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# THE EQUIVALENCE OF TWO MATRICES AS BOUNDED LINEAR OPERATORS ON $l^p$

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We obtain sufficient conditions for $(\bar{N}, p_n)$ and $(\bar{N}, q_n)$ weighted mean matrices to be equivalent, when considered as bounded operators on $l^p$, $1 < p < \infty$.

1.1 Let $\Sigma a_n$ be an infinite series with partial sum $s_n$. If $p_n \geq 0, p_0 > 0$,

$$\Sigma p_n = \infty$$

(so that $P_n = p_0 + p_1 + \dots\dots\dots\dots + p_n \to \infty$ as $n \to \infty$), and $t_n = \frac{1}{P_n} \sum_{v=0}^{n} P_v s_v \to s$

when $n \to \infty$, then we say that $s_n \to s\ (\bar{N}, p_n)$

It is evident that the method $(\bar{N}, p_n)$ is regular.

**Key Words: Bounded Linear Operators**

A theorem concerning the relations between the methods corresponding to two different sequences $\{p_n\}$ and $\{q_n\}$ have been given in Hardy[3] [Theorem 14]

***Theorem*** 1[3] — *If* $p_n > 0, q_n > 0, \Sigma p_n = \infty, \Sigma q_n = \infty$, *and either (a)*

$$q_{n+1}/q_n \leq p_{n+1}/p_n \qquad \dots (1)$$

*or* (b) $$p_{n+1}/p_n \leq q_{n+1}/q_n \qquad \dots (2)$$

*and also* $$P_n/p_n \leq HQ_n/q_n, \qquad \dots (3)$$

*then* $\Sigma a_n = s\ (\bar{N}, p_n)$ ***implies*** $\Sigma a_n = s\ (\bar{N}, q_n)$.

Borwein and Cass[2] have proved under certain conditions that $(\bar{N}, p_n)$ and $C$, the cesaro matrix of order 1 are equivalent on $c$, the space of convergent sequences. Equivalence over $c$ means that $\sigma_n \to s$ if and only if $t_n \to s$, where $$\sigma_n = \frac{1}{(n+1)} \sum_{k=0}^{n} s_k \qquad \text{... (4)}$$

and $$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \qquad \text{... (5)}$$

***Theorem*** $2^2$ — *If* $(\overline{N}, p_n)$ *is regular,* $p_n > 0$ *for all* $n$ *and either*

$$\{p_n\} \text{ is non decreasing and } n \le H_1 \frac{P_n}{p_n} \qquad \text{... (6)}$$

$$\text{or } \{p_n\} \text{ is non-increasing and } \frac{P_n}{p_n} \le H_2 (n+1) \qquad \text{... (7)}$$

*then* $(\overline{N}, p_n) \Leftrightarrow (\overline{N}, 1)$.

Recently in 1994, Rhoades[4] has obtained sufficient conditions for certain weighted mean matrices to be equivalent to $C$, the cesoro matrix of order 1, considered as bounded operators on $l^p$, $1 < p < \infty$.

***Theorem*** $3^4$ — *Let* $\{a_n\}$ *be a positive sequence satisfying the condition*

$$(n + 1)\,(a_{n+1} - a_n) \approx (n+1)^{\alpha} \text{ for some } \alpha \ge 0. \qquad \text{... (8)}$$

*Then* $(\overline{N}, a)$ *and* $C$ *are equivalent over* $l^p$ *for* $1 < p < \infty$.

The object of this paper is to obtain sufficient conditions for $(\overline{N}, p_n)$ and $(\overline{N}, q_n)$ weighted mean matrices to the equivalent, when considered as bounded operators on $l^p$, $1 < p < \infty$.

