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OSCILLATION PROPERTIES OF SOLUTIONS OF SECOND ORDER DIFFERENCE EQUATIONS

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The authors obtain results on the oscillation properties of solutions of second order non-linear difference equations. Examples illustrating the results are inserted.

Key Words : Second Order Difference Equation; Oscillation

INTRODUCTION

Recently there has been a growing interest towards the study of oscillatory and asymptotic behaviour of solutions of difference equations, for example see [1, 2, 4] and the references cited therein. Following this trend, in this paper we consider the difference inequality (*I*) and the difference equation (*E*) :

$$\Delta (r_n \Delta (a_n y_n) + f(n, y_{n+1}) \leq e_n, n \in IN(n_0) \qquad \text{... (I)}$$

and

$$\Delta (r_n \Delta a_n y_n) + f(n, y_{n-1}) = e_n, n \in IN(n_0), \qquad \text{... (E)}$$

where $IN(n_0) = \{n_0, n_0 + 1, ...\}$, n_0 is a non-negative integer, Δ denotes the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\{r_n\}$, $\{a_n\}$ and $\{e_n\}$ are sequences of real numbers, and obtain some new oscillation results for (*I*) and (*E*).

By a solution of (*I*) (or (*E*)) we mean a sequence of real numbers which is defined for $n \geq n_0 \geq 0$ and which satisfies (*I*) (or (*E*)) for $n \geq n_0$. A solution $\{y_n\}$ of (*I*) or (*E*) is called eventually positive (or eventually negative) if there exists an integer $n_1 \geq n_0$ such that $y_n > 0$ (or $y_n < 0$) should hold for any $n \geq n_1$. A solution $\{y_n\}$ of (*E*) is said to be oscillatory if the terms y_n are neither eventually positive nor eventually negative.

The results obtained in this paper are partially motivated by that of in[3, 4].

2. MAIN RESULTS

Assume that the following conditions are hold without further mention:

(c_1) $\{r_n\}$ and $\{a_n\}$ are positive real sequences.

(c_2) $f : N(n_0) \times IR \rightarrow I R_+ (I R_+ = (0, \infty))$ is continuous and non-decreasing in the second variable.

Now we consider two possible cases for $\{r_n\}$.

Case 1 — $\sum_{n=n_0}^{\infty} \frac{1}{r_n} = \infty.$

Theorem 1 — *Assume that (i) there exists a real sequence $\{\delta_n\}$ such that*

$$\Delta (r_n \Delta (a_n \delta_n)) = e_n \quad \dots (1)$$

and $\{\delta_n\}$ *is of alternate sign;*

$$(ii) \liminf_{n \to \infty} \frac{1}{Q(n, N)} \sum_{s=N}^{n-1} Q(n, s) (e_s - f(s, \delta_{s+1}^{+})) = -\infty \quad \dots (2)$$

for all sufficiently large integer $N > n_0$, where

$$Q(n, s) = \sum_{t=s}^{n-1} \frac{1}{r_t}, n > s \in IN(n_0)$$

and $\delta_s^{+} = \max \{\delta_s, 0\}.$

Then inequality (I) has no eventually positive solution for $n \geq N$ for any $N \in IN(n_0)$.

PROOF : Assume to the contrary that there is an eventually positive solution $\{y_n\}$ of (*I*), then there exists an integer $n_1 \geq n_0$ such that $y_n > 0$ for all $n \geq n_1$. From (*I*) and (1), we have

$$\Delta (r_n \Delta (a_n y_n) - r_n \Delta (a_n \delta_n)) \leq 0,$$

which implies that either $\Delta (a_n y_n - a_n \delta_n) \leq 0$ or $\Delta (a_n y_n - a_n \delta_n) \geq 0$ eventually. Thus there are two possible cases: either $a_n y_n - a_n \delta_n \geq 0$ or $a_n y_n - a_n \delta_n \leq 0$ eventually. Since $\{\delta_n\}$ is oscillatory and $\{y_n\}$ is positive, we must have

$$y_n \geq \delta_n^{+}. \quad \dots (3)$$

From condition (c_2) and (3), we have

$$f(n, y_{n+1}) \geq f(n, \delta_{n+1}^{+}). \quad \dots (4)$$

Substituting (4) into (I), we have

$$a_n y_n \leq d_1 + d_2 Q(n, n_1) + \sum_{s=n_1}^{n-1} Q(n, s) (e_s - f(s, \delta_{s+1}^{+})). \quad \dots (5)$$

In view of (2), (5) implies,

$$\liminf_{n \to \infty} \frac{a_n y_n}{Q(n, n_1)} = -\infty$$

a contradiction. The proof is now complete.

