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A UNICITY THEOREM FOR ENTIRE FUNCTIONS CONCERNING DIFFERENTIAL POLYNOMIALS

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In this paper, we study the value distribution of entire functions and prove the following theorem : Let $f(z)$ and $g(z)$ be two transcendental entire functions, $n \geq 11$ be a positive integer. If $f^n(z)(f(z)-1)f'(z)$ and $g^n(z)(g(z)-1)g'(z)$ share 1 CM, then $f(z) \equiv g(z)$.

Key Words : Entire Function; Sharing Value; Differential Polynomial

1. INTRODUCTION AND MAIN RESULTS

Let $f(z)$ be a nonconstant meromorphic function in the whole complex plane. We shall use the following standard notations of value distribution theory,

$$T(r,f),\ m(r,f),\ N(r,f),\ \overline{N}(r,f),\ \ldots$$

(see Hayman[1], Yang[2]). We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\},$$

as $r \to +\infty$, possibly outside of finite measure. For any constant a, we define

$$\Theta(a,f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r,f)}.$$

Let $g(z)$ be a meromorphic function. If $f(z) - a$ and $g(z) - a$ assume the same zeros with the same multiplicities, then we call that $f(z)$ and $g(z)$ share the value a *CM*, where a is a complex number.

Hayman[3] and Clunie[4] proved the following result.

Theorem A — *Let $f(z)$ be an entire function, $n \geq 1$ a positive integer. If $f^n f' \neq 1$, then $f(z)$ is constant.*

Fang and Hua[5] obtained a unicity theorem corresponding to above result.

Theorem B — *Let $f(z)$ and $g(z)$ be two transcendental entire functions, $n \geq 6$ be a positive integer. If $f^n(z) f'(z)$ and $g^n(z) g'(z)$ share 1 CM, then either $f^n f' g^n g' = 1$ or $f = cg$ for a constant c with $c^{n+1} = 1$.*

By using the same argument as did in[3] we prove the following result.

Proposition 1 — Let $f(z)$ be an entire function, $n \geq 2$ a positive integer. If $f^n(f-1)f' \neq 1$, then $f(z)$ is a constant.

Naturally, we ask by Theorem A and Theorem B whether there exists a corresponding unicity theorem to Proposition 1. In this paper, we give a positive answer to above question by proving

Theorem 1 — *Let $f(z)$ and $g(z)$ be two transcendental entire functions, $n \geq 11$ be a positive integer. If $f^n(z)(f(z)-1)f'(z)$ and $g^n(z)(g(z)-1)g'(z)$ share 1 CM, then $f(z) \equiv g(z)$.*

2. Some Lemmas

For the proof of our result we need the following lemmas.

Lemma 1 ([1, 2]) — Let $f(z)$ be a nonlinear entire function, k is a positive integer, and c non-zero finite complex number. Then

$$T(r, f)$$

$$\leq N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f) \quad \text{... (2.1)}$$

$$\leq (k+1)\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f) \quad \text{... (2.2)}$$

here $N_0(r, 1/f^{(k+1)})$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f \neq 0$.

Lemma 2 ([1, 2]) — Let $f(z)$ be a meromorphic function. If there exist two functions $a_i(z)$ such that $T(r, a_i) = S(r, f)$, $i = 1, 2$, then

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f).$$

Lemma 3 ([6]) — Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$, where $a_n (\neq 0)$, $a_{n-1}, \ldots, a_0$ are constants. If $f(z)$ is a meromorphic function, then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 4 — Let f and g be two transcendental entire functions, k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 *CM* and

$$\Theta(0,f)+\Theta(0,g)>\frac{2k+3}{k+2}, \qquad \text{... (2.3)}$$

then either $f^{(k)}g^{(k)}\equiv 1$ or $f\equiv g$.

PROOF : Let

$$\phi(z)=\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)}-2\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1}-\frac{g^{(k+2)}(z)}{g^{(k+1)}(z)}+2\frac{g^{(k+1)}(z)}{g^{(k)}(z)-1}. \qquad \text{... (2.4)}$$

If z_0 is a common simple 1-point of $f^{(k)}(z)$ and $g^{(k)}(z)$, substituting their Taylor series at z_0 into (2.4), we see that z_0 is a zero of $\phi(z)$. Thus we have

$$N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)=N_{1)}\left(r,\frac{1}{g^{(k)}-1}\right)$$

$$\leq\overline{N}\left(r,\frac{1}{\phi}\right)\leq T(r,\phi)+O(1)$$

$$\leq N(r,\phi)+S(r,f)+S(r,g), \qquad \text{... (2.5)}$$

here $N_{1)}(r,1/(f^{(k)}-1))$ is the counting function which only counts these points such that $f^{(k)}-1=0$ but $f^{(k+1)}\neq 0$.

