Indian J. pure appl. Math., **32**(9) : 1309-1316, September 2001

ON PARA SASAKIAN MANIFOLDS

INDU KALA MISHRA AND RAM HIT OJHA

Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi 221 005

(*Received* 23 *February* 2000; *after final revision* 2 *March* 2000; *accepted* 8 *January* 2001)

In this paper we have introduced D-conformal transformation in a P-Sasakian manifold. Some properties of curvature tensors are also obtained.

1. PRELIMINARIES

An n-dimensional Riemannian manifold Mn on which there are defined a tensor field F of type (1, 1), a vector field ξ,a 1-form η and a metric tensor g satisfying for arbitrary vectors field X, Y, Z, ...

(a) $F^2 X = X - \eta (X) \xi$

(b) $F \xi = 0$

(c) $\eta (FX) = 0$

and (d) $\eta (\xi) = 1$ ... (1.1)

(a) $g (FX, FY) = g(X, Y) - \eta (X) \eta (Y)$

and (b) $'F (X, Y) = g(FX, Y) = g(X, FY) = 'F (Y, X)$

where (c) $g (\xi, X) = \eta (X)$. ... (1.2)

Then (M_n, g) is called Para-contact metric manifold [1, 2]. If in M_n the relations

(a) $(D_X \eta) (Y) - (D_Y \eta) (X) = 0$ ... (1.3)

(b) $(d \eta) (X, Y) = 0$ ... (1.3)

i.e. η is closed.

$$(D_X 'F) (Y, Z) = - g (X, Z) \eta (Y) - g(X, Y) \eta (Z) + 2 \eta (X) \eta (Y) \eta (Z). \quad ... (1.4)$$

$$(D_X \eta) (y) + D_Y \eta (X) = 2 'F(X, Y) \quad ... (1.5)$$

and $$(D_X \xi) = FX \qquad \text{... (1.6)}$$

hold-good, then (M_n, g) is called Para-Sasakian manifold or briefly P-Sasakian manifold [1]-[2]. Where D denotes the Riemanniam connection in M_n.

Further if in $(M_n\, g)$ the following relation also holds good.

$$D_X \eta(Y) = -g(X, Y) + \eta(X)\,\eta(Y) \qquad \text{... (1.7)}$$

along with (1.1), (1.2) and (1.6), then it is called Special Para Sasakian or briefly $S.P.$ Sasakian manifold.

In a P-Sasakian manifold (Mn, g) the following relations also hold :

$$\text{(a)}\ R(X, Y)\,\xi = R(X, Y, \xi) = \eta(X)\,Y - \eta(Y)\,X \qquad \text{... (1.8)}$$

$$\text{(b)}\ Ric\ (X, \xi) = -(n-1)\,\eta(X)$$

$$\text{(c)}\ R(\xi, X, \xi) = X - \eta(X)\,\xi$$

and $$\text{(d)}\ \eta(R(X, Y, Z)) = g(X, Z)\,\eta(Y) - g(Y, Z)\,\eta(X)$$

$$\text{(e)}\ 'R(\xi, X, Y, Z) = g(X, Z)\,\eta(Y) - g(X, Y)\,\eta(Z),$$

where $$\text{(f)}\ 'R(X, Y, Z, U) = g(R(X, Y, Z), U)$$

$$\text{(a)}\ (D_X{}'F)(\overline{Y}, \overline{Z}) + (D_X{}'F)(\overline{Y}, Z) = -(D_X \eta)(Y)\,\eta(Z) - (D_X \eta)(Z)\,\eta(Y)$$

$$\text{(b)}\ (D_X{}'F)(Y, \overline{Z}) + (D_X{}'F)(Y, Z) = -\eta(Y)\,(D_X \eta)(FZ) - \eta(Z)\,(D_X \eta)(FY) \qquad \text{... (1.9)}$$

2. D-Conformal Transformation

Let the corresponding Jacobian map B of the transformation b transforms the structure (F, ξ, η, g) to the structure (F, V, v, h) such that

$$\text{(a)}\ BFX = F\ (BX) \qquad \text{... (2.1)}$$

$$\text{(b)}\ h(BX, BY)\ ob = e^{\sigma} g(FX, FY) + e^{2\sigma}\,\eta(X)\,\eta(Y)$$

$$\text{(c)}\ V = e^{-\sigma} B\,\xi$$

and $$\text{(d)}\ \upsilon(BX)\ ob = e^{\sigma}\,\eta(X),$$

where σ is a differentiable function on M_n. Then the transformation is said to be D-conformal transformation. If σ is constant, the transformation is known as D-homothetic.