***Theorem*** — *Let* $p_n$ *and* $q_n$ *be two positive sequences satisfying the following conditions*

$$(n+1)\left(\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right) \approx (n+1)^{\alpha}, \text{ for some } \alpha \ge 0, \qquad \text{... (9)}$$

$$\frac{p_n}{q_n} \cdot \frac{Q_n}{P_n} \le H \qquad \text{... (10)}$$

$$\frac{Q_n}{n+1} \text{ is non decreasing} \qquad \text{... (11)}$$

$$Q_n = q_0 + q_1 + \ldots\ldots + q_n \to \infty \text{ as } n \to \infty.$$

*Then* $(\overline{N}, p_n)$ *and* $(\overline{N}, q_n)$ *matrices are equivalent over* $l^p$ *for* $1 < p < \infty$ *as bounded linear operators.*

When we take $p_n = a_n$ and $q_n = 1$ in our theorem, then condition (9) reduces to condition

(8) of Rhoades, condition (10) is implied by (9) and condition (11) of our theorem is automatically satisfied.

Thus Theorem 3 is a particular case of our Theorem.

*Remarks* : In view of our Theorem, Theorem 1[3], Theorem 14] is proved for $l^p$ spaces, $1 < p < \infty$.

*Definition* 1.2 — A triangular matrix $A = (a_{nk})$ is said to be factorable if $a_{nk} = c_n d_k$, $0 \le k \le n$.

For the proof of our Theorem we need the following lemmas.

*Lemma* 1[1] — Let $1 < p < \infty$, $q$ the conjugate index of $p$.

If $c_n, d_k \ge 0$ and

$$c_n \sum_{k=0}^{n} d_k^q \le K d_n^{1/(p-1)}$$

for $n = 0, 1, ....$ and some constant $K$, then $A$ is a bounded operator on $l^p$.

*Lemma* 2 — Let $p_n$ and $q_n$ be two positive sequences satisfying the conditions (9), (10) and (11) of our theorem.

Then, for $\alpha = 0$,

(i) $P_n \approx \sum_{k=0}^{n} q_k \log(k+1)$,

(ii) $P_n \to \infty$, as $n \to \infty$,

(iii) $P_n \ge HQ_n \log(n+1)$, for some constant $H$.

*The constant $H$ may be different at different occurrences

For $\alpha > 0$,

(iv) $P_n \approx \sum_{k=0}^{n} q_k k^{\alpha}$

(v) $P_n \to \infty$, as $n \to \infty$,

and (vi) $P_n \ge H n^{\alpha} Q_n$, for some constant $H$.

(vii) $\frac{q_n}{p_n} \cdot \frac{P_n}{Q_n}$ is bounded for $\alpha \ge 0$.

PROOF : For $\alpha = 0$, by condition (9)

$$(k+1)\left(\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right) \approx 1$$

$$\Rightarrow \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} \approx \frac{1}{k+1}$$

Now taking summation

$$\sum_{k=0}^{n-1} \left( \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} \right) \approx \sum_{k=0}^{n-1} \frac{1}{k+1} \approx \log(n+1)$$

$$\Rightarrow \frac{p_n}{q_n} - \frac{p_0}{q_0} \approx \log(n+1)$$

This implies

$$\frac{p_n}{q_n} \approx \log(n+1) \Rightarrow p_n \approx q_n \log(n+1)$$

$$\Rightarrow \sum_{k=0}^{n} p_k \approx \sum_{k=0}^{n} q_k \log(k+1)$$

$$\Rightarrow P_n \approx \sum_{k=0}^{n} q_k \log(k+1)$$

$$\sum_{k=0}^{n} q_k \log(k+2)$$

$$\geq \log 2 \sum_{k=0}^{n} q_k$$

$$= \log 2\, Q_n.$$

Since $(\overline{N}, q_n)$ method is regular, so $Q_n \to \infty$ as $n \to \infty$.

Thus

$$\sum_{k=0}^{n} q_k \log(k+2) \to \infty \text{ as } n \to \infty$$

$$\Rightarrow P_n \to \infty \text{ as } n \to \infty.$$

This proves (i) and (ii).

