon_n \|. \quad \ldots (2.10)$$

Substituting (2.9), (2.10) into (2.8) and simplifying we have

$$\|y_n - q \|^2 - \|x_n - q \|^2 \leq \beta_n \{ \|T^n x_n - q \|^2 - \|x_n - q\|^2 \} + 2r \|v_n\|$$

$$\leq \beta_n (k_n^2 - 1) \|x_n - q\|^2 + 2r\|v_n\|. \quad \ldots (2.11)$$

Substituting (2.11) into (2.7) and simplifying we have

$$\| x_{n+1} - q \|^2$$

$$\leq \| x_n - q \|^2 + \alpha_n (k_n^2 - 1) \| y_n - q \|^2 + \alpha_n \{ \beta_n (k_n^2 - 1) \| x_n - q \|^2) + 2r \| \upsilon_n \| \}$$

$$+ 2r \| u_n \| - \alpha_n (1 - \alpha_n) g(\| x_n - T^n y_n \|).$$

$$\leq \| x_n - q \|^2 + \alpha_n (k_n^2 - 1) \{ \| y_n - q \|^2 + \| x_n - q \|^2 \}$$

$$+ 2r (\| u_n \| + \| \upsilon_n \|) - \alpha_n (1 - \alpha_n) g (\| x_n - T^n y_n \|)$$

Since $\{x_n - q\}$, $\{y_n - q\}$ both belong to $B(0, r)$, $\| x_n - q \| \leq r$ and $\| y_n - q \| \leq r$. Besides by condition (1.6), $0 < \varepsilon \leq \alpha_n$ and $\varepsilon \leq 1 - \alpha_n$ for all $n \geq n_0$. Hence we have

$$\| x_{n+1} - q \|^2 \leq \| x_n - q \|^2 + 2 \alpha_n (k_n^2 - 1) r^2$$

$$+ 2r (\| u_n \| + \| \upsilon_n \|) - \varepsilon^2 g (\| x_n - T^n y_n \|) \text{ for all } n \geq n_0. \quad \ldots (2.12)$$

Therefore we have

$$\varepsilon^2 g (\| x_n - T^n y_n \|) \leq \| x_n - q \|^2 - \| x_{n+1} - q \|^2 + 2(k_n^2 - 1) r^2$$

$$+ 2r (\| u_n \| + \| \upsilon_n \|) \text{ for all } n \geq n_0.$$

For any $m > n_0$ we have

$$\varepsilon^2 \sum_{n=n_0}^{m} g(\| x_n - T^n y_n \|) \le \| x_{n_0} - q \|^2 - \| x_{m+1} - q \|^2 + 2r^2 \sum_{n=n_0}^{m} (k_n^2 - 1)$$

$$+ 2r \sum_{n=n_0}^{m} (\| u_n \| + \| v_n \|)$$

$$\le \| x_{n_0} - q \|^2 + 2r^2 \sum_{n=n_0}^{m} (k_n^2 - 1)$$

$$+ 2r \sum_{n=n_0}^{m} (\| u_n \| + \| v_n \|)$$

Letting $m \to \infty$, by condition (iii) we have

$$\varepsilon^2 \sum_{n=n_0}^{\infty} g(\| x_n - T^n y_n \|) \le \| x_{n_0} - q \|^2 + 2r^2 \sum_{n=n_0}^{\infty} (k_n^2 - 1)$$

$$+ 2r \sum_{n=n_0}^{\infty} (\| u_n \| + \| v_n \|) < \infty. \qquad \text{... (2.13)}$$

This implies that

$$g(\| x_n - T^n y_n \|) \to 0. \qquad \text{... (2.14)}$$

Since $g : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing with $g(0) = 0$, it follows from (2.14) that

$$\| x_n - T^n y_n \| \to 0 \ (n \to \infty). \qquad \text{... (2.15)}$$

From (1.7)′ we have

$$\| x_n - y_n \| = \| \beta_n (x_n - T^n x_n) - v_n \|$$

$$\le \beta_n \{\| x_n - T^n y_n \| + \| T^n y_n - T^n x_n \|\} + \| v_n \| \qquad \text{... (2.16)}$$

$$\le \beta_n \{\| x_n - T^n y_n \| + L \| y_n - x_n \|\} + \| v_n \|.$$

Simplifying, we have

$$(1 - L\beta_n) \| x_n - y_n \| \le \beta_n \| x_n - T^n y_n \| + \| v_n \|$$

$$\le \| x_n - T^n y_n \| + \| v_n \|.$$

By condition (ii), we have $1 - L \cdot \beta_n > 0$ for all $n \geq n_1$. Therefore from (2.15) we have

$$\lim_{n \to \infty} \| x_n - y_n \| = 0. \qquad \text{... (2.17)}$$

It follows from (2.17) and (2.15) that

$$\| T^n x_n - x_n \| \leq \| T^n x_n - T^n y_n \| + \| T^n y_n - x_n \|$$

$$\leq L \| x_n - y_n \| + \| T^n y_n - x_n \|$$

$$\to 0 \ (n \to \infty). \qquad \text{... (2.18)}$$

By Lemma 1.3 we know that

$$\| Tx_n - x_n \| \to 0 \ (n \to \infty). \qquad \text{... (2.19)}$$

Since T is semi-compact, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$x_{n_i} \to x^* \in D \ (n_i \to \infty). \qquad \text{... (2.20)}$$