By our assumptions, $\phi(z)$ have poles only at zeros of $f^{(k+1)}$ and $g^{(k+1)}$. We deduce from (2.4) that

$$N(r,\phi)\leq\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+N_0\left(r,\frac{1}{f^{(k+1)}}\right)+N_0\left(r,\frac{1}{f^{(k+1)}}\right), \qquad \text{... (2.6)}$$

here $N_0(r,1/f^{(k+1)})$ has the same meaning as in Lemma 1. Obviously,

$$\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)$$

$$=2\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\leq N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)+N\left(r,\frac{1}{f^{(k)}-1}\right). \qquad \text{... (2.7)}$$

By lemma 1 we have

$$T(r,f)\leq(k+1)\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)-N_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f), \qquad \text{... (2.8)}$$

$$T(r,g)\leq(k+1)\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)-N_0\left(r,\frac{1}{g^{(k+1)}}\right)+S(r,g), \qquad \text{... (2.9)}$$

Thus we deduce from (2.5)-(2.9) that

$$T(r,f)+T(r,g)\le(k+2)\,\overline{N}\left(r,\frac{1}{f}\right)+(k+2)\,\overline{N}\left(r,\frac{1}{g}\right)$$

$$+\,N\left(r,\frac{1}{f^{(k)}-1}\right)+S(r,f)+S(r,g). \qquad \dots (2.10)$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f)\le T(r,g)$ for $r\in I$. Since

$$N\left(r,\frac{1}{f^{(k)}-1}\right)\le T\,(r,f^{(k)})+O(1)\le T(r,f)+S(r,f).$$

we obtain from (2.10) that

$$T(r,g)\le(k+2)\,\overline{N}\left(r,\frac{1}{f}\right)+(k+2)\,\overline{N}\left(r,\frac{1}{g}\right)+S(r,g)$$

$$\le\{(k+2)\,[2-\Theta(0,f)-\Theta(0,g)]+\varepsilon\}\;T(r,g)+S(r,g),$$

for $r\in I$. Thus we obtain from (2.3) and (2.11) that $T(r,g)\le S(r,g)$ for $r\in I$, a contradiction. Hence we have $\phi(z)\equiv 0$, that is

$$\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)}-2\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1}\equiv\frac{g^{(k+2)}(z)}{g^{(k+1)}(z)}-2\frac{g^{(k+1)}(z)}{g^{(k)}(z)-1}. \qquad \dots (2.12)$$

By solving this we obtain

$$\frac{1}{f^{(k)}-1}=\frac{bg^{(k)}+a-b}{g^{(k)}-1}, \qquad \dots (2.13)$$

for two constants a and b. Next we consider three cases.

Case 1 — $b\ne 0$ and $a=b$. From (2.13) we obtain that $g^{(k)}\ne 0$. Thus there exists an entire function $h(z)$ such that $g^{(k)}(z)=e^{h(z)}$ and

$$f^{(k)}=1+\frac{1}{b}-\frac{1}{b}e^{-h}.$$

If $b=-1$, then $f^{(k)}(z)\,g^{(k)}(z)\equiv 1$. If $b\ne -1$, then $f^{(k)}-(1+1/b)=-1/be^{-h}\ne 0$, thus we deduce from lemma 1 that

$$T(r,f)\le(k+1)\,\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)$$

$$\le(k+1)\,[1-\Theta(0,f)+1-\Theta(0,g)]\,T(r,f)+S(r,f),$$

that is

$$\left[\Theta(0,f)+\Theta(0,g)-\frac{2k+1}{k+1}\right]T(r,f)\le S(r,f).$$

Hence by (2.3) we deduce that $T(r,f) \leq S(r,f)$, a contradiction.