Example 1 — Consider the difference inequality

$$\Delta^2 y_n + n\, y_{n+1} \le (-1)^n. \qquad \text{... (6)}$$

Here $e_n = (-1)^n,\ \delta_n = \frac{(-1)^n}{4},\ r_n = a_n = 1$. It is easy to see that

$$\liminf_{n \to \infty} \frac{1}{Q(n, N)} \sum_{s=N}^{n-1} Q(n, s)\,(e_s - f(s, \delta^+_{s+1}))$$

$$= \liminf_{n \to \infty} \frac{1}{n-N} \sum_{s=N}^{n-1} (n-s) \left((-1)^s - \frac{s\,((-1)^{s+1})^+}{4} \right)$$

$$= -\infty.$$

Therefore by Theorem 1, the inequality (6) has no eventually positive solution.

Case 2 — $\sum_{n=n_0}^{\infty} \frac{1}{r_n} < \infty$.

Theorem 2 — *In addition to condition (i) of Theorem 1, assume that*

$$\sum_{n=N}^{\infty} \frac{1}{r_n} \sum_{s=N}^{n-1} f\left(s, \left(\frac{d\,\lambda_{s+1}}{a_{s+1}} + \delta^+_{s+1} \right) \right) = \infty \ \text{for all } d > 0, \qquad \text{... (7)}$$

where $\lambda_n = \sum_{s=n}^{\infty} \frac{1}{r_s}$. *Then inequality (I) has no eventually positive solution for* $n \ge IN$ *for any* $N \in IN\,(n_0)$.

PROOF : If no, let $\{y_n\}$ be an eventually positive solution of (*I*) for $n \ge N$. Let $z_n = y_n - \delta_n$ then,

$$\Delta\,(r_n\, \Delta\,(a_n z_n)) \le -f(n, y_{n+1}) \le 0 \ \text{for } n \ge N. \qquad \text{... (8)}$$

Hence, $$r_n\, \Delta\,(a_n z_n) \le r_{N_1}\, \Delta\,(a_{N_1} z_{N_1}) \ \text{for } n \ge N_1 \ge N. \qquad \text{... (9)}$$

Dividing (9) by r_n and summing it we have

$$a_n z_n - a_{N_1} z_{N_1} \le r_{N_1} \Delta (a_{N_1} z_{N_1}) \sum_{s=N_1}^{n-1} \frac{1}{r_s}, n \ge N_1 \ge N.$$

This implies that $\{a_n z_n\}$ is bounded above and

$$a_{N_1} z_{N_1} \ge - r_{N_1} \Delta (a_{N_1} z_{N_1}) \lambda_{N_1} \text{ for } n \ge N_1 \ge N; \qquad \text{.. (10)}$$

since, by the same reason as in the proof of Theorem 1, we must have $z_n > 0$ eventually.

Now there are two possible cases: First we consider the case $\Delta (a_n z_n) > 0$ for $n \ge N_1$. From (8), we obtain

$$\sum_{n=N_1}^{\infty} f(n, y_{n+1}) < \infty. \qquad \text{... (11)}$$

Since $a_n z_n \ge a_{N_2} z_{N_2} = d > 0$ for $n \ge N_2 N_2 \ge N_1$

we obtain $\quad z_n \ge \dfrac{d}{a_n}$

and $\quad y_n = z_n + \delta_n \ge \left(\dfrac{d}{a_n} + \delta_n^+ \right).$ ... (12)