Again using condition (10) of our theorem and the fact $\frac{p_n}{q_n} \approx \log(n+1)$, we have

$$\log(n+1)\frac{Q_n}{P_n} \le K$$

$$\Rightarrow P_n \ge HQ_n \log(n+1)$$

This proves (iii)

For $\alpha > 0$, by condition (9)

$$(k+1)\left(\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right) \approx (k+1)^{\alpha}$$

$$\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} \approx (k+1)^{\alpha-1}$$

Now taking summation

$$\sum_{k=0}^{n}\left(\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right) \approx \sum_{k=0}^{n} (k+1)^{\alpha-1}$$

$$\Rightarrow \frac{p_{n+1}}{q_{n+1}} - \frac{p_0}{q_0} \approx (n+1)^{\alpha}$$

This implies

$$\frac{p_n}{q_n} \approx n^{\alpha} \Rightarrow p_n \approx q_n n^{\alpha}$$

$$\Rightarrow \sum_{k=0}^{n} p_k \approx \sum_{k=0}^{n} q_k k^{\alpha}$$

$$\Rightarrow P_n \approx \sum_{k=0}^{n} q_k k^{\alpha}$$

$$\sum_{k=0}^{n} q_k (k+1)^{\alpha} \ge 1^{\alpha} \sum_{k=0}^{n} q_k = Q_n$$

since $(\overline{N}, q_n)$ method is regular, so $Q_n \to \infty$ as $n \to \infty$.

Thus
$$\sum_{k=0}^{n} q_n (k+1)^{\alpha} \to \infty \text{ as } n \to \infty.$$

$\Rightarrow P_n \to \infty$ as $n \to \infty$. THis proves (iv) and (v).

Again using condition (10) and $\frac{p_n}{q_n} \approx n^{\alpha}$,

we have $P_n \geq Hn^{\alpha} Q_n$. This proves (vi)

For $\alpha = 0$ we have

$$\frac{p_n}{q_n} \approx \log (n+1)$$

$$\Rightarrow p_k \approx q_k \log (k+1)$$

$$\Rightarrow \sum_{k=0}^{n} p_k \approx \sum_{k=0}^{n} q_k \log (k+1)$$

$$\Rightarrow P_n \leq HQ_n \log (n+1) \Rightarrow P_n \leq HQ_n \cdot \frac{p_n}{q_n}$$

This gives $\frac{q_n}{p_n} \cdot \frac{P_n}{Q_n} \leq K.$

For $\alpha > 0$, we get

$$\frac{p_n}{q_n} \approx n^{\alpha}$$

$$\Rightarrow p_k \approx q_k k^{\alpha}$$

$$\Rightarrow \sum_{k=0}^{n} p_k \approx \sum_{k=0}^{n} q_k k^{\alpha}$$

$$\Rightarrow P_n \leq Hn^{\alpha} Q_n \Rightarrow P_n \leq H \frac{p_n}{q_n} \cdot Q_n$$

This implies $\frac{q_n}{p_n} \cdot \frac{P_n}{Q_n} \leq K$

so $\frac{q_n}{p_n} \cdot \frac{P_n}{Q_n}$ is bounded.

This proves (vii).

1.3 Proof of the Theorem : Define

$$t_{n,q} = \frac{1}{Q_n} \sum_{k=0}^{n} q_k s_k \qquad \text{... (12)}$$

$$t_{n,p} = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \qquad \text{... (13)}$$

solving (12) for $s_n$ and then substituting into (13) we have

$$t_{n,p} = \frac{1}{P_n} \sum_{k=0}^{n} \frac{p_k}{q_k} (Q_k t_{k,q} - Q_{k-1} t_{k-1,q})$$

By summation by parts,

$$t_{n,p} = \frac{1}{P_n} \left[ \frac{p_n}{q_n} Q_n t_{n,q} + \sum_{k=0}^{n-1} \left( \frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \right) Q_k t_{k,q} \right]$$