Case 2 — $b \neq 0$ and $a \neq b$. Then from (2.13) we have $g^{(k)} + (a-b)/b \neq 0$. From lemma 1 we deduce

$$T(r,g) \leq (k+1)\,\overline{N}\left(r, \frac{1}{g}\right) + S(r,g).$$

Next by using the argument as in case 1, we get a contradiction.

Case 3 — $b = 0$ and $a \neq 0$. From (2.13) we obtain

$$f = \frac{1}{a}g + p(z), \qquad \text{... (2.14)}$$

where $p(z)$ is a polynomial. If $p(z) \not\equiv 0$, then by lemma 2 we have

$$T(r,f) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-p(z)}\right) + S(r,f) = \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + S(r,f)$$

$$\leq [1 - \Theta(0,f)]\,T(r,f) + [1 - \Theta(0,g)]\,T(r,g) + S(r,f). \qquad \text{... (2.15)}$$

Obviously, by (2.14) we have $T(r,f) = T(r,g) + S(r,f)$. Hence substituting this into (2.15) we get

$$[\Theta(0,f) + \Theta(0,g) - 1]\,T(r,f) \leq S(r,f). \qquad \text{... (2.16)}$$

Thus by (2.3) and (2.16) we deduce that $T(r,f) \leq S(r,f)$, a contradiction. Therefore we deduce that $p(z) \equiv 0$, that is

$$f = \frac{1}{a}g. \qquad \text{... (2.17)}$$

If $a \neq 1$, then by $f^{(k)}$ and $g^{(k)}$ sharing the value 1 CM we deduce from (2.17) that $g^{(k)} \neq 1$. Next we can deduce a contradiction as in case 2. Thus we get that $a = 1$, that is $f \equiv g$. The proof of the lemma is complete.

3. Proof of Proposition 1

By (2.1) and lemma 3 we have

$$(n+2)\,T(r,f) = T\left(r, f^{n+1}\left(\frac{1}{n+2}f - \frac{1}{n+1}\right)\right) + S(r,f)$$

$$\leq N\left(r, \frac{1}{f^{n+1}\left(\frac{1}{n+2}f - \frac{1}{n+1}\right)}\right) + N\left(r, \frac{1}{f^n(f-1)f' - 1}\right)$$

$$- N\left(r\frac{1}{\left(f^{n+1}\left(\frac{1}{n+2}f - \frac{1}{n+1}\right)\right)'}\right) + S(r,f)$$

$$\leq 3T(r,f) + H\left(r, \frac{1}{f^n(f-1)f' - 1}\right) + S(r,f),$$

thus we get

$$(n-1)\,T(r,f) \le N\left(r, \frac{1}{f^n (f-1) f' - 1}\right) + S(r,f) \le S(r,f).$$

Hence we obtain that f is a constant.

4. Proof of Theorem 1

Let $F = f^{n+1}(1/(n+2)f - 1/(n+1))$ and $G = g^{n+1}(1/(n+2)\,g - 1/(n+1))$.

Then by $n \ge 11$ and lemma 3 we obtain

$$\Theta(0, F) + \Theta(0, G) \ge \frac{n}{n+2} + \frac{n}{n+2} > \frac{5}{3}.$$

Obviously, $F' = f^n (f-1) f'$, $G' = g^n (g-1) g'$.

Thus we obtain that F' and G' share the value 1 CM. Hence by Lemma 4 we deduce that either $F' G' \equiv 1$ or $F \equiv G$. If $F' G' \equiv 1$, that is

$$f^n (f-1) f' g^n (g-1) g' \equiv 1.$$

Since f and g are entire functions, we deduce that $f \ne 0, 1, \infty$, a contradiction. Hence $F \equiv G$, that is

$$f^{n+1}\left(\frac{1}{n+2} f - \frac{1}{n+1}\right) = g^{n+1}\left(\frac{1}{n+2} g - \frac{1}{n+1}\right) \qquad \text{... (4.1)}$$

Let $f/g = h$. If $h \not\equiv 1$, then by (4.1) we have

$$g = \frac{(n+2)(1 + h + \ldots + h^n)}{(n+1)(1 + h + \ldots + h^{n+1})}.$$

Thus we deduce by Picard's theorem that $h(z)$ is a constant. Hence g is a constant, a contradiction. Therefore we deduce that $h(z) \equiv 1$, that is $f(z) \equiv g(z)$.

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