***Theorem* 2.1** — *The structure (F, V, υ, h) is P-Sasakian.*

PROOF : By virtue of (1.1) (a), (c) and (2.1) (b), (d) we have

$$h(BFX, BFY)\ ob = e^{\sigma} g\,(F^2 X, F^2 Y)$$

$$= e^{\sigma} \{g(X, Y) - \eta(X)\eta(Y)\}$$

$$= e^{\sigma} g(FX, FY)$$

$$= h(BX, BY)\, ob - e^{2\sigma} \eta(X)\eta(Y)$$

$$= h(BX, BY)\, ob - \{v(BX)ob\}\{v(BY)\, ob\}$$

or (a) $h(BFX, BFY) = h(BX, BY) - v(BX)\, v(BY)$. ... (2.2)

Making the use of (1.1), (2.1) (a), (2.1) (e) and (2.1) (d), we obtain

(b) $F^2 BX = BF^2F = BX - \eta(X) B\xi = BX = \upsilon(BX)\, obV$

also (c) $FV = e^{-\sigma} FB\xi = 0$

(2.2) prove the statement.

Theorem 2.2 — *Let D and E be Riemannian connections with respect to g and h such that*

(a) $E_{BX} BY = B D_X Y = BH(X, Y)$ ... (2.3)

(b) $'H(X, Y, Z) = g(H(X, Y), Z)$.

Then $$2E_{BX} BY = 2BD_X Y + B[2e^{\sigma}\{(X\sigma)\eta(Y)\xi + (Y\sigma)\eta(X)\xi - (^{-1} G\nabla\sigma)\eta(Y)\}$$

$$+ 2(e^{\sigma} - 1)\{(D_X \eta)(Y) - \eta(H(X, Y))\}\xi]. \quad ... (2.4)$$

PROOF : From (2.1) (b), we have

$$BX(h(BY, BZ) = ob = X(e^{\sigma} g(FX, FZ) + e^{2\sigma}\eta(Y)\eta(Z)).$$

Consequently

$$h(E_{BX} BY, BZ)\, ob + h(BY, E_{BX} BZ)\, ob$$

$$= (X\sigma) e^{\sigma} g(FY, FZ) + e^{\sigma} g(D_X FY, FZ) + e^{\sigma} g(FY, D_X FZ)$$

$$+ 2(X\sigma) e^{2\sigma}\eta(Y)\eta(Z) + 2^{2\sigma}(D_X\eta)(Y)\eta(Z) + e^{2\sigma}\eta(D_X Y)\eta(X) + e^{2\sigma}\eta(Y)(D_X\eta)(Z)$$

$$+ e^{2\sigma}\eta(Y)\eta(D_X Z); \quad ... (2.5)$$

also $$h(E_{BX} BY, BZ)ob + h(BY, E_{BX} BZ)ob$$

$$= h(BD_X Y + BH(X, Y), BZ)\, ob + h(BY, BD_X Z + BH(X, Z))$$

$$= e^{\sigma} g(FD_X Y, FZ) + e^{2\sigma}\eta(D_X Y)\eta(Z) + e^{\sigma} g(FH(X, Y), FZ) + e^{2\sigma}\eta(H(X, Y))\eta(Z)$$

$$+ e^{\sigma} g(FY, FD_X Z) + e^{2\sigma} \eta(Y) \eta(D_X Z) + e^{\sigma} g(FY, FH(X, Z))$$

$$+ e^{2\sigma} \eta(Y) \eta(H(X, Z)). \qquad \dots (2.6)$$

From (2.5) and (2.6), we have

$$(X\sigma) g(FY, FZ) + 2(X\sigma) e^{\sigma} \eta(Y) \eta(Z) + e^{\sigma}[(D_X \eta)(Y) \eta(Z) + \eta(Y)(D_X \eta)(Z)]$$

$$+ [g(D_X FY, FZ) + g(FY, D_X FZ)]$$

$$= 'H(X, Y, Z) + 'H(X, Z, Y) + (e^{\sigma} - 1)[\eta(H(X, Y)) \eta(Z) + \eta(H(X, Z)) \eta(Y)]$$

$$+ [g(FD_X Y, FZ) + g(FY, FD_X Z)]$$

The above equation implies

$$(X\sigma) g(FY, FZ) + 2(X\sigma) e^{\sigma} \eta(Y) \eta(Z) + e^{\sigma}[(D_X \eta)(Y) \eta(Z) + \eta(Y)(D_X \eta)(Z)]$$

$$+ [g(D_X F) Y, FZ) + g(FY, (D_X F) Z]$$

$$= 'H(X, Y, Z) + 'H(X, Z, Y) + (e^{\sigma} - 1)[\eta(H(X, Y)) \eta(Z) + \eta(H(X, Z)) \eta(Y)]$$