Then $t_{n,p} = A_n (t_{n,q})$, where $A$ is the lower triangular matrix with entries

$$a_{nk} = \begin{cases} \dfrac{Q_k}{P_n} \left( \dfrac{p_k}{q_k} - \dfrac{p_{k+1}}{q_{k+1}} \right) & 0 \le k < n, \\[2ex] \dfrac{p_n}{q_n} \dfrac{Q_n}{P_n} & k = n, \\[2ex] 0 & k > n. \end{cases}$$

Obviously, $A = B + D$, where

$$b_{nk} = \frac{Q_k}{P_n} \left( \frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \right), \; 0 \le k \le n \qquad \text{... (14)}$$

and $D$ is the diagonal matrix with diagonal entries

$$d_{nn} = \frac{p_{n+1}}{q_{n+1}} \frac{Q_n}{P_n}$$

To show that $A \in B(l^p)$ it is sufficient to show that $B \in B(l^p)$ and that $D$ is bounded. In view of condition (10), $D$ is bounded.
In order to show that $B$ is bounded, we use Lemma 1.

$$\text{Choose } c_n := \frac{1}{P_n} \text{ and } d_k := Q_k \left( \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} \right) \ge 0.$$

Let $\alpha = 0$. Then condition (9) of the Theorem implies that $d_k \approx \dfrac{Q_k}{(k+1)}$ and by condition (iii) Lemma 2, we have

$$\frac{Q_n}{P_n} \le \frac{H}{\log (n+1)}$$

By Lemma 1, if we show that

$$c_n \sum_{k=0}^{n} d_k^q \le K d_n^{1/(p-1)}, \text{ then } B \text{ is bounded.}$$

$$\text{Now } \frac{c_n}{d_n^{1/(p-1)}} \sum_{k=0}^{n} d_k^q \approx \frac{1}{P_n \left[ \frac{Q_n}{n+1} \right]^{1/(p-1)}} \sum_{k=0}^{n} \left( \frac{Q_k}{k+1} \right)^q$$

$$\approx \frac{1}{P_n \left[ \frac{Q_n}{n+1} \right]^{q-1}} \sum_{k=0}^{n} \left( \frac{Q_k}{k+1} \right)^q$$

$$\le \frac{1}{P_n} \cdot \frac{(n+1)^{q-1}}{Q_n^{q-1}} \cdot \frac{Q_n^q}{(n+1)^q} \sum_{k=0}^{n} 1 \quad \text{(using condition (11))}$$

$$= \frac{Q_n}{P_n} \cdot \frac{1}{(n+1)} \cdot (n+1) = \frac{Q_n}{P_n} \le \frac{H}{\log (n+1)} \le K,$$

for some constant $K$.

Let $\alpha > 0$. Then condition (9) of theorem implies that $d_k \approx Q_k (k+1)^{\alpha - 1}$. By Lemma 1, if we show that $c_n \sum_{k=0}^{n} d_k^q \le K d_n^{1/p-1}$, then $B$ is bounded.

$$\text{Now } \frac{c_n}{d_n^{1/(p-1)}} \sum_{k=0}^{n} d_k^q \approx \frac{1}{P_n [Q_n (n+1)^{\alpha-1}]^{1/(p-1)}} \sum_{k=0}^{n} [Q_k (k+1)^{(\alpha-1)}]^q$$

$$\le \frac{K_1}{P_n Q_n^{q-1}} (n+1)^{(\alpha-1)(q-1)} \cdot \frac{Q_n^q}{(n+1)^q} \sum_{k=0}^{n} (k+1)^{\alpha q},$$

using (11), for some constant $K_1$,

$$\le \frac{K_1 Q_n}{P_n (n+1)^{\alpha q - \alpha + 1}} \cdot (n+1)^{\alpha q + 1} \le K_1 \frac{Q_n}{P_n} (n+1)^{\alpha}$$

$$\le \frac{H}{n^{\alpha}} (n+1)^{\alpha} \text{ by condition (vi) of Lemma 2.}$$

$$\le K.$$

Thus $t_{n,p} \in l^p$, whenever $t_{n,q} \in l^p$

Now we show that $t_{n,q} \in l^p$ whenever $t_{n,p} \in l^p$

By equations (12) and (13), we have

$$t_{n,q} = \frac{1}{Q_n} \sum_{k=0}^{n} \frac{q_k}{p_k} (P_k t_{k,p} - P_{k-1} t_{k-1,p})$$

