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ON A GEOMETRIC PROPERTY OF A CLASS OF LINEAR SYSTEMS

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Given a linear time-invariant system satisfying appropriate dynamics in state space, it is shown that an initial state defined with respect to the system forced response exists with the following property : The system dynamics with respect to this initial state can be expressed as a sum of two vector fields, one associated with the system transient response and the other with the system forced response. At any desired instant, using a simple procedure, the system can be switched to one governed by the transient vector field alone. The system then converges to a single point on the trajectory of the forced response. The point of convergence is specified by a single parameter, the time of switching.

Key Word : Differential Geometry; Linear System; State Space

1. INTRODUCTION

The solution to the state-variable model of linear time-invariant systems involves terms dependent on the initial state and the input. The system state response can be expressed as the sum of a transient response and a forced response. The transient response decays to zero for stable systems and the forced response is sustained by the input. The forced response is not a function of the intial state. In particular, the initial value of the transient response results from the additive contribution of two aspects. One, the initial state and the other is the value of the forced response at time zero. It therefore follows that if the forced response is known, then with hindsight, the initial state may be so chosen as to make system transient response identically zero. In other words, the system response is identically the forced response.

This paper deals in a sense with a counterpart to the above property. What this paper shows is that for a system satisfying appropriate dynamics, an initial state exists with the following property: The system dynamics can then be expressed as the sum of two specific vector fields. One of these may be described as a "transient" and the other a "forced" vector field. Then with the system evolving from the initial state, at any desired instant of time, it is possible to switch the system dynamics to that described by the "transient" vector field alone. This results in a system whose response converges to a single point on the forced response of the original system. The desired initial state and the switching procedure depends on a priori knowledge of the forced response.

This paper is a study of a system property dependent on the location in state-space of the initial state.

2. Geometric Concepts

The material in this section is briefly adapted from Schutz[1] and Spivak[2].

A manifold is essentially a continuous space which locally looks like Euclidean space. Any sufficiently small region of the manifold can be mapped one-to-one onto a region of a Euclidean space. The dimension of the Euclidean space defines the dimension of the manifold. A coordinate system on the region of the manifold results from the inverse map of the cartesian coordinates of the Euclidean space. The ith coordinate in coordinate system x is denoted x^i (the superscript denotes an index). Choosing a different mapping results in a different coordinate system and the mathematical equations relating the two mappings define a coordinate transformation.

A curve is defined as a continuous one to one mapping between points on a real line which define the curve parameter and points on the manifold. By definition, a curve includes the specification of its parameter.

In some coordinate system x, let $x(\alpha)$ be the equation of a curve on the manifold with curve parameter α. The tangent vector field along the curve is denoted by $d/d\alpha$ and can be expressed using the chain rule as

$$d/d\alpha = \sum_{i=1}^{n} (dx^i/d\alpha)\, \partial/\partial x^i. \qquad \text{... (1)}$$

At a point on the curve, the set $dx^i/d\alpha$ are the components of the tangent vector to the curve and the set $\partial/\partial x^i$ are the coordinate basis vectors. A change in the coordinate system results in the tangent vector field having a different set of components and coordinate basis vectors at each point. Given a vector field on a region of the manifold, a set of non-intersecting curves which fill the region and whose tangent vector field is the given vector field define the integral curves of the vector field.

On a region of the manifold, consider two vector fields $d/d\alpha$ and $d/d\beta$. The Lie bracket of $d/d\alpha$ and $d/d\beta$ denoted $[d/d\alpha, d/d\beta]$, is a vector field and is defined as the commutator of the two vector fields which expressed in component form in some coordinate system x is

$$[d/d\alpha, d/d\beta] = \sum_{i=1}^{n} \left(d/d\alpha\,(dx^i/d\beta) - d/d\beta\,(dx^i/d\alpha)\right) \partial/\partial x^i \qquad \text{... (2)}$$

With reference to eq. (2), the Lie bracket components expressed as a column will be denoted $[d/d\alpha, d/d\beta]\,|_x$. If the vector fields $d/d\alpha$ and $d/d\beta$ are linearly independent and their Lie bracket vanishes then α is constant along an integral curve of $d/d\beta$ and β is constant along an integral curve of $d/d\alpha$.

3. Geometric Property

For convenience, this section is divided into three parts. In Part I the system model along with its associated definitions are presented. In Part II the vector fields associated with the system model are defined and their properties stated. Part III describes the procedure to realise the objective of the paper.