From which we obtain

$$(X\sigma) g(FY, FZ) + 2(X\sigma) e^{\sigma} \eta(Y) \eta(Z) + e^{\sigma}[(D_X \eta)(Y) \eta(Z) + (D_X \eta)(Z) \eta(Y)]$$

$$+ [(D_X F)(Y, FZ) + (D_X' F)(FY, Z)]$$

$$= 'H(X, Y, Z) + 'H(X, Z, Y) + (e^{\sigma} - 1)[\eta(HX, Y)) \eta(Z) + \eta[H(X, Z)) \eta(Y)$$

Using (1.9) (a) in the above equation, we have

$$(X\sigma) g(FY, FZ) + 2(X\sigma) e^{\sigma} \eta(Y) \eta(Z) + (e^{\sigma} - 1)[(D_X \eta)(Y) \eta(Z)$$

$$+ (D_X \eta)(Z) \eta(Y)]$$

$$= 'H(X, Y, Z) + 'H(X, Z, Y) + (e^{\sigma} - 1)[\eta H(X, Y)) \eta(z) + \eta(H(X, Z)) \eta(Y)]. \qquad \dots (2.7)$$

Writing two other equations by cyclic permutation of X, Y, Z and subtracting the third equation from the sum of the first two equations and using the symmetry of $'H$ in the first two slots, we get,

$$2\,'H(X, Y, Z) = 2e^{\sigma}[(X\sigma) \eta(Y) \eta(Z) + (Y\sigma) \eta(X) \eta(Z)$$

$$- (Z\sigma) \eta(X) \eta(Y)] + (e^{\sigma} - 1)$$

$$[\eta(Z)\ \{(D_X\eta)(Y)+(D_Y\eta)(X)-2\,\eta(H(X,Y))\}\ +\ \eta(Y)\ \{(D_X\eta)(Z)\ -\ (D_Z\eta)(X)\}$$

$$+\ \eta(X)\ \{(D_Y\eta)(Z)-(D_Z\eta)(Y)\}]. \qquad \ldots (2.8)$$

From eqs. (1.3) and (2.8), we have

$$2\,'H(X,Y,Z)=2e^{\sigma}[X\sigma)\,\eta(Y)\,\eta(Z)+(Y\sigma)\,\eta(X)\,\eta(Z)-(Z\sigma)\,\eta(X)\,\eta(Y)]$$

$$+\ 2\,(e^{\sigma}-1)\,[\eta(Z)\ \{D_X\eta)(Y)-\eta(H(X,Y))\}] \qquad \ldots (2.9)$$

which gives

$$2\,H(X,Y)=2e^{\sigma}[X\sigma)\,\eta(Y)\,\xi+(Y\sigma)\,\eta(X)\,\xi-(^{-1}G\nabla\sigma)\,\eta(X)\,\eta(Y)]$$

$$+\ 2(e^{\sigma}-1)\,[(D_X\eta)(Y)-\eta(H(X,Y))\}\xi]. \qquad \ldots (2.10)$$

Substitution of (2.10) into (2.3) (a), gives (2.4).

3. Curvature Tensors

Projective curvature tensor W, concircular curvature tensor C and conharmonic curvature tensor L are defined as follows.

$$W(X,Y,Z)=R(X,Y,Z)+\frac{1}{n-1}\,[Ric\,(X,Z)\,Y-Ric\,(Y,Z)\,X] \qquad \ldots (3.1)$$

$$L(X,Y,Z)=R(X,Y,Z)+\frac{1}{n-2}\,[Ric\,(Y,Z)\,Y-Ric\,(X,Z)\,Y]$$

$$+\ g(Y,Z)\,RX-g\,((X,Z)\,RY] \qquad \ldots (3.2)$$

$$C(X,Y,Z)=R(X,Y,Z)-\frac{r}{n(n-1)}\,[g(Y,Z)\,X-g\,(X,Z)\,Y] \qquad \ldots (3.3)$$