By partial summation formula,

$$t_{n,q} = \frac{1}{Q_n} \left[ \frac{q_n}{p_n} P_n t_{n,p} + \sum_{k=0}^{n-1} \left( \frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}} \right) P_k t_{k,p} \right]$$

$$= E_n (t_{n,p}),$$

where $E$ is lower triangular matrix with entries

$$e_{nk} = \begin{cases} \dfrac{P_k}{Q_n} \left( \dfrac{q_k}{p_k} - \dfrac{q_{k+1}}{P_{k+1}} \right) & 0 \le k < n, \\ \dfrac{q_n}{p_n} \dfrac{P_n}{Q_n} & k = n, \\ 0 & k > n \end{cases}$$

Alternatively, we can write $E = F + G$,

where $$f_{nk} = \frac{P_k}{Q_n} \left( \frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}} \right), \quad 0 \le k \le n. \qquad \text{... (15)}$$

and $G$ is the diagonal matrix with diagonal entries

$$g_{nn} = \frac{q_{n+1}}{p_{n+1}} \frac{P_n}{Q_n}$$

By condition (vii) of Lemma 2,

$\dfrac{q_n}{p_n} \cdot \dfrac{P_n}{Q_n}$ is bounded and consequently $G$ is bounded.

Next we show that $F \in B(l^p)$

Let $c_n := \dfrac{1}{Q_n}$,

$$d_k := P_k \left( \frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}} \right)$$

By condition (9), $d_k \geq 0$.

Let $\alpha = 0$. Using condition (9), $\frac{p_n}{q_n} \approx \log(n+1)$

Now
$$d_k = P_k \left( \frac{p_{k+1} q_k - q_{k+1} p_k}{p_k p_{k+1}} \right)$$

$$\approx \frac{P_k}{k+1} \cdot \frac{q_k}{p_k} \cdot \frac{q_{k+1}}{p_{k+1}}, \text{ using condition (9).}$$

Thus
$$d_k \approx \frac{P_k}{(k+1)\log(k+1)\log(k+2)}$$

By Lemma 1, $F \in B(l^p)$ if $c_n \sum_{k=0}^{n} d_k^q \leq K d_n^{1/(p-1)}$ i.e. $\frac{c_n}{d_n^{1/(p-1)}} \sum_{k=0}^{n} d_k^q \leq K$.

First of all we estimate

$$d_k^q \approx \left[ \frac{P_k}{(k+1)\log(k+1)\log(k+2)} \right]^q$$

$$\approx \left[ \frac{\sum_{\nu=0}^{k} q_\nu \log(\nu+1)}{(k+1)\log(k+1)\log(k+2)} \right]^q, \text{ by condition (i) of Lemma 2.}$$

$$\leq \frac{K_2 Q_k^q \log^q(k+1)}{(k+1)^q \cdot \log^q(k+1)\log^q(k+2)} = \frac{K_2 Q_k^q}{(k+1)^q \cdot \log^q(k+2)}$$

for some constant $K_2$.

$$\frac{c_n}{d_n^{1/p-1}} \sum_{k=0}^{n} d_k^q$$

$$\leq \frac{K_2}{Q_n \dfrac{P_n^{q-1}}{(n+1)^{q-1} \cdot \log^{q-1}(n+1) \cdot \log^{q-1}(n+2)}} \sum_{k=0}^{n} \frac{Q_k^q}{(k+1)^q \log^q(k+2)}$$

$$= \frac{K_2 (n+1)^{q-1} \cdot \log^{q-1}(n+1) \cdot \log^{q-1}(n+2)}{Q_n P_n^{q-1}} \sum_{k=0}^{n} \frac{Q_k^q}{(k+1)^q \log^q (k+2)}$$

$$\leq H \frac{(n+1)^{q-1} \cdot \log^{q-1}(n+1) \cdot \log^{q-1}(n+2)}{Q_n Q_n^{q-1} \log^{q-1}(n+1)} \sum_{k=0}^{n} \frac{Q_k^q}{(k+1)^q \log^q (k+2)},$$

by condition (iii) of Lemma 2.