Substituting (12) in (11) we have

$$+ \sum_{n=N_1}^{\infty} f\left(n, \left(\frac{d}{a_{n+1}} + \delta_{n+1}^+ \right)\right) < \infty. \qquad \text{... (13)}$$

Since $\lambda_n \to 0$ as $n \to \infty$ and

$$\sum_{n=N}^{\infty} \frac{1}{r_n} \sum_{s=N}^{n-1} f\left(s \left(\frac{d}{a_{s+1}} + \delta_{s+1}^+ \right)\right) \le \sum_{n=N}^{\infty} \frac{1}{r_n} \sum_{n=N}^{\infty} f\left(n, \left(\frac{d}{a_{n+1}} + \delta_{n+1}^+ \right)\right)$$

we obtain $\quad \displaystyle\sum_{n=N}^{\infty} \frac{1}{r_n} \sum_{s=N}^{n-1} f\left(s, \left(\frac{d}{a_{s+1}} + \delta_{s+1}^+ \right)\right) < \infty,$

which contradicts (7).

Now, we consider the second case that $\Delta (a_n z_n) < 0$ eventually. In view of (8) and (10) we have

$$a_n z_n \ge - r_n \Delta (a_n z_n) \lambda_n \ge l \lambda_n$$

where l is a positive constant. Therefore,

$$z_n \geq \frac{l\,\lambda_n}{a_n}$$

and
$$y_n = z_n + \delta_n \geq \left(\frac{l\,\lambda_n}{a_n} + \delta_n^+ \right). \qquad (14)$$

From (8) and (14) we have

$$\sum_{s=N}^{n-1} \frac{1}{r_s} \sum_{j=N}^{s-1} f\left(j, \left(\frac{l\,\lambda_{j+1}}{a_{j+1}} + \delta_{j+1}^+ \right) \right) \leq a_N z_N < \infty$$

which contradicts (7), the proof of this theorem is now complete.

Example 2 — Consider the difference inequality

$$\Delta\,(n(n+1)\,\Delta\,y_n) + n^3\,y_{n+1} \leq \frac{4(-1)^n}{n(n+2)}. \qquad \text{... (15)}$$

Here $r_n = n(n+1)$, $a_n = 1$, $f(n, y_{n+1}) = n^3\,y_{n+1}$, $\delta_n = (-1)^n$ and $\lambda_n = 1/n$. It is easy to see that all conditions of Theorem 2 are satisfied and therefore (15) has no positive solution for $n \geq N$.

Theorem 3 — *In addition to condition (i) of theorem 1, assume that*

(c_3) $f(n, -u) = -f(n, u)$ *for all* $n \in IN(n_0)$ *and* $u \in IR_+$ *then either*

$$\sum_{n=n_0}^{\infty} \frac{1}{r_n} = \infty \text{ and condition (2)} \qquad \text{... (16)}$$

or
$$\sum_{n=n_0}^{\infty} \frac{1}{r_n} < \infty \text{ and condition (7)} \qquad \text{... (17)}$$

implies that every solution of equation (E) is oscillatory.

PROOF : Let $\{y_n\}$ be a non-oscillatory solution of (E), that is, $\{y_n\}$ is either eventually positive or eventually negative. If $y_n > 0$ for $n \geq IN(n_0)$ then $\{y_n\}$ is an eventually positive solution of (I), which contradicts Theorem 1 if condition (16) holds and Theorem 2 if condition (17) holds. If $y_n < 0$ for $n \geq N$ then $u_n = -y_n$ is a positive solution of the equation.

$$\Delta\,(r_n\,\Delta\,(a_n\,u_n)) + f(n, u_{n+1}) = -e_n,\ n \geq N.$$

As in the case $y_n > 0$, we are led to a contradiction. Then the theorem is proved.

We conclude this paper with the following example.

Example 3 — The difference equation

$$\Delta^2 y_n + 3 y_{n+1} = (-1)^n \qquad \ldots (18)$$

satisfies condition (16) of Theorem 3 and hence every solution of (18) is oscillatory. In fact $\{y_n\} = \{(-1)^n\}$ is one such solution of eq. (18).

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