Part I : Consider a linear time-invariant system modeled via

$$dx(\tau)/d\tau = Ax(\tau) + Bu(\tau). \quad \text{... (3)}$$

In eq. (3), x is a $n \times 1$ column of state-variables, A is a $n \times n$ constant system matrix, B is a $n \times m$ constant input matrix and u is a $m \times 1$ column of inputs.

For some initial state x (0), let the system response $x(\tau)$ for $\tau \geq 0$, that is, the response satisfying eq. (3) and the initial condition, be expressed as

$$x(\tau) = x_{tr}(\tau) + x_f(t(\tau)). \quad \text{... (4)}$$

In eq. (4), $x_{tr}(\tau)$ is the transient response and $x_f(t(\tau))$ is the forced response. By definition, along the forced response, $t(\tau) = \tau$. The reason for expressing the independent variable in the forced response as $t(\tau)$ has to do with the definition of a vector field based on the forced response dynamics requiring its own integral curve parameter. The forced response satisfies the equation $dx_f/d\tau = Ax_f + Bu$. It then follows from eqs. (3) and (4) that the transient response satisfies the equation $dx_{tr}/d\tau = Ax_{tr}$ with initial condition $x(0) - x_f(0)$. With A a stability matrix, the transient response decays to zero with increasing τ.

It is assumed that the forced response satisfies

$$dx_f(t)/dt = A_f x_f(t). \quad \text{... (5)}$$

In eq. (5), A_f is a $n \times n$ constant matrix and it is assumed that $A - A_f$ is a stability matrix. It is also assumed that the matrices A and A_f commute, that is, $AA_f = A_f A$. This condition is essential towards developing the geometric arguments presented below.

From eq. (4) it follows that

$$dx/d\tau = dx_{tr}/d\tau + dx_f/d\tau \quad \text{... (6)}$$

Using eq. (5) and the definition of the transient and forced responses it follows that

$$dx(\tau)/d\tau = A(x(tr(\tau)) + A_f x_f(t(\tau)). \quad \text{... (7)}$$

In state-space, the various responses associated with the system define respective trajectories. In particular, let the trajectory associated with the forced response be called the forced trajectory.

Part II : The state-space can be considered to be a n dimensional differentiable manifold. The state-variables define a coordinate system on this region.

Define a vector field *d/dt* on the manifold whose components in coordinate system x are

$$dx(t)/dt = A_f x(t). \quad \text{... (8)}$$

Thus the forced trajectory $x_f(t)$ lies on an integral curve of *d/dt*. This follows from eq. (5).

Through a point on the manifold, there exists an integral curve of d/dt. On this curve, the point defines a unique value of the curve parameter t. In coordinate system x, let the label x_t attached to a point represent both the coordinates x of the point and the curve parameter t. In a given situation, the particular value of t will be understood from the context. The label thus specifies two pieces of information about the point.

Define a vector field $d/d\mu$ on the manifold whose components in coordinate system x are

$$dx_t(\mu)/d\mu = (A - A_f)(x_t(\mu) - x_f(t)). \quad \text{... (9)}$$

In eq. (9), note the use of the label x_t as defined in the previous paragraph. The value of t in $x_t(\mu)$ is the same as the value of t in $x_f(t)$ by definition. For points along the forced trajectory, that is, $x_t = x_f(t)$, the components of $d/d\mu$ vanish.

The vector field $d/d\mu$ has the following important property. For a initial point, as μ increases, the integral curve $d/d\mu$ converges to a point on the integral curve of d/dt corresponding to the forced trajectory. It does so in a manner in which the parameter t remains constant along the curve. This aspect is now considered.

The vector fields $d/d\mu$ and d/dt are linearly independent. This will be true if A is not a scale multiple of A_f, which is assumed. Since A and A_f commute, it is easy to verify that the Lie bracket of $d/d\mu$ and d/dt vanishes. This follows since in coordinate system x

$$[d/d\mu, d/dt]\,|_x = A_f(A - A_f)(x_t - x_f(t)) - (A - A_f)A_f(x_t - x_f(t)) = 0. \quad \text{... (10)}$$

It then follows that t is constant along an integral curve of $d/d\mu$ and μ is constant along an integral curve of d/dt.

For initial condition $x_t(\mu)\,|_{\mu=0}$, the solution to eq. (9) is

$$x_t(\mu) = \exp(\mu(A - A_f))(x_t(\mu)\,|_{\mu=0} - x_f(t)) + x_f(t)). \quad \text{... (11)}$$

Since $A - A_f$ is a stability matrix, then as μ increases along an integral curve of $d/d\mu$, $x_t(\mu)$ approaches $x_f(t)$.