$$\leq H \frac{(n+1)^{q-1} \cdot \log^{q-1}(n+2)}{Q_n^q} \frac{Q_n^q}{(n+1)^q} \sum_{k=0}^{n} \frac{1}{\log^q (k+2)}, \text{ by condition (11)}$$

$$= H \frac{\log^{q-1}(n+2)}{(n+1)} \sum_{k=0}^{n} \frac{1}{\log^q (k+2)}$$

$$= H \frac{\log^{q-1}(n+2)}{n+1} \sum_{k=0}^{n} \frac{k+2}{(k+2) \log^q (k+2)}$$

$$= \frac{HO(1) \log^{q-1}(n+2)}{\log^{q-1}(n+2)}$$

$\leq K$, for some constant $K$.

Let $\alpha > 0$. Then $c_n := \frac{1}{Q_n}$

and $$d_k := P_k \left( \frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}} \right)$$

Using condition (9) $d_k \geq 0$ and $\frac{p_n}{q_n} \approx n^\alpha$

Now $$d_k = P_k \left( \frac{p_{k+1} q_k - q_{k+1} p_k}{p_k p_{k+1}} \right)$$

$$\approx P_k \frac{q_k q_{k+1} (k+1)^{\alpha-1}}{p_k p_{k+1}}, \text{ using condition (9)}$$

$$d_k \approx \frac{P_k}{k^\alpha (k+1)}$$

By Lemma 1,

$$\frac{c_n}{d_n^{1/p-1}} \sum_{k=0}^{n} d_k^{q} \approx \frac{1}{Q_n \left[ \dfrac{P_n}{n^{\alpha}(n+1)} \right]^{q-1}} \sum_{k=0}^{n} \left[ \frac{P_k}{k^{\alpha}(k+1)} \right]^{q}$$

$$\approx \frac{n^{\alpha(q-1)}(n+1)^{q-1}}{Q_n P_n^{q-1}} \sum_{k=0}^{n} \frac{P_k^q}{k^{\alpha q}(k+1)^q}$$

$$\leq \frac{Hn^{\alpha(q-1)}(n+1)^{q-1}}{Q_n (n^{\alpha} Q_n)^{q-1}} \sum_{k=0}^{n} \frac{P_k^q}{k^{\alpha q}(k+1)^q}, \text{ by condition (vi) of lemma 2}$$

$$\approx \frac{H(n+1)^{q-1}}{Q_n^q} \sum_{k=0}^{n} \frac{\left[ \sum_{\nu=0}^{k} q_{\nu} \nu^{\alpha} \right]^q}{k^{\alpha q}(k+1)^q}, \text{ by condition (iv) of lemma 2}$$

$$\leq \frac{H(n+1)^{q-1}}{Q_n^q} \sum_{k=0}^{n} \frac{Q_k^q k^{\alpha q}}{k^{\alpha q}(k+1)^q}$$

$$\leq \frac{H(n+1)^{q-1}}{Q_n^q} \frac{Q_n^q}{(n+1)^q} \sum_{k=0}^{n} 1 \quad \text{by condition (11) of the theorem}$$

$$\leq K.$$

Therefore $F \in B(l^p)$, $1 < p < \infty$.

This proves the theorem.

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## REFERENCES

1. G. Bennett, *Quart. J. Math.*, **38** (1987) pp 401-25.
2. D. Borwein and F. P. A. Cass, *Math. Z.* **103** (1968) pp. 94-111.
3. G. H. Hardy, *Divergent series*, Oxford, 1949.
4. B. E. Rhoades, *Indian J. Math.*, **36**, (1994) pp 1-6.