Theorem 3.1 — *Let Mn be a P-Sasakian manifold, then the following conditions are equivalent.*

(*i*) M_n *is symmetric*,

(*ii*) M_n *is of constant curvature* 1,

(*iii*) $R(X,Y).\,R=0$

PROOF : (i) ⇒ (ii) ⇒ (iii) Obviously. Now we take

$$R(X,Y).\ R\ =\ 0\ \Rightarrow\ R(X,\xi)\cdot R=0.$$

which is equivalent to

$$R(X,\xi,R(U,V,W))-R(R(X,\xi,U),V,W)-R(U,R(X,\xi,V),W)$$

$$-\ R\,(U,V,R(X,\xi,W))=0$$

or
$$g(U, W)\, R(X, \xi, V) - g(V, W)\, R(X, \xi, U) - g(X, U)\, R(\xi, V, W)$$
$$+ \eta(U)\, R(X, V, W) - g(X, V)\, R(U, \xi, W) + \eta(V)\, R(U, X, W) \quad \dots (3.4)$$
$$- g(X, W)\, R(U, V, \xi) + \eta(W)\, R(U, V, X) = 0.$$

Now putting ξ for U in (3.4), we have

$$R(X, V, W) = g(X, W)\, V - g(V, W)\, X.$$

Theorem 3.2 : *Let M_n be a P-Sasakian manifold, the the following conditions are equivalent.*

(*i*) *M_n is concircularly symmetric,*

(*ii*) *M_n is of constant curvature* 1,

(*iii*) $R(X,Y) \cdot C = 0$.

PROOF : (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) obviously. Now we take

$$R(X,Y) \cdot C = 0 \Rightarrow R(X, \xi) \cdot C = 0$$

which is equivalent to

$$R(X, \xi, C(U,V,W)) - C(R(X, \xi, U), V, W) - C(U, R(X, \xi, V), W)$$
$$- C(U, V, R(X, \xi, V)) = 0$$
$$\text{or } R(X, \xi, C(U, V, W)) - g(X, U)\, C(\xi, V, W) + \eta(U)\, C(X, V, W) - g(X, V)\, C(U, \xi, W)$$
$$- \eta(V)\, C(U, X, W) - g(X, W)\, C(U, V, \xi) + \eta(W)\, C(U, V, X) = 0.$$

Using (3.3) in this equation, we get, after a long computations

$$g(U, W)\, R(X, \xi, V) - g(V, W)\, R(X, \xi, U)$$
$$- g(X, U)\, R(\xi, V, W) + \eta(U)\, R(X, V, W) - g(X, V)\, R(U, \xi, W) + \eta(V)\, R(U, X, W)$$
$$- g(X, W)\, R(U, V, \xi) + \eta(W)\, R(U, V, X) = 0. \quad \dots (3.5)$$

Putting ξ for U in (3.5) and using (1.8), we have

$$R(X, V, W) = g(X, W)\, V - g(V, W)\, X.$$

which prove the statement.

Theorem 3.3 — *If in P-Sasakian manifold any two of the three conditions hold. Then third also holds.*

(*i*) *M_n is projectively symmetric,*

(*ii*) *M_n is an Einstein manifold,*

(*iii*) *M_n is of constant curvature* 1.

PROOF : (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) obviously. Now we take

$$R(X, Y).\ W = 0 \Rightarrow R(X, \xi).\ W = 0$$

which is equivalent to

$$R(X, \xi, W(Z, U, V)) - W(R(X, \xi, Z), U, V) - W(Z, R(X, \xi, U), V)$$
$$- W(Z, U, R(X, \xi V)) = 0$$

or

$$R(X, \xi, W(Z, U, V)) - g(X, Z) W(\xi, U, V) + \eta(Z) R(X, U, V)$$
$$- g(X, U) R(Z, \xi, V) + \eta(U) W(Z, X, V) - g(X, V) W(Z, U, \xi) + \eta(V) R(Z, U, X) = 0$$

Using (3.2) in this equation, we have, after a long computations

$$g(Z, V) R(X, \xi, U) - g(U, V) R(X, \xi, Z) - g(X, Z) R(\xi, U, V) + \eta(Z) R(X, U, V)$$
$$-g(X, U) R(Z, \xi, V) + \eta(U) R(Z, X, V) - g(X, V) R(Z, U, \xi) + \eta(V) R(Z, U, X)$$
$$+ \frac{1}{n-1} [(n-1) g(X, Z) \eta(V) U + \eta(Z) + Ric(X, V) U - (n-1) g(X, U) \eta(V) Z$$
$$- \eta(U) Ric(X, V) Z + (n-1) g(X, V) \eta(Z) U - (n-1) g(X, V) \eta(U) Z$$
$$+ \eta(V) Ric(Z, X) U - \eta(V) Ric(U, X) Z] = 0. \quad \ldots (3.6)$$