The vector fields $d/d\mu$ and d/dt may be classified as "transient" and "forced" vector fields respectively, the reason behind the classification resting on the properties of their integral curves as indicated above.

Define a vector fields $d/d\tau$ on the manifold whose components in coordinate system x are

$$dx(\tau)/d\tau = Ax(\tau) + Bu(\tau). \quad \text{... (12)}$$

The vector field $d/d\tau$ describes system dynamic as modeled in eq. (3).

Part III — With respect to eq. (3), that is, for a given initial state and input, consider the following question. Can the vector field $d/d\tau$ be expressed as the sum of the vector fields $d/d\mu$ and d/dt :

$$d/d\tau = d/d\mu + d/dt. \quad \text{... (13)}$$

The answer in general is in the negative. The reason for this is that in general, the integral curve of $d/d\mu$ through $x(0)$ will not converge to the point $x_f(0)$ which it must do if eq. (13) is to hold. In other words, the value of t in the label x_t attached to the initial state $x(0)$ does not equal zero. For different choices of the initial state, the integral curve of $d/d\mu$ through the initial state will in general converge to different points on the integral curve of d/dt corresponding to the forced trajectory.

Since the vector fields $d/d\mu$ and d/dt are linearly independent and commute and if eq. (13) holds, then the system state is constrained to evolve on a two-dimensional coordinate grid defined by integral curves of $d/d\mu$ and d/dt. Any point on the system trajectory can be given two independent coordinates μ and t. It is also specified by the parameter τ which fixes its position along the system trajectory. If the initial state is characterized by the values $\tau=0$, $\mu=0$ and $t = 0$, then for any point on the system trajectory specified by $\tau, \mu(\tau)=\tau$ and $t(\tau)=\tau$. This follows essentially from the way in which $d/d\mu$ has been defined.

In order to make eq. (13) valid, the options is to deliberately setup an initial state satisfying the criterion stated in the last paragraph.

For an arbitrary initial state $x(0)$, the first option is accomplished by moving along the integral curve of d/dt through the initial state by an appropriate amount. There will exist a point on this curve with the property that the integral curve of $d/d\mu$ through this point will converge to the point $x_f(0)$. Let this point be $x_0(0)$. Then there will exist a constant α such that

$$x_0(0) = \exp(\alpha A_f)\, x(0). \qquad \text{... (14)}$$

Eq. (14) results from the equation of an integral curve of d/dt.

Using this option, let eq. (13) describe system dynamics. Let the system evolve from $\tau = 0$ and consider any particular time τ_0. The system state is $x_t(\tau_0)$. At this point, if the system dynamics can be switched from $d/d\tau$ to $d/d\mu$, the system will converge to the point $x_f(t(\tau_0)) = x_f(\tau_0)$. The point of switching and convergence is specified by a single parameter, time. What has to be now considered is the way in which the switching is accomplished. On the manifold, the switching process involves a change in both the magnitude and direction of the vector field $d/d\tau$. What is needed is a way to subtract out the vector field d/dt at the point of switching so that the system dynamics will then be governed by the vector field $d/d\mu$.

Expanding eq. (13) results in

$$dx_t(\tau)/d\tau = Ax_t(\tau) + Bu(\tau) = Ax_t(\tau) + (A_f - A)\, x_f(t(\tau)). \qquad \text{... (15)}$$

It therefore follows that

$$Bu(\tau) = (A_f - A)\, x_f(t(\tau)) = (A_f - A)\, x_f(\tau) \qquad \text{... (16)}$$

From eq. (16) it is evident that to keep the forced response fixed at $x_f(\tau_0)$, the input should be fixed at $\upsilon(\tau_0)$.

After the system evolves over $\tau=0$ to $\tau=\tau_0$, the switching action at τ_0 is accomplished by using the terms $A_f x_t(\tau_0)$ and $u(\tau_0)$ as follows. For $\tau \geq \tau_0$, the system dynamics is modified to

$$dx_t(\tau)/d\tau = Ax_t(\tau) + Bu(\tau_0) - A_f x_t(\tau_0). \qquad \text{... (17)}$$

Note that the input is held fixed at the value $\upsilon(\tau_0)$. From eq. (16) it then follows that the force response is held fixed at $x_f(\tau_0)$. Eq. (18) evolves along the direction of $d/d\mu$. It specifies the components of the tangent vector to the integral curve of $d/d\mu$ passing through the point $x_t(\tau_0)$ with $\mu = \tau$. The system of eq. (17) then converges to the point $x_f(\tau_0)$ as desired.

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