By the continuity of T, it follows from (2.20) that

$$\lim_{n_i \to \infty} \| x_{n_i} - Tx_{n_i} \| = \| x^* - Tx^* \| = 0,$$

i.e., x^* is a fixed point of T in D. Again from (2.18) we have

$$\| T^{n_i} x_{n_i} - x^* \| \leq \| T^{n_i} x_{n_i} - x_{n_i} \| + \| x_{n_i} - x^* \| \to 0 \ \ (n_i \to \infty).$$

Hence from (1.7)', (2.18) we have

$$y_{n_i} = x_{n_i} - \beta_{n_i} (x_{n_i} - T^{n_i} x_{n_i}) + \upsilon_{n_i} \to x^* \ (n_i \to \infty).$$

Again since

$$\| T^{n_i} y_{n_i} - x^* \| \leq L \| y_{n_i} - x^* \|$$

we have $$T^{n_i} y_{n_i} \to x^* \ (n_i \to \infty). \qquad \text{... (2.21)}$$

Now in (2.12) taking $q = x^*$, we have

$$\| x_{n_{i+1}} - x^* \|^2 \leq \| x_{n_i} - x^* \|^2 + 2 \, \alpha_n \, (k_{n_i}^2 - 1) \, r^2$$

$$+ \ 2r \, (\| u_{n_i} \| + \| \upsilon_{n_i} \|) - \varepsilon^2 \, g(\| x_{n_i} - T^{n_i} y_{n_i} \|) \ \text{for all} \ n_i \geq n_0.$$

It follows from (2.15), (2.20) and the continuity of g that

$$\lim_{n_i \to \infty} \| x_{n_i+1} - x^* \|^2 = 0.$$

i.e.,

$$x_{n_i+1} \to x^* \; (n_i \to \infty). \qquad \text{... (2.22)}$$

Therefore we have

$$\| T^{n_i+1} x_{n_i+1} - x^* \| \le L \| x_{n_i+1} - x^* \| \to 0 \; (n_i \to \infty). \qquad \text{... (2.23)}$$

By (1.7)′, (2.22) and (2.23), we know that

$$y_{n_i+1} = x_{n_i+1} - \beta_{n_i+1} (T^{n_i+1} x_{n_i+1} - x_{n_i+1}) + v_{n_i+1} \to x^* \;\; (n_i \to \infty).$$

Therefore we have

$$\| T^{n_i+1} y_{n_i+1} - x^* \| \le L \| y_{n_i+1} - x^* \| \to 0 \; (n_i \to \infty). \qquad \text{... (2.25)}$$

Continuing in this way, by induction we can prove that for any $m \ge 0$,

$$x_{n_i+m} \to x^*, y_{n_i+m} \to x^* \; (n_i \to \infty),$$

$$T^{n_{i+m}} x_{n_i+m} \to x^*, T^{n_i+m} \to x^* \;\; (n_i \to \infty).$$

Next we prove that $x_n \to x^* \; (n \to \infty)$.

In fact, it is easy to see that

$$\{x_n\}_{n=n_1}^{\infty} = \lim_{k \to \infty} \bigcup_{m=0}^{k} \{x_{n_i+m}\}_{i=1}^{\infty} . \qquad \text{... (2.26)}$$

Since the sequences $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ both converge to x^* (as $n_i \to \infty$), for any given $\varepsilon > 0$, there exist positive integers j_0 and j_1 such that

$$\| x_{n_i} - x^* \| < \varepsilon, \;\; \forall \; i \ge j_0,$$

$$\| x_{n_i+1} - x^* \| < \varepsilon, \;\; \forall \; i \ge j_1.$$

Therefore for $i \ge \max\{j_0, j_1\}$, we have

$$\| x_{n_i+1} - x^* \| < \varepsilon, \; l = 1 \text{ or } 2.$$

This implies that the sequence $\bigcup_{m=0}^{1} \left\{ x_{n_i+m} \right\}_{i=1}^{\infty} \to x^*$

By induction, we can prove that for any positive integer k, the sequence

$$\bigcup_{m=0}^{k} \left\{ x_{n_i+m} \right\}_{i=1}^{\infty} \to x^*.$$

Letting $k \to \infty$, it follows from (2.26) that the sequence $\{x_n\}$ converges to x^*.

Similarly, we can also prove that $y_n \to x^*$.

This completes the proof of Theorem 2.1.

Remark 2.1 : Theorem 1.1, 1.2 not only improve and extend the corresponding results of Goebel-Kirk[3], Huang[4], Rhoades[5] and Schu[6] but also correct the mistakes in[4]

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