Putting $Z = \xi$ in (3.6) and using (1.8), we have,

$$R(X, U, V) + g(U, V) X - g(X, V) U + \frac{1}{n-1} [Ric(X, V) U - (n-1) g(X, U) \eta(V) \xi$$
$$\eta(U) Ric(X, V) \xi + (n-1) g(X, V) U - (n-1) g(X, V) \eta(U) \xi$$
$$- \eta(V) Ric(U, X) \xi] = 0.$$

Taking Ric $(X, Y) = -(n-1) g(X, Y)$ in the above equation we have,

$$R(X, U, V) = g(X, V) U - g(U, V) X.$$

which proves the statement.

Theorem 3.4 — *If in a P-Sasakian manifold any two of the three conditions hold. Then third also holds.*

(i) M_n *is conharmonically symmetric,*

(ii) M_n *is an Einstein manifold,*

(iii) M_n *is of constant curvature* 1.

PROOF : (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) obviosuly, Now we take

$$R(X, Y). L = 0 \Rightarrow R(X, \xi). L = 0$$

which is equivalent to

$$R(X, \xi, L(U, V, W)) - L(R, X, \xi, U), V, W) - L(U, R(X, \xi, V) W)$$
$$- L(U, V, R(X, \xi, W)) = 0.$$

or
$$R(X, \xi, L(U, V, W)) - g(X, U)\, L(\xi, V, W) + \eta(U)\, L(X, V, W) - g(X, V)\, L(U, \xi, W)$$
$$+\ \eta(V)\, L(U, X, W) - g(X, W)\, L(U, V, \xi) + \eta(W)\, L(U, V, X) = 0.$$

Using (3.2) in this equation, we have, after a long computations

$$g(U, W)\, R(X, \xi, V) - g(V, W)\, R(X, \xi, U) - g(X, U)\, R(\xi, V, W) + \eta(U)\, R(X, V, W)$$
$$- g(X, V)\, R(U, \xi, W) + \eta(V)\, R(U, X, W) - g(X, W)\, R(U, V, \xi) + \eta(W)\, R(U, V, X)$$
$$- \frac{1}{n-2}\, [g(V, W)\, Ric\,(X, U)\, \xi + (n-1)\, g(V, W)\, \eta(U)\, X - g(U, W)\, Ric\,(X, V)\, \xi$$
$$- (n-1)\, g(U, W)\, \eta(V)\, X - (n-1)\, g(X, U)\, \eta(W)\, V - g(V, W)\, g(X, U)\, R\, \xi$$
$$- \eta(U)\, Ric\,(X, W)\, V + g(V, W)\, \eta(U)\, RX - g(X, V)\, Ric\,(\xi, W)\, U$$
$$+\ g(X, V)\, g(U, W)\, R\, \xi - \eta(V)\, Ric\,(X, W)\, U - g(U, W)\, \eta(V)\, RX$$
$$+\ (n-1)\, g(X, W)\, \eta(V)\, U - (n-1)\, g(X, W)\, \eta(U)\, V + \eta(W)\, Ric\,(V, X)\, U$$
$$- \eta(W)\, Ric\,(U, X)\, V] = 0.$$

Putting ξ for U using and (1.8) in this equation, we have

$$R(X, V, W) + g(V, W)\, X - g(X, W)\, V - \frac{1}{n-2}\, [- Ric\,(X, W)\, V + \eta(V)\, Ric\,(X, W)\, \xi$$
$$+\ (n-1)\, g(X, W)\, \eta(V)\, \xi - (n-1)\, g(X, W)\, V] = 0.$$

Taking Ric $(X, Y) = -(n-1)\, g(X, Y)$ in the above equation, we have

$$R\,(X, V, W) = g(X, W)\, V - g(V, W)\, X$$

which proves the statement.

REFERENCES

1. I. Sato and K. Matsumoto: *Tensor N. S.* **33** (1979) 173-78.
2. S. N. Pandey and Km Savita Verma: *Indian J. Pure Appl. Math.* **30**(1) (1999) 15-22.