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APPLICATIONS OF KKM-MAP PRINCIPLE

A. CARBONE

Departmento de Matematics, Universiá Degij Studi Della Calabra 87036, *Arcayacata Di Rende (Cosenza), Italy; E-mail: Carbonea@unical.it*

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An extension of a result due to Prolla[5] is given and a few results are derived as corollaries

Key Words : KKM Maps; Almost Affine Maps; Fixed Points

The following important result due to Ky Fan[1] has applications in Fixed Point Theory, Approximation Theory, Minimax Theory and Variational Inequalities.

Theorem 1 — *Let K be a nonempty subset of a Hauadorff topological vector space E. For each* $x \in K$ *let F(x) be a closed subset of K such that*

1) *the convex hull of any finite subset* $\{x_1, x_2, \ldots, x_n\}$ *of K is contained in* $\bigcup_{i=n}^{n} Fx_i$

2) $F(x)$ *is compact for some* $x \in K$

Then $\bigcap_{x \subset K} F(x) \neq 0$.

Prolla[5] proved the following theorem using tools from approximation theory and the generalized form of the Kakutani fixed point theorem.

Theorem 2 — *Let* $X \subset E$ *be a non-empty, compact, convex subset of a normed linear space E and let g be a continuous almost affine self map of X onto X. Then for each continuous map* $f : X \to E$ *there is some* $x \in X$ *such that*

$$\| g(x) - f(x) \| = d(f(x), X).$$

The aim of this paper is to extend the result of Prolla and derive corollaries.

Our proof depend on Ky Fan's Theorem 1.

We need the following definitions.

A function f is said to be *strongly continuous* if $x_n \to x$ weakly implies that $fx_n \to fx$ strongly. A function $g : X \to E$, where X is non-empty convex set, is said to be *almost affine* if

$$\| g(t) - y \| \leq \lambda \| g(t_1) - y \| + (1 - \lambda) \| g(t_2) - y \|$$

for all $t_1, t_2 \in X, 0 < \lambda < 1, t = \lambda t_1 + (1 - \lambda) t_2$ and $y \in E$.

A function $F : X \to 2^X$ is called a *KKM-map* if co $\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} Fx_i$ for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of X.

For details on KKM-maps, see Granas[3].

We state our main result.

***Theorem* 3** — *Let X be a non-empty, convex, weakly compact subset of a normed linear space E and let $g : X \to X$ be a strongly continuous, almost affine, onto map.*

Then for each strongly continuous map $f : X \to E$, there exists a $y_0 \in X$ such that

$$\| g(y_0) \cdot f(y_0) \| = d(f(y_0), x).$$

PROOF : Define $G : X \to 2^E$ by

$$G(x) = \{y : y \in X, \| g(y) - f(y) \| < \| (g(x) - f(y) \| \}.$$

Since f and g are strongly continuous functions $G(x)$ is weakly-closed in X and since X is weakly compact $G(x)$ is weakly-compact. Since g is an almost affine map so G is a KKM-map.

Therefore $\bigcap_{x \in X} G(x) \neq 0$ by Theorem 1 using the weak topology.

Hence, there is a $y_0 \subset X$ such that

$$\| g(y_0) - f(y_0) \| = \inf_{x \in X} \| g(x) - f(y_0) \|.$$

It is given that $g : X \to X$ is onto so

$$\| g(y_0) - f(y_0) \| = d(f(y_0), X).$$

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Note: In case $f(y_0) \subset X$ then we get $f(y_0) = g(y_0)$, a coincidence theorem.

We give the following

***Theorem* 4** — *Suppose X is a non-empty, convex, weakly-compact subset of a normed space E and let $f : X \to E$ be a strongly continuous, almost affine onto map. Let $f : X \to E$ be a strongly continuous map such that for each $x \subset X$ with $g(x) \neq f(x)$, the line segment*

$$[g(x), f(x)] = \{\lambda g(x) + (1 - \lambda) f(x), 0 < \lambda < 1\}$$

contains at least two points of X.

Then there is a $y_0 \in X$ such that $g(y_0) = f(y_0)$.

PROOF : By Theorem 3, we have

$$\| f(y_0) - g(y_0) \| = d(f(y_0), X).$$

If $f(y_0) \neq g(y_0)$ then we get a contradiction. For example, let $x = \lambda g(y_0) + (1 + \lambda) f(y_0)$ for some $\lambda \in (0, 1)$.

Then

$$\| g(y_0) - f(y_0) \| \leq \| \lambda g(y_0) + (1 - \lambda) f(y_0) \| = \lambda \| g(y_0) - f(y_0) \| < \| g(y_0) - f(y_0) \|,$$

a contradiction.

This implies that $g(y_0) = f(y_0)$.

Remarks : In case $g = I$ an identity function, f is continuous and X is a compact, convex subset of E, then we get the result due to Ky Fan[2] stated below.

Theorem 5 — *Let X be a non-empty compact, convex subset of a normed linear space E and let $f: X \rightarrow E$ be a continuous function.*

Then there exists a $y_0 \in X$ such that $\| y_0 - f(y_0) \| = d(f(y_0), X)$.

PROOF : It is suffices to show that f is strongly continuous. Let A be a closed subset of E. Since f is continuous $f^{-1}(A)$ is closed in X. As X is a compact set so $f^{-1}(A)$ is compact implies that it is weakly compact and therefore $f^{-1}(A)$ is weakly closed.

If X is compact, convex and f and g are continuous functions as in Theorem 2 then we get the result due to Prolla[5].

In case $g = 1$, an identity function, then we get the result due to Kapoor[4] given below.

Theorem 6 — *Let X be a non-empty, convex, weakly-compact subset of a normed linear space E and let $f: X \rightarrow E$ be a strongly continuous map.*

Then there is a $y_0 \in X$ such that

$$\| y_0 \cdot f(y_0) \| = \min_{x \in \lambda} \| x - f(y_0) \|.$$

The following result in approximation theory follows easily from Theorem 6.

Let X be a weakly-compact, convex ans subset of a normed linear space E.

Then for each point $x \in E$ there exists a $y_0 \in X$ such that

$$\| x - y_0 \| = d(x, X).$$

For a further extension of Theorem 6, see